ON $(J,\ p_n)$-SUMMABILITY OF FOURIER SERIES

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(Received 13th April 1971)

1. Let $p_n > 0$ be such that $\sum_{n=0}^{\infty} p_n$ diverges, and the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \tag{1.1}$$

is 1. Given any series Σa_n with partial sums s_n, we shall use the notation

$$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n, \tag{1.2}$$

and

$$J_s(x) = p_s(x)/p(x). \tag{1.3}$$

If the series on the right of (1.2) is convergent in the right open interval [0, 1), and if

$$\lim_{x \to 1-0} J_s(x) = s,$$

we say that the series Σa_n or the sequence $\{s_n\}$ is summable (J, p_n) to s, where s is finite ((**1**); (**2**), page 80).

Particular cases of this method of summability are

(*a*) the Abel method: when $p_n = 1$, for all n;

(*b*) the (A_k)-method: when p_n is given by

$$(1-x)^{-k-1} = \sum_{n=0}^{\infty} p_n x^n, \text{ for } k > -1, (|x| < 1);$$

(*c*) the logarithmic method (L): when p_n is given by

$$x^{-1} \log (1-x)^{-1} = \sum_{n=0}^{\infty} p_n x^n.$$

2. Suppose that $f(x)$ is a Lebesgue integrable function, periodic with period 2π. Let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Fixing x_0, we write

$$\phi(t) = \phi_{x_0}(t) = \tfrac{1}{2}\{f(x_0+t) + f(x_0-t) - 2s\}.$$

https://doi.org/10.1017/S0013091500026109 Published online by Cambridge University Press

In a recent paper Hsiang (3) has applied the (L)-method to Fourier series and has proved the following theorems.

Theorem A. *A necessary and sufficient condition for the Fourier series of $f(x)$ to be summable (L) to the sum s, at the point x_0, is that*

$$\int_0^\pi \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t}\, dt = o(|\log(1-x)|),$$

as $x \to 1-0$.

Theorem B. *The (L)-summability of the Fourier series of $f(x)$, at x_0, is a local property of $f(x)$ near x_0.*

Theorem C. *If*

$$\text{(i)} \int_0^t |\phi(u)|\, du = o\left(t \log \frac{1}{t}\right), \quad (t \to +0),$$

$$\text{(ii)} \int_t^\delta (|\phi(u)|/u) du = o\left(\log \frac{1}{t}\right),$$

as $t \to +0$ for any arbitrary δ, $0 < \delta < \pi$, then the Fourier series of $f(x)$ is summable (L) to s at x_0.

The object of this note is to generalise the above results by proving corresponding theorems for (J, p_n)-summability. We establish the following theorem.

Theorem 1. *A necessary and sufficient condition for the Fourier series of $f(x)$ to be summable (J, p_n) to the sum s, at the point x_0, is that*

$$\int_0^\delta \frac{\phi(t)}{t} \operatorname{Im} p(xe^{it}) dt = o(p(x)),$$

for any arbitrary δ, $0 < \delta < \pi$, as $x \to 1-0$.

3. Proof of the Theorem 1

Let

$$s_n(x_0) = \tfrac{1}{2} a_0 + \sum_{v=1}^{n} (a_v \cos v x_0 + b_v \sin v x_0)$$

be the nth partial sum of the Fourier series of $f(x)$ at x_0. Then we have

$$s_n(x_0) - s = \frac{2}{\pi} \int_0^\pi \frac{\phi(t)}{t} \sin nt\, dt + o(1).$$

https://doi.org/10.1017/S0013091500026109 Published online by Cambridge University Press

Thus

$$\sum_{n=0}^{\infty} p_n\{s_n(x_0)-s\}x^n = \frac{2}{\pi}\sum_{n=0}^{\infty} p_n x^n \int_0^{\pi} \phi(t)\frac{\sin nt}{t}\,dt + o\left(\sum_{n=0}^{\infty} p_n x^n\right)$$

$$= \frac{2}{\pi}\sum_{n=0}^{\infty} p_n x^n \int_0^{\delta} \frac{\phi(t)}{t}\sin ntdt$$

$$+\frac{2}{\pi}\sum_{n=0}^{\infty} p_n x^n \int_{\delta}^{\pi} \frac{\phi(t)}{t}\sin ntdt + o(p(x))$$

$$= \frac{2}{\pi}\int_0^{\delta}\frac{\phi(t)}{t}\sum_{n=0}^{\infty} p_n x^n \sin ntdt + o(p(x))$$

since

$$\int_{\delta}^{\pi}\phi(t)\frac{\sin nt}{t}\,dt \to 0, \text{ as } n\to\infty,$$

by the Riemann-Lebesgue theorem, and hence

$$\sum_{n=0}^{\infty} p_n x^n \int_{\delta}^{\pi}\frac{\phi(t)}{t}\sin ntdt = o(p(x)),$$

by the regularity of the method (J, p_n).

Now,

$$\sum_{n=0}^{\infty} p_n\{s_n(x_0)-s\}x^n = \sum_{n=0}^{\infty} s_n(x_0)p_n x^n - sp(x)$$

$$= p_s(x) - sp(x).$$

Hence, the sequence $\{s_n(x_0)\}$ is summable (J, p_n) to s if and only if

$$\int_0^{\delta}\frac{\phi(t)}{t}\,\mathrm{Im}\,p(xe^{it})dt = o(p(x)),$$

for any arbitrary δ, $0<\delta<\pi$, as $x\to 1-0$.

This establishes Theorem 1.

4. From the proof of Theorem 1, we get the following almost self evident result.

Theorem 2. *The (J, p_n)-summability of the Fourier series of $f(x)$ at x_0, is a local property of $f(x)$ near x_0,* i.e.

$$p_s(x) = \frac{2}{\pi}\int_0^{\delta}\frac{\phi(t)}{t}\,\mathrm{Im}\,p(xe^{it})dt + o(p(x)),$$

for any arbitrary δ, $0<\delta<\pi$, as $x\to 1-0$.

5. Next, we derive a criterion for (J, p_n)-summability for the Fourier series of $f(x)$ at x_0 as follows.

Theorem 3. *Let the sequence $\{p_n\}$ be positive and decreasing steadily to zero, such that $\{np_n\}$ is bounded. If*

(i) $\int_0^t |\phi(u)|\, du = o(tp(1-t)),\ (t \to +0),$

(ii) $\int_t^\delta \frac{|\phi(u)|}{u}\, du = o(p(1-t)),$

as $t \to +0$, for any arbitrary δ, $0 < \delta < \pi$, then the Fourier series of $f(x)$ is summable (J, p_n) to s at x_0.

6. Proof of Theorem 3

We write

$$\int_0^\delta \frac{\phi(t)}{t} \operatorname{Im} p(xe^{it})dt = \int_0^{1-x} + \int_{1-x}^\delta$$
$$= J_1(x) + J_2(x),$$

say. Then, since

$$\lim_{t \to +0} \frac{1}{t} \sum_{n=0}^\infty p_n x^n \sin nt = \lim_{t \to +0} \Sigma np_n x^n \frac{\sin nt}{nt}$$
$$= O(\Sigma np_n x^n)$$
$$= O\left(\frac{1}{1-x}\right),$$

by hypothesis, we have by (i)

$$J_1(x) = O\left(\frac{1}{1-x}\right) \int_0^{1-x} |\phi|\, dt$$
$$= O\left(\frac{1}{1-x}\right) o\{(1-x)p(x)\}$$
$$= o(p(x)),$$

as $x \to 1-0$. Observing that

$$\sum_{n=0}^\infty p_n x^n \sin nt = O(1),$$

uniformly for $0 \leqq x < 1$ and $0 < t \leqq \pi$, which follows from an example of Titchmarsh ((4), p. 5), since $\{p_n x^n\}$ is positive and decreases steadily to zero uniformly for $0 \leqq x < 1$, we find from (ii)

$$J_2(x) = O\left(\int_{1-x}^\delta \frac{|\phi|}{t}\, dt\right)$$
$$= o(p(x)),$$

as $x \to 1-0$. Hence, Theorem 3 follows from Theorem 1.

The author expresses his sincerest thanks to Dr. Z. U. Ahmad for his generous encouragement and valuable advice, and to the referee for his valuable suggestions for the improvement of this note.

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