A CHARACTERIZATION OF THE ESSCHER-TRANSFORMATION

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1. INTRODUCTION

One of the central problems in risk theory is the calculation of the distribution function F of aggregate claims of a portfolio. Whereas formerly mainly approximation methods could be used, nowadays the increased speed of the computers allows application of iterative methods of numerical mathematics (see Bertram (1981), Kupper (1971) and Strauss (1976)). Nevertheless some of the classical approximation methods are still of some interest, especially a method developed by Esscher (1932).

The idea of this so called Esscher-approximation (see Esscher (1932), Grandell and Widaeus (1969) and Gerber (1980)) is rather simple:

In order to calculate $1-F(x)$ for large x one transforms F into a distribution function $\bar{F}$ such that the mean value of $\bar{F}$ is equal to x and applies the Edgeworth expansion to the density of $\bar{F}$. The reason for applying the transformation is the fact that the Edgeworth expansion produces good results for x near the mean value, but poor results in the tail (compare also Daniels (1954)).

Nevertheless at first sight the transformation $F \to \bar{F}$, introduced by Esscher, looks a little bit artificial and one would like to have a characterization of this transformation, showing its significance. In the present rather short note we state the desired characterization.

For those familiar with information theoretical statistics the following few lines might appear to be unnecessary, but we think that they are of interest for the theorists among the actuaries.

2. NOTATIONS

Let F be a distribution function of a random variable X (the total claims amount of a risk) with bounded range. Under a premium principle one understands a functional which assigns a real number to each F.

Especially the functional:

$$\mathrm{NRP}\,(F) := E(X)$$

is the so called *net premium principle* and the functional:

$$\mathrm{EXP}_c\,(F) := \frac{1}{c} \cdot \ln E(\exp\,(c \cdot X))$$

the *exponential premium principle* with parameter $c > 0$ (see Gerber (1980), pp. 66–68). For $c > 0$ define F_c as distribution function with:

$$dF_c = \frac{\exp\,(ct)}{\int \exp\,(ct) F(dt)}\, dF. \tag{2.1}$$

In risk theory the transformation $F \to F_c$ is usually denoted as *Esscher transformation* (with parameter c) (see GERBER (1980), p. 62). Finally define for distribution functions F, G:

$$K(G, F) := \int \ln \left(\frac{g}{f}\right) g \, d\mu$$

where f, g are densities of the distributions corresponding to F, G with respect to a dominating measure μ. The quantity K plays an important role in information theory and mathematical statistics and is usually called *Kullback Leibler information number* (see KULLBACK (1959)). It can be interpreted as a measure for discrimination between G and F, e.g., as a sort of distance of G from F. Especially one has:

$$K(G, F) \geq 0$$

$$K(G, F) = 0 \quad \text{iff } F = G$$

(see KULLBACK (1959), pp. 14–15).

3. THE CHARACTERIZATION

Now let x be a real number with

$$x > \text{NRP}\,(F)$$

and let $c = c(x) > 0$ be a solution of (for existence see DANIELS (1954), p. 638):

(3.1) $$\text{NRP}\,(F_c) = x$$

(notice NRP (F_c) is the so called *Esscher premium principle* with parameter c applied to F (see BÜHLMANN (1980))). The transformation $F \to F_{c(x)}$ is just the transformation used in the Esscher-approximation for calculating $1 - F(x)$. One has:

Theorem

Let μ be a measure dominating (the distribution of) F. Denote by $\mathcal{R}(\mu)$ the set of all distribution functions dominated by μ. Then holds with F_c of (3.1):

$$K(F_c, F) = \min \{K(G, F) \colon G \in \mathcal{R}(\mu), \text{NRP}\,(G) \geq x\}$$

more exactly:

(3.2) $$K(F_c, F) = c(x - \text{EXP}_c\,(F)).$$

Remark

This theorem is nothing else than a slightly modified reformulation of theorem 2.1 in KULLBACK (1959, p. 38). For sake of completeness we give an easy proof.

Proof

Denote the μ-densities of F, F_c, G by f, f_c, g. Assume the distribution function G is dominated by F (one has otherwise $K(G, F) = \infty$). Jensen's inequality implies:

$$\int \ln\left(\frac{f_c}{g}\right) g \, d\mu \leq \ln\left(\int_{\{g>0\}} f_c \, d\mu\right) \leq 0,$$

consequently:

$$\int \ln(f_c) g \, d\mu \leq \int \ln(g) g \, d\mu. \tag{3.3}$$

By (2.1) one has:

$$f_c = \frac{\exp(ct)}{\int \exp(ct) F(dt)} f,$$

yielding with (3.3) and the definition of K:

$$K(G, F) \geq c(\text{NRP}(G) - \text{EXP}_c(F)).$$

Now the statement follows by proving (3.2).

Consequently the Esscher transformation yields among all distribution functions with net premium x that distribution function which is most similar to F. The distance of this distribution function from F is proportional to the difference of the Esscher premium and the exponential premium applied to F (see (3.1), (3.2)).

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