A NOTE ON GV-MODULES WITH KRULL DIMENSION

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Abstract. Extending a result of Boyle and Goodearl in **[1]** on V-rings it was shown in Yousif **[11]** that a generalized V-module (GV-module) has Krull dimension if and only if it is noetherian. Our note is based on the observation that every GV-module has a maximal submodule (Lemma 1). Applying a theorem of Shock **[6]** we immediately obtain that a GV-module has *acc* on essential submodules if and only if for every essential submodule $K \subset M$ the factor module M/K has finitely generated socle. Yousif's result is obtained as a corollary.

Let R be an associative ring with unity and R-*Mod* the category of unital left R-modules. *Soc* M denotes the *socle* of an R-module M. If $K \subset M$ is an essential submodule we write $K \trianglelefteq M$.

An R-module M is called *co-semisimple* or a *V-module,* if every simple module is M-injective (**[2]**, **[7]**, **[9]**, **[10]**). According to Hirano **[3]** M is a *generalized V-module* or *GV-module,* if every singular simple R-module is M-injective. This extends the notion of a *left GV-ring* in Ramamurthi-Rangaswamy **[5]**.

It is easy to see that submodules, factor modules and direct sums of co-semisimple modules (GV-modules) are again co-semisimple (GV-modules) (e.g. **[10**, § 23]).

1. Lemma. *Every GV-module has a maximal submodule.*

Proof. If M is semisimple it has a maximal submodule. If M is not semisimple there is an $m \in M$ with Rm not semisimple. Then Rm contains an essential maximal submodule K. Since M is a GV-module the factor module Rm/K is M-injective and hence a direct summand in M/K. It follows that M/K, and hence M, has a maximal submodule.

With this result we can easily prove the following theorem.

2. Theorem. *For a GV-module M the following conditions are equivalent*:

(a) *M has acc on essential submodules*;

(b) *M/K has finitely generated socle for every $K \trianglelefteq M$*;

(c) *M/K has finite uniform dimension for every $K \trianglelefteq M$*;

(d) *$M/Soc\,M$ has Krull dimension*;

(e) *$M/Soc\,M$ is noetherian.*

Proof. (a)$\Leftrightarrow$(e) This is shown for arbitrary modules in [**8**, Lemma 2] and [**4**, Corollary 2.6].

(e)$\Rightarrow$(d)$\Rightarrow$(c)$\Rightarrow$(b) are obvious.

(b)$\Rightarrow$(a) We have to show that for every $K \trianglelefteq M$ the factor module $\bar{M} = M/K$ is noetherian: Since submodules of factor modules of $\bar{M}$ are again GV-modules they all have maximal submodules by Lemma 1. By (b) all factor modules of $\bar{M}$ have finitely generated socle and hence $\bar{M}$ is noetherian by Theorem 3.8 of Shock **[6]**.

The modules considered above are obviously noetherian if their socles are finitely generated and we get the following result.

3. Corollary. *For a GV-module M the following assertions are equivalent*:

(a) *M is noetherian*;

(b) *M has Krull dimension*;

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(c) *every factor module of M has finite uniform dimension*;

(d) *every factor module of M has finitely generated socle.*

The equivalence of (a) and (b) for GV-modules was proved in Yousif [**11**, Theorem 3]. Setting $M=R$ the Corollary yields characterizations of left GV-rings with Krull dimension including Proposition 13 in [**1**] on left V-rings with Krull dimension.

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