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Maximal Operators Associated with Vector Polynomials of Lacunary Coefficients

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Abstract. We prove the $L^p(\mathbb{R}^d)$ $(1 < p \leq \infty)$ boundedness of the maximal operators associated with a family of vector polynomials given by the form $\{(2^{k_1}\mathfrak{p}_1(t), \dots, 2^{k_d}\mathfrak{p}_d(t)) : t \in \mathbb{R}\}$. Furthermore, by using the lifting argument, we extend this result to a general class of vector polynomials whose coefficients are of the form constant times 2^k.

1 Introduction

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and consider a multi-parameter maximal function defined by

$$\mathcal{M}_d^{\mathrm{lac}} f(x) = \sup_{K \in \mathbb{Z}^d, r>0} \frac{1}{2r} \int_{-r}^{r} \Big| f\big(x_1 - 2^{k_1}\mathfrak{p}_1(t), \dots, x_d - 2^{k_d}\mathfrak{p}_d(t)\big) \Big| \, dt,$$

where $K = (k_1, \dots, k_d) \in \mathbb{Z}^d$, and $\mathfrak{p}_\tau$ is a polynomial of the form

$$\mathfrak{p}_\tau(t) = \sum_{\ell=1}^{q} a_\ell^\tau t^\ell \text{ for each } \tau = 1, \dots, d. \tag{1.1}$$

We are interested in the L^p boundedness of this operator. The history of our problem goes back to the case where $\mathfrak{p}_1(t) = \cdots = \mathfrak{p}_d(t) = t$, which has been studied by many authors [1, 2, 4, 7]. Cordoba, Fefferman, and Strömberg developed the L^p $(p \geq 2)$ theory for $\mathcal{M}_2^{\mathrm{lac}}$ by using a suitable geometric argument [2, 7]. Nagel, Stein, and Wainger [4] used the Littlewood–Paley decompositions to prove the remaining range $1 < p \leq \infty$. Furthermore, Carbery [1] extended this result to arbitrary dimension $d \geq 3$.

Recently, Hare and Ricci [3] showed the L^p boundedness of the operator $\mathcal{M}_2^{\mathrm{lac}}$ when the density of the measure on the line is changed. They considered the case that the line $L_K = \{(2^{k_1}t, 2^{k_2}t) : t \in \mathbb{R}\}$ is replaced by the polynomial curve $\{(2^{k_1}\mathfrak{p}_1(t), 2^{k_2}\mathfrak{p}_2(t)) : t \in \mathbb{R}\} \subset L_K$ with $\mathfrak{p}_1(t) = \mathfrak{p}_2(t)$.

In this paper we consider a general polynomial curve on $\mathbb{R}^d$. The purpose of this paper is to show the L^p boundedness of $\mathcal{M}_d^{\mathrm{lac}}$ for all $d \geq 1$ and arbitrary polynomials $\{\mathfrak{p}_\tau(t)\}_{\tau=1}^d$.

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Theorem 1.1 *For $1 < p \le \infty$, there exists a constant $C_p > 0$, independent of the coefficients of $\mathfrak{p}_1, \dots, \mathfrak{p}_d$, such that*

$$\|\mathcal{M}_d^{\mathrm{lac}} f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

Next we consider more general maximal operators. We set a vector polynomial

$$P_K(t) = \Big(\sum_{\ell=1}^{q} 2^{k_{1,\ell}} a_\ell^1 t^\ell, \dots, \sum_{\ell=1}^{q} 2^{k_{d,\ell}} a_\ell^d t^\ell \Big),$$

where $K = (k_{\tau,\ell}) \in \mathbb{Z}^{qd}$. Associated with P_K, we define a multi-parameter maximal operator $\mathcal{S}$ by

$$\mathcal{S} f(x) = \sup_{K \in \mathbb{Z}^{qd}, r > 0} \frac{1}{2r} \int_{-r}^{r} \big| f\big(x - P_K(t)\big) \big| \, dt. \tag{1.2}$$

In the last section of this article we use an appropriate lifting lemma and the result of Theorem 1.1 to obtain the following.

Theorem 1.2 *For $1 < p \le \infty$, there exists a constant $C_p > 0$ independent of the coefficients of P_K such that $\|\mathcal{S} f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$.*

In proving Theorem 1.1, we develop the idea of [4] to reduce the parameters used for defining $\mathcal{M}_d^{\mathrm{lac}}$ one by one. In Section 2, we briefly sketch the proof of Theorem 1.1 by introducing the reduction scheme. In Section 3, we give some vector valued and Littlewood–Paley inequalities which will be used frequently in the proof of Theorem 1.1. In Section 4, we discuss the angular decomposition of the frequency part which enables us to control the bad region having no decay property. The angular decomposition is crucial in performing the reduction process successively. In Sections 5 and 6, we handle the operators corresponding to bad and good decay properties respectively. In Section 7, we finish the proof of Theorem 1.1 by using the similar bootstrap argument used in [4]. Finally in Section 8, we give a proof of Theorem 1.2.

Throughout the remainder of this paper, we shall use the notation $A \lesssim B$ when $A \le CB$ with a constant C depending only on the dimension d and the degrees of polynomials $\mathfrak{p}_1, \dots, \mathfrak{p}_d$. We also write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2 Sketch for the Proof of Theorem 1.1

Let $\mathcal{P}$ be a collection of vector polynomials of the form $\mathfrak{p}(t) = (\mathfrak{p}_1(t), \dots, \mathfrak{p}_d(t))$ where $\mathfrak{p}_\tau$ is a polynomial of degree at most q of the form (1.1). Put

$$\varphi_j(u) = \varphi(u/2^j)/2^j,$$

where φ is a nonnegative smooth function such that $\operatorname{supp} \varphi \in [1/4, 4]$ and $\varphi(x) = 1$ for all $x \in [1/2, 2]$. Now for each vector polynomial $\mathfrak{p} \in \mathcal{P}$ and $(j, K) \in \mathbb{Z}^{d+1}$ we define a measure $\mu_{j,K}^{\mathfrak{p}}$ on $\mathbb{R}^d$ by

$$\mu_{j,K}^{\mathfrak{p}}(f) = \int f\big(2^{k_1} \mathfrak{p}_1(t), \dots, 2^{k_d} \mathfrak{p}_d(t)\big) \varphi_j(t) \, dt.$$

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We express the region of integration $[0, r]$ in the definition of $\mathcal{M}_d^{\mathrm{lac}}$ as the union of $[2^{j-1}, 2^j]$'s. Then we can write the average of f over the line as

$$\sum_{2^j \le r} \frac{2^j}{r} \Big(\frac{1}{2^j} \int_{2^{j-1}}^{2^j} |f\big(x_1 - 2^{k_1}\mathfrak{p}_1(t), \dots, x_d - 2^{k_d}\mathfrak{p}_d(t)\big)|\, dt \Big).$$

Thus we see that $\mathcal{M}_d^{\mathrm{lac}} f$ with t integral restricted on $[0, \infty)$, is majorized by the maximal function defined by $\sup_{j,K} \mu_{j,K}^{\mathfrak{p}} * |f|$, Therefore in showing Theorem 1.1, we prove that for $1 < p \le 2$

$$\big\| \sup_{j,K} \mu_{j,K}^{\mathfrak{p}} * |f| \big\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.1}$$

The other range is obtained by the interpolation with $p = \infty$ and $p = 2$.

2.1 Reduction Scheme

By using the class of measures $\{\mu_{j,K}^{\mathfrak{p}} : \mathfrak{p} \in \mathcal{P} \text{ and } (j, K) \in \mathbb{Z}^{d+1}\}$, we shall define $\mathcal{A}_r$ as a family of maximal operators for each $r = 0, 1, \dots, d$. We consider a set of r integers $\{d(1), \dots, d(r)\}$ satisfying $1 \le d(1) < d(2) < \cdots < d(r) = d$. To each collection of r integers $\{d(1), \dots, d(r)\}$, we assign a set

$$\begin{aligned} &\mathbb{Z}(d(1), \dots, d(r)) \\ &\qquad = \big\{ K = (k_1, \dots, k_d) \in \mathbb{Z}^d : k_n = k_{d(\nu)} \text{when } d(\nu - 1) < n \le d(\nu) \big\}, \end{aligned}$$

where $\nu = 1, \dots, r$ and $d(0) = 0$. Associated with each $\{d(1), \dots, d(r)\}$, we define the maximal operator

$$M_{(d(1),\dots,d(r))}^{\mathfrak{p}} f(x) = \sup_{\substack{j \in \mathbb{Z} \\ K \in \mathbb{Z}(d(1),\dots,d(r))}} |\mu_{j,K}^{\mathfrak{p}} * f(x)|. \tag{2.2}$$

Note that the maximal operator $M_{(d(1),\dots,d(r))}^{\mathfrak{p}}$ is determined by $r + 1$ parameters

$$k_{d(1)}, \dots, k_{d(r)} \text{ and } j.$$

Definition 2.1 We define $\mathcal{A}_r$ as the family of maximal operators given by

$$\mathcal{A}_r = \{M_{(d(1),\dots,d(r))}^{\mathfrak{p}} : \mathfrak{p} \in \mathcal{P} \text{ and } 0 = d(0) < d(1) < \cdots < d(r) = d\}.$$

Example We see that the maximal operator M defined by

$$Mf(x) = \sup_{j,k} \int |f(x_1 - 2^k \mathfrak{p}_1(t), \dots, x_d - 2^k \mathfrak{p}_d(t))| \varphi_j(t)\, dt$$

belongs to the class $\mathcal{A}_1$. The maximal function M defined by

$$Mf(x) = \sup_{j,k,\ell} \int |f(x_1 - 2^k \mathfrak{p}_1(t), \dots, x_{d-1} - 2^k \mathfrak{p}_{d-1}(t), x_d - 2^\ell \mathfrak{p}_d(t))| \varphi_j(t)\, dt$$

is in the class $\mathcal{A}_2$. The maximal operator $f \mapsto \sup_{j,K} \mu_{j,K}^{\mathfrak{p}} * |f|$ is in $\mathcal{A}_d$.

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Definition 2.2 We also define a number $\|\mathcal{A}_r\|_p$ associated with $\mathcal{A}_r$ as follows:

$$\|\mathcal{A}_r\|_p = \sup\{\|M\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} : M \in \mathcal{A}_r\}.$$

We see that $\|\mathcal{A}_r\|_p \le \|\mathcal{A}_{r+1}\|_p$, since the supremum over the larger index set is greater than that over the smaller index set.

In order to prove (2.1), we show the following estimates.

Proposition 2.3 *For $1 < p \le 2$, $\|\mathcal{A}_1\|_p \le B(p)$. For $1 < p \le 2$ and $r \in \{2, \dots, d\}$, there exists a constant $B(p, r-1) > 0$ such that*

$$\|\mathcal{A}_r\|_p \le B(p, r-1) \tag{2.3}$$

where $B(p, r-1)$ is of the form

$$C_p \prod_{i=1}^{N_p} (\|\mathcal{A}_{r-1}\|_{a^i(p)} + 1)^{c_p^i}, \tag{2.4}$$

where $C_p > 0$, $N_p \ge 1$, $c_p^i \ge 0$ and $1 < a^i(p) \le 2$.

In what follows, $B(p, r-1)$ will be chosen to be different constants of the form (2.4) line by line. Hence Theorem 1.1 which is the case $r = d$ follows inductively from Proposition 2.3.

Remark 1 For fixed K, the maximal operator defined by $\sup_{j\in\mathbb{Z}} \mu^{\mathfrak{p}}_{j,K} * |f|(x)$ is known to be bounded in $L^p(\mathbb{R}^d)$. For the proof, see [5, pp. 477–486].

2.2 Sketch for the Proof of Proposition 2.3

Let us consider $r = m$ in Proposition 2.3. Here we show how the reduction is performed by using the angular decomposition in the spirit of [4]. To simplify the notation, we write $\mu_{j,K}$ instead of $\mu^{\mathfrak{p}}_{j,K}$. We write the Fourier transform of the measure $\mu_{j,K}$ as

$$\widehat{\mu_{j,K}}(\xi_1, \dots, \xi_d) = \int e^{2\pi i P_{j,K}(\xi,t)} \varphi(t)\, dt,$$

where

$$\begin{aligned}
P_{j,K}(\xi, t) &= 2^{k_1}\xi_1 \mathfrak{p}_1(2^j t) + \cdots + 2^{k_d}\xi_d \mathfrak{p}_d(2^j t)\\
&= 2^{k_1}\xi_1 \big(a_1^1 2^j t + a_2^1 2^{2j} t^2 + \cdots + a_\ell^1 2^{\ell j} t^\ell + \cdots + a_q^1 2^{qj} t^q\big)\\
&\qquad\vdots\\
&\quad + 2^{k_\tau}\xi_\tau \big(a_1^\tau 2^j t + a_2^\tau 2^{2j} t^2 + \cdots + a_\ell^\tau 2^{\ell j} t^\ell + \cdots + a_q^\tau 2^{qj} t^q\big)\\
&\qquad\vdots\\
&\quad + 2^{k_d}\xi_d \big(a_1^d 2^j t + a_2^d 2^{2j} t^2 + \cdots + a_\ell^d 2^{\ell j} t^\ell + \cdots + a_q^d 2^{qj} t^q\big).
\end{aligned}$$

We note that if $d(\nu-1) < \tau \le d(\nu)$, then $k_\tau = k_{d(\nu)}$. Thus we can write

$$P_{j,K}(\xi,t) = \sum_{\ell=1}^{q}\Big(\sum_{\nu=1}^{m} 2^{k_{d(\nu)}}\xi(\ell,\nu)\Big)2^{j\ell}t^{\ell},$$

where $\xi(\ell,\nu) = (a_\ell^{d(\nu-1)+1}\xi_{d(\nu-1)+1} + \cdots + a_\ell^{d(\nu)}\xi_{d(\nu)})$. By using the Van der Corput lemma in [5, Chapter 8], we obtain that for some $\sigma > 0$,

$$|\widehat{\mu_{j,K}}(\xi_1,\ldots,\xi_d)| \le C\min\Big\{\Big|\sum_{\nu=1}^{m} 2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)\Big|^{-\sigma} : \ell = 1,\ldots,q\Big\}.$$

Since we do not have good decay property in the set

$$\Big\{\xi : \Big|\sum_{\nu=1}^{m} 2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)\Big| = 0\Big\}, \tag{2.5}$$

we need to split the frequency domain into two regions: one is the bad region containing (2.5) and the other is the good region away from (2.5). Precisely, the bad region is defined as the set

$$B(K) = \bigcup_{\ell=1}^{q}\bigcup_{\nu\neq\mu} B(k_{d(\nu)},k_{d(\mu)},\ell),$$

where $K = (k_1,\ldots,k_d)$. The first union above is taken over $\nu,\mu \in \{1,\ldots,m\}$ and

$$B(k_{d(\nu)},k_{d(\mu)},\ell) = \{\xi : 2^{k_{d(\nu)}+j\ell}|\xi(\ell,\nu)| \sim 2^{k_{d(\mu)}+j\ell}|\xi(\ell,\mu)|\},$$

where in what follows we denote $a \sim b$ if $(2^{d+1}d!)^{-1} \le |\frac{a}{b}| \le 2^{d+1}d!$.

The good region is defined as the set

$$G(K) = \bigcap_{\ell=1}^{q}\bigcap_{\nu\neq\mu} G(k_{d(\nu)},k_{d(\mu)},\ell),$$

where

$$G(k_{d(\nu)},k_{d(\mu)},\ell) = \{\xi : 2^{k_{d(\nu)}+j\ell}|\xi(\ell,\nu)| \nsim 2^{k_{d(\mu)}+j\ell}|\xi(\ell,\mu)|\}.$$

Let χ_A be the characteristic function of the set A. On the good region we have good decay properties such as

$$\begin{aligned}|\widehat{\mu_{j,K}}(\xi_1,\ldots,\xi_d)|\chi_{G(K)}(\xi)&\\ &\le C\min\{|2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)|^{-\sigma} : \nu = 1,\ldots,m, \text{ and } \ell = 1,\ldots,q\}.\end{aligned}$$

This leads us to the L^p boundedness of the maximal operator

$$f \mapsto \sup_{\substack{j\in\mathbb{Z}\\ K\in\mathbb{Z}(d(1),\ldots,d(m))}} |\mu_{j,K} * \chi^{\vee}_{G(K)} * f|, \tag{2.6}$$

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where $g^{\vee}$ denotes an inverse Fourier transform of g.

Before considering the bad part, we note that $B(k_{d(\nu)}, k_{d(\mu)}, \ell)$ defined above does not depend on the parameter j. Moreover, we can observe that it is determined by the parameter $k_{d(\mu)} - k_{d(\nu)}$.

For the bad region, we need to handle the maximal operator for each $\nu \neq \mu$ and ℓ

$$f \mapsto \sup_{\substack{j\in\mathbb{Z} \\ K\in\mathbb{Z}(d(1),\dots,d(m))}} |\mu_{j,K} * \chi^{\vee}_{B(k_{d(\nu)},k_{d(\mu)},\ell)} * f|. \tag{2.7}$$

By using a new parameter k, we write $k_{d(\mu)} = k_{d(\nu)} + k$ and put $A_k = \chi^{\vee}_{B(k_{d(\nu)},k_{d(\mu)},\ell)}$. We then replace the supremum over one parameter k by the square summation

$$\sup_{\substack{j\in\mathbb{Z} \\ K\in\mathbb{Z}(d(1),\dots,d(m))}} |\mu_{j,K} * \chi^{\vee}_{B(k_{d(\nu)},k_{d(\mu)},\ell)} * f(x)| \lesssim \Big(\sum_{k\in\mathbb{Z}} |\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}(A_k * f)(x)|^2\Big)^{\frac{1}{2}},$$

where

$$\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k} f(x) = \sup_{\substack{j\in\mathbb{Z} \\ K\in\mathbb{Z}(d(1),\dots,d(m)) \\ k_{d(\mu)}=k_{d(\nu)}+k}} |\mu_{j,K} * f(x)|.$$

We can check that the operator $\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}$ belongs to the class $\mathcal{A}_{m-1}$. If we have the following vector valued inequality

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}(A_k * f)|^2\Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim B(p, m-1) \Big\| \Big(\sum_{k\in\mathbb{Z}} |A_k * f|^2\Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)}, \tag{2.8}$$

where $B(p, m-1)$ is of the form (2.4), then the desired L^p estimate is obtained from the inequality

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |A_k * f|^2\Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

This is a variant of the Littlewood–Paley inequality on the angular sectors and will be shown in Lemma 3.4 in the next section.

We can easily see that (2.8) holds for $p = 2$ with the bound $\|\mathcal{A}_{m-1}\|_2$, which leads us to the L^2 boundedness of the operator defined in (2.7). Combined with the L^2 boundedness of the operator in (2.6), we obtain that the maximal operator defined by $\sup_{j\in\mathbb{Z},K\in\mathbb{Z}(d(1),\dots,d(m))} |\mu_{j,K} * f|$ is bounded in $L^2(\mathbb{R}^d)$. In order to treat the range $1 < p < 2$, we shall use this L^2 estimate and the vector valued inequalities (2.8) in Lemma 3.6, as we shall see.

3 Preliminary Lemmas

Let us choose an even nonnegative function $\psi \in C_0^\infty(-1,1)$ such that $\psi \equiv 1$ on $[-1/2,1/2]$. Set $\chi(t) = \psi(t/2) - \psi(t)$. We define a function Ω by

$$\Omega(t) = \sum_{\ell=-2^d d!}^{2^d d!} \chi(t/2^\ell).$$

For $\tau = 1,\dots,d$, we define a dyadic decomposition on each τ-th coordinate of $\mathbb{R}^d$ by $\widehat{L_j^\tau}(\xi) = \chi(2^j\xi_\tau)$ where $\xi = (\xi_1,\dots,\xi_d)$. Let us define Ω_j on $\mathbb{R}^2$ away from $\mathbb{R}^1 \times \{0\}$ by

$$\Omega_j(s,t) = \Omega(s/(2^j t)). \tag{3.1}$$

Then Ω_j is supported in $\{2^{j-2^d d!-1} \le |s/t| \le 2^{j+2^d d!+1}\}$. We define a dyadic decomposition on the angular sectors by using the following measures in $\mathbb{R}^d$:

$$\widehat{A_j^{(\alpha,\beta)}}(\xi) = \Omega_j(\xi_\alpha, \xi_\beta).$$

Let $K = (k_1,\dots,k_m)$ in what follows. By the Marcinkiewicz multiplier theorem in [6] we have the following Littlewood–Paley type inequalities.

Lemma 3.1 *Let $1 \le \tau_i, \alpha_i \ne \beta_i \le d$ with $i = 1,\dots,m$. Then we have for $1 < p < \infty$,*

$$\Big\| \Big(\sum_{K\in\mathbb{Z}^m} |L_{k_1}^{\tau_1} * \cdots * L_{k_m}^{\tau_m} * f|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

$$\Big\| \Big(\sum_{K\in\mathbb{Z}^m} |A_{k_1}^{(\alpha_1,\beta_1)} * \cdots * A_{k_m}^{(\alpha_m,\beta_m)} * f|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof We use the Rademacher functions to switch the square sums above into linear sums.

Then, by applying the multiparameter Marcinkiewicz multiplier theorem, we obtain the above desired inequalities. ■

For a nonzero vector $\mathfrak{a} = (a_1,\dots,a_d) \in \mathbb{R}^d$, we define a measure $P_j^{\mathfrak{a}}$ which restricts the frequency variable $2^j|\xi\cdot\mathfrak{a}| \lesssim 1$ so that

$$\widehat{P_j^{\mathfrak{a}} * f}(\xi) = \psi(2^j\xi\cdot\mathfrak{a})\widehat{f}(\xi),$$

where $\xi\cdot\mathfrak{a} = \xi_1 a_1 + \cdots + \xi_d a_d$.

Lemma 3.2 *For each $x \in \mathbb{R}^d$, $\sup_{j\in\mathbb{Z}} |P_j^{\mathfrak{a}} * f(x)| \lesssim M_{\mathfrak{a}} f(x)$, where $M_{\mathfrak{a}}$ is a directional maximal function along the line $\{t\mathfrak{a} : t \in \mathbb{R}\}$ in $\mathbb{R}^d$.*

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Proof The proof is obvious from the following inequality

$$P_j^{\mathfrak{a}} * f(x) = \int f(x_1 - a_1 t, \dots, x_d - a_d t) \frac{1}{2^j} |\psi^\vee| \Big(\frac{t}{2^j}\Big)\, dt \lesssim M_{\mathfrak{a}} f(x),$$

because $\psi^\vee$ is a Schwartz function in $\mathbb{R}$. ■

Let $\{\mathfrak{a}^1, \dots, \mathfrak{a}^m\}$ be a collection of nonzero vectors in $\mathbb{R}^d$ and define a measure by

$$\widehat{L_k^{\mathfrak{a}^i}}(\xi) = \chi(2^k \mathfrak{a}^i \cdot \xi).$$

Then we have the Littlewood–Paley type inequality corresponding to the above measures.

Lemma 3.3 *For $1 < p < \infty$, we have*

$$\Big\| \Big(\sum_{k_i \in \mathbb{Z}} |L_{k_i}^{\mathfrak{a}^i} * f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

$$\Big\| \Big(\sum_{K \in \mathbb{Z}^m} |L_{k_1}^{\mathfrak{a}^1} * \cdots * L_{k_m}^{\mathfrak{a}^m} * f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof We may assume that $a_\nu^i \neq 0$ for some $\nu = 1, \dots, d$. Then the first inequality follows from Lemma 3.1 and the fact that

$$\widehat{L_{k_i}^{\mathfrak{a}^i}}(\xi) = \widehat{L_{k_i}^{\nu}}(G_i \xi),$$

where $\widehat{L_{k_i}^{\nu}}(\xi) = \chi(2^{k_i} \xi_\nu)$ where G_i is the invertible $d \times d$ matrix defined by

$$G_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & & 0 & 0 \\ a_1^i & a_2^i & \cdots & a_\nu^i & \cdots & a_{d-1}^i & a_d^i \\ 0 & 0 & & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We obtain the second inequality by switching it to linear sums via Rademacher functions. ■

For each $i = 1, \dots, m$, let $\mathfrak{a}^i$ and $\mathfrak{b}^i$ be two nonzero vectors in $\mathbb{R}^d$ and set

$$\widehat{A_k^{\mathfrak{a}^i,\mathfrak{b}^i}}(\xi) = \Omega_k(\xi \cdot \mathfrak{a}^i, \xi \cdot \mathfrak{b}^i). \tag{3.2}$$

Lemma 3.4 *For* $1 < p < \infty$,

$$\Big\| \Big(\sum_{k_i \in \mathbb{Z}} |A_{k_i}^{\mathfrak{a}^i,\mathfrak{b}^i} * f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

$$\Big\| \Big(\sum_{K \in \mathbb{Z}^m} |A_{k_1}^{\mathfrak{a}^1,\mathfrak{b}^1} * \cdots * A_{k_m}^{\mathfrak{a}^m,\mathfrak{b}^m} * f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof Suppose that $\mathfrak{a}^i$ and $\mathfrak{b}^i$ are linearly dependent. Then there exists a nonzero c such that $\xi \cdot \mathfrak{a}^i = c\,\xi \cdot \mathfrak{b}^i$ for $\xi \in \mathbb{R}^d$. Then by (3.1), for all $x \in \mathbb{R}^d$ we have

$$\Big(\Big| \sum_k |A_k^{\mathfrak{a}^i,\mathfrak{b}^i} f(x) \Big|^2 \Big)^{1/2} \lesssim |f(x)|.$$

Thus we assume that $\mathfrak{a}^i$ and $\mathfrak{b}^i$ are linearly independent vectors. Without loss of generality we may assume that (a_1^i, a_2^i) in $\mathfrak{a}^i$ and (b_1^i, b_2^i) in $\mathfrak{b}^i$ are linearly independent. Then the $d \times d$ matrix

$$R_i = \begin{pmatrix} a_1^i & a_2^i & a_3^i & \cdots & a_d^i \\ b_1^i & b_2^i & b_3^i & \cdots & b_d^i \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \vdots & 0 & \ddots & \vdots \\ 0 & \vdots & 0 & \cdots & 1 \end{pmatrix}$$

is invertible. For each $i = 1, \ldots, d$, we use the fact $\widehat{A_{k_i}^{\mathfrak{a}^i,\mathfrak{b}^i}}(\xi) = \widehat{A_{k_i}^{1,2}}(R_i\xi)$ and the known corresponding estimate for $A_{k_i}^{1,2}$ in Lemma 3.1 to obtain the first inequality. We obtain the second inequality by switching the square sum into a linear sum. ■

For the L^p estimate, we shall use the following vector valued inequality.

Lemma 3.5 *Suppose that σ_J with each $J \in \mathbb{Z}^\ell$ is an operator satisfying the positivity condition such that $|\sigma_J(f)(x)| \le |\sigma_J(g)(x)|$ if $|f(x)| \le |g(x)|$ for all $x \in \mathbb{R}^d$. We assume that $\| \sup_{J \in \mathbb{Z}^\ell} |\sigma_J(f)| \|_{L^q(\mathbb{R}^d)} \le C_1 \|f\|_{L^q(\mathbb{R}^d)}$ for some $q \le 2$, and $\|\sigma_J(f)\|_{L^r(\mathbb{R}^d)} \le C_2 \|f\|_{L^r(\mathbb{R}^d)}$ for all $r > 1$. Then for $\frac{1}{p} < \frac{1}{2}(1 + \frac{1}{q})$ we have*

$$\Big\| \Big(\sum_{J \in \mathbb{Z}^\ell} |\sigma_J(f_J)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)} \le C \Big\| \Big(\sum_{J \in \mathbb{Z}^\ell} |f_J|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^d)},$$

where $C = C_1^{1-\frac{r}{2}} C_2^{r/2}$ with r satisfying $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}(1 - \frac{r}{2})$, that is, $r = 2[1 - (\frac{1}{p} - \frac{1}{2})q]$.

Proof This lemma follows from the interpolation of $L^r(l^r(\mathbb{R}^d))$ with any $r > 1$ and $L^q(l^\infty(\mathbb{R}^d))$ for the vector valued operator $\mathfrak{T}$ given by $\{f_J\} \to \{\sigma_J(f_J)\}$. We can see that

$$\|\mathfrak{T}\|_{L^q(l^\infty(\mathbb{R}^d)) \to L^q(l^\infty(\mathbb{R}^d))} \le C_1, \quad \|\mathfrak{T}\|_{L^r(l^r(\mathbb{R}^d)) \to L^r(l^r(\mathbb{R}^d))} \le C_2.$$

By using $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{r}$ and $\frac{1}{2} = \frac{1-\theta}{\infty} + \frac{\theta}{r}$, we obtain the desired bound. ■

This lemma is applied to the following situation. We recall the maximal operator $\mathcal{M}_{\nu,\mu,k}^{(d(1),\dots,d(m))}$ that appeared in Section 1,

$$\mathcal{M}_{\nu,\mu,k}^{(d(1),\dots,d(m))} f(x) = \sup_{\substack{j\in\mathbb{Z} \\ K\in\mathbb{Z}(d(1),\dots,d(m)) \\ k_{d(\mu)}=k_{d(\nu)}+k}} |\mu_{j,K} * f(x)|.$$

Thus we can write

$$\mathcal{M}^{(d(1),\dots,d(m))} f(x) = \sup_{\substack{j\in\mathbb{Z} \\ K\in\mathbb{Z}(d(1),\dots,d(m))}} |\mu_{j,K} * f(x)| = \sup_{k\in\mathbb{Z}} \mathcal{M}_{\nu,\mu,k}^{(d(1),\dots,d(m))} f(x).$$

Lemma 3.6 *Suppose that $\mathcal{M}^{(d(1),\dots,d(m))}$ is bounded in $L^q(\mathbb{R}^d)$ for some $q > 1$ and $\mathcal{M}_{\nu,\mu,k}^{(d(1),\dots,d(m))}$ is bounded in $L^r(\mathbb{R}^d)$ for all $r > 1$. Then for $\frac{1}{p} < \frac{1}{2}(1+\frac{1}{q})$, we have*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mathcal{M}_{\nu,\mu,k}^{(d(1),\dots,d(m))} f_k|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \lesssim C \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)},$$

where the constant C is chosen to be

$$C = \|\mathcal{M}^{(d(1),\dots,d(m))}\|_{L^q(\mathbb{R}^d)\to L^q(\mathbb{R}^d)}^{1-\frac{r}{2}} \|\mathcal{M}_{\nu,\mu,k}^{(d(1),\dots,d(m))}\|_{L^r(\mathbb{R}^d)\to L^r(\mathbb{R}^d)}^{r/2},$$

where r satisfies $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}(1-\frac{r}{2})$, that is, $r = 2[1-(\frac{1}{p}-\frac{1}{2})q]$.

4 Reduction and Angular Decomposition

In this section we set up the reduction process and angular decompositions which are used for the proof of (2.3) with $r = m$ in Proposition 2.3.

4.1 Reduction

We recall that the Fourier transform of the measure $\mu_{j,K}$ is

$$\widehat{\mu_{j,K}}(\xi_1,\dots,\xi_d) = \int e^{2\pi i P_{j,K}(\xi,t)} \varphi(t)\,dt,$$

where

$$P_{j,K}(\xi,t) = \sum_{\ell=1}^{q} \Big(\sum_{\nu=1}^{m} 2^{k_{d(\nu)}} \xi(\ell,\nu) \Big) 2^{j\ell} t^{\ell}.$$

Let $\Lambda_0 = \{(\ell,\nu) : (a_\ell^{d(\nu-1)+1},\dots,a_\ell^{d(\nu)}) \neq 0$ where $\nu = 1,\dots,m$, and $\ell = 1,\dots,q\}$. We take an arbitrary subset $\Lambda \subset \Lambda_0$ and define

$$\widehat{M_{j,K}^{\Lambda}}(\xi) = \int e^{2\pi i P_{j,K}^{\Lambda}(\xi,t)} \varphi(t)\,dt, \qquad P_{j,K}^{\Lambda}(\xi,t) = \sum_{(\ell,\nu)\in\Lambda} \xi(\ell,\nu) 2^{k_{d(\nu)}+j\ell} t^{\ell}.$$

Let us define a maximal operator $\mathcal{M}^{\Lambda}$ associated with $\Lambda \subset \Lambda_0$ as

$$\mathcal{M}^{\Lambda} f(x) = \sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\dots,d(m))} |M^{\Lambda}_{j,K} * f(x)|.$$

In showing (2.3), we shall prove the following lemmas in Sections 4 through 6.

Lemma 4.1 *For $1 < p \le 2$,*

$$\|\mathcal{M}^{\Lambda}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim B(p, m-1) \sum_{D\in\mathcal{P}(\Lambda)} \|\mathcal{M}^{\Lambda - D}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)},$$

where $\mathcal{P}(\Lambda)$ is a family of all nonempty subsets of Λ, and $B(p, m-1)$ is of the form (2.4).

By applying Lemma 4.1 finitely many times, we see that the set Λ will be exhausted down to $\varnothing$. Therefore Lemma 4.1 leads to the following.

Lemma 4.2 *For $1 < p \le 2$, $\|\mathcal{M}^{\Lambda}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim B(p, m-1)$ where $B(p, m-1)$ is of the form* (2.4).

Note that Lemma 4.2 implies Proposition 2.3, since Λ is an arbitrary subset of Λ_0.

4.2 Angular Decomposition

Since $\widehat{M^{\Lambda}_{j,K}}(\xi)$ is of the form of oscillatory integrals, we split $\widehat{M^{\Lambda}_{j,K}}(\xi)$ into a good part and a bad part based on whether the size of the phase function is dominated by one term or not. The bad region is defined as the set of frequency variables such that for some (ℓ, ν) and (ℓ, μ) in Λ with $\nu \ne \mu$

$$2^{k_{d(\nu)}+j\ell}|\xi(\ell,\nu)| \sim 2^{k_{d(\mu)}+j\ell}|\xi(\ell,\mu)|. \tag{4.1}$$

The good region is defined as the set of frequency variables such that

$$2^{k_{d(\nu)}+j\ell}|\xi(\ell,\nu)| \nsim 2^{k_{d(\mu)}+j\ell}|\xi(\ell,\mu)|, \tag{4.2}$$

for any (ℓ, ν) and (ℓ, μ) in Λ with $\nu \ne \mu$. We define a good maximal function by

$$\mathfrak{M}^{\Lambda} f(x) = \sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\dots,d(m))} |\mathfrak{M}^{\Lambda}_{j,K} * f(x)|.$$

The Fourier transform of the measure $\mathfrak{M}^{\Lambda}_{j,K}$ is restricted to the region (4.2), and written as

$$\widehat{\mathfrak{M}^{\Lambda}_{j,K}}(\xi) = \widehat{M^{\Lambda}_{j,K}}(\xi) \cdot \prod_{\substack{(\ell,\nu),(\ell,\mu)\in\Lambda\\ \nu\ne\mu}} \left(1 - \Omega_{k_{d(\mu)}-k_{d(\nu)}}(\xi(\ell,\nu), \xi(\ell,\mu))\right).$$

On the other hand, a bad maximal function $\mathfrak{N}^\Lambda f$ is also defined by

$$\mathfrak{N}^\Lambda f(x) = \sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\ldots,d(m))} |\mathfrak{N}^\Lambda_{j,K} * f(x)|.$$

The Fourier transform of the measure $\mathfrak{N}^\Lambda_{j,K}$ is restricted to the region (4.1) and written as

$$\widehat{\mathfrak{N}^\Lambda_{j,K}}(\xi) = \widehat{M^\Lambda_{j,K}}(\xi) \cdot \Big(1 - \prod_{\substack{(\ell,\nu),(\ell,\mu)\in\Lambda\\ \nu\neq\mu}} \big(1 - \Omega_{k_{d(\mu)}-k_{d(\nu)}}(\xi(\ell,\nu),\xi(\ell,\mu))\big)\Big).$$

From the fact that $\mathcal{M}^\Lambda f \le \mathfrak{M}^\Lambda f + \mathfrak{N}^\Lambda f$ we show that the L^p bounds of $\mathfrak{N}^\Lambda$ and $\mathfrak{M}^\Lambda$ are majorized by certain powers of $\|\mathcal{A}_{m-1}\|_{a(p)}$ and $\sum_{D\in\mathcal{P}(\Lambda)} \|\mathcal{M}^{\Lambda-D}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}$, respectively, for the proof of Lemma 4.1.

5 L^2 Estimate for the Bad Maximal Operator $\mathfrak{N}^\Lambda$

In this section, as a part of the proof of Lemma 4.1, we show that for $p = 2$,

$$\|\mathfrak{N}^\Lambda\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim B(p, m-1), \tag{5.1}$$

where $B(p, m-1)$ is of the form (2.4). The case $p \neq 2$ will be treated in Section 7.

Proof of (5.1) **for** $p = 2$ Our proof is based on the angular Littlewood–Paley decomposition. We can write $\widehat{\mathfrak{N}^\Lambda_{j,K}}(\xi)$ as the sum of

$$\widehat{M^\Lambda_{j,K}}(\xi) \cdot \prod_{(\ell,\nu),(\ell,\mu)\in G\times G',\nu\neq\mu} \Omega_{k_{d(\mu)}-k_{d(\nu)}}(\xi(\ell,\nu),\xi(\ell,\mu)), \tag{5.2}$$

where G and G' are subsets of Λ. Let us consider the simple case

$$\widehat{M^\Lambda_{j,K}}(\xi)\Omega_{k_{d(\mu)}-k_{d(\nu)}}(\xi(\ell,\nu),\xi(\ell,\mu)) \tag{5.3}$$

with fixed ν, μ, and ℓ. Let

$$\mathfrak{a}^\nu_\ell = (0,\ldots,0,a_\ell^{d(\nu-1)+1},\ldots,a_\ell^{d(\nu)},0,\ldots,0), \tag{5.4}$$

where nonzero terms are located from the $(d(\nu-1)+1)$-th entry to the $d(\nu)$-th entry. We also let

$$\mathfrak{a}^\mu_\ell = (0,\ldots,0,a_\ell^{d(\mu-1)+1},\ldots,a_\ell^{d(\mu)},0,\ldots,0), \tag{5.5}$$

where nonzero terms are located from the $(d(\mu-1)+1)$-th coordinate to $d(\mu)$-th coordinate. Then it is immediate thatthe two vectors defined above are linearly independent. In view of (5.3), (5.4), (5.5) and (3.2) the maximal function $\mathfrak{N}^{\Lambda}_{j,K}f$ can be written as

$$\mathfrak{N}^{\Lambda}_{j,K}f(x)=\sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\dots,d(m))}|M^{\Lambda}_{j,K}*A^{\mathfrak{a}^{\nu}_{\ell},\mathfrak{a}^{\mu}_{\ell}}_{k_{d(\mu)}-k_{d(\nu)}}*f(x)|.$$

Let us define a new parameter k as $k=k_{d(\mu)}-k_{d(\nu)}$. Then

$$\mathfrak{N}^{\Lambda}_{j,K}f(x)\le\Big(\sum_{k}\Big|\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}\big(A^{\mathfrak{a}^{\nu}_{\ell},\mathfrak{a}^{\mu}_{\ell}}_{k}*f(x)\big)\Big|^{2}\Big)^{\frac{1}{2}},$$

where

$$\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}f(x)=\sup_{(j,K)\in R_k}|M^{\Lambda}_{j,K}*f(x)|, \tag{5.6}$$

with $R_k=\{(j,K)\in\mathbb{Z}^{d+1}:(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\dots,d(m))\text{ and }k_{d(\mu)}=k_{d(\nu)}+k\}$. Note that the maximal operator $\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}$ is the operator in the class $\mathcal{A}_{m-1}$, as we have seen in (2.2). In this way our maximal function corresponding to (5.2) is majorized by

$$\Big(\sum_{k\in\mathbb{Z}}\Big|\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}(G_kf(x))\Big|^{2}\Big)^{\frac{1}{2}},$$

where

$$G_kf(x)=\Big(\sup_{K=(k_{\ell,\tau,\sigma})\in\mathbb{Z}^{\operatorname{card}(H)}}|A^{\mathfrak{a}^{\nu}_{\ell^*},\mathfrak{a}^{\mu}_{\ell^*}}_{k}*\Big(\circledast_{[(\ell,\tau),(\ell,\sigma)]\in H}A^{\mathfrak{a}^{\tau}_{\ell},\mathfrak{a}^{\sigma}_{\ell}}_{k_{\ell,\tau,\sigma}}\Big)*f(x)|\Big)$$

and $H=G\times G'-[(\ell^*,\nu),(\ell^*,\mu)]$ with the notation $\circledast_{i=1}^{N}A_i=A_1*\cdots*A_N$. By changing the supremum into square summation and using Lemma 3.4, we obtain

$$\Big\|\Big(\sum_{k}|G_kf|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^d)}\lesssim\|f\|_{L^p(\mathbb{R}^d)}.$$

In view of the bound $B(2,m-1)=\|\mathcal{A}_{m-1}\|_2$, we obtain that for $p=2$,

$$\Big\|\Big(\sum_{k}\Big|\mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k}(G_kf)\Big|^{2}\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^d)}\lesssim B(p,m-1)\|f\|_{L^p(\mathbb{R}^d)}, \tag{5.7}$$

which implies (5.1) when $p=2$. ■

Remark 2 For the case $\mathcal{A}_1$, we do not deal with the bad part and regard the operator $\sup_{j\in\mathbb{Z},K\in\mathbb{Z}(d(1))}|M_{j,K}*f|$ as the good maximal function $\mathfrak{M}^{\Lambda}f$.

6 L^2 Estimate for the Good Maximal Operator $\mathfrak{M}^\Lambda$

In this subsection we prove that for $p = 2$,

$$\|\mathfrak{M}^\Lambda\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim B(p, m-1) + \sum_{D\in\mathcal{P}(\Lambda)} \|\mathcal{M}^{\Lambda-D}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}, \tag{6.1}$$

where $\mathcal{P}(\Lambda)$ is a family of nonempty subset of Λ and $B(p, m-1)$ is of the form (2.4). We shall treat the case $p \neq 2$ in Section 6. Let $\psi^c = 1 - \psi$ and decompose $\widehat{\mathfrak{M}^\Lambda_{j,K}}(\xi)$ as

$$\begin{aligned}\widehat{\mathfrak{M}^\Lambda_{j,K}}(\xi) &= \prod_{(\ell,\nu)\in\Lambda} (\psi(2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)) + \psi^c(2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)) \cdot \widehat{\mathfrak{M}^\Lambda_{j,K}}(\xi)\\ &= \sum_{A\cup B=\Lambda, A\cap B=\varnothing} \widehat{\mathfrak{M}^{A,B}_{j,K}}(\xi),\end{aligned}$$

where $\widehat{\mathfrak{M}^{A,B}_{j,K}}(\xi)$ is defined by

$$\widehat{\mathfrak{M}^{A,B}_{j,K}}(\xi) = \prod_{(\ell,\nu)\in A} \psi(2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)) \prod_{(\ell,\nu)\in B} \psi^c(2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)) \cdot \widehat{\mathfrak{M}^\Lambda_{j,K}}(\xi).$$

We define a maximal operator $\mathfrak{M}^{A,B}$ by

$$\mathfrak{M}^{A,B} f(x) = \sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\ldots,d(m))} |\mathfrak{M}^{A,B}_{j,K} * f(x)|.$$

In order to prove (6.1), it suffices to show that for any choice of two subsets A and B of Λ such that $A \cup B = \Lambda$ and $A \cap B = \varnothing$,

$$\|\mathfrak{M}^{A,B}\|_{L^p(\mathbb{R}^d)\mapsto L^p(\mathbb{R}^d)} \lesssim B(p, m-1) + \sum_{D\in\mathcal{P}(\Lambda)} \|\mathcal{M}^{\Lambda-D}\|_{L^p(\mathbb{R}^d)\mapsto L^p(\mathbb{R}^d)}. \tag{6.2}$$

Let us define a measure $T^{A,B}_{j,K}$ whose Fourier transform $\widehat{T^{A,B}_{j,K}}(\xi)$ is given by

$$\widehat{T^{A,B}_{j,K}}(\xi) = \int e^{i\sum_{(\ell,\nu)\in B}\xi(\ell,\nu)2^{k_{d(\nu)}+j\ell}t^\ell} \prod_{(\ell,\nu)\in A} (e^{i\xi(\ell,\nu)2^{k_{d(\nu)}+j\ell}t^\ell} - 1)\varphi(t)\, dt. \tag{6.3}$$

We note that

$$\widehat{T^{A,B}_{j,K}}(\xi) = \widehat{M^\Lambda_{j,K}}(\xi) + \sum_{D\in\mathcal{P}(A)} \pm\widehat{M^{\Lambda-D}_{j,K}}(\xi). \tag{6.4}$$

In view of (6.4) and Lemma 3.2, we obtain

$$\begin{aligned}\|\mathfrak{M}^{A,B} f\|_{L^p(\mathbb{R}^d)} \lesssim \|&\sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\ldots,d(m))} |\mathfrak{T}^{A,B}_{j,K} * f|\|_{L^p(\mathbb{R}^d)}\\ &+ \sum_{D\in\mathcal{P}(A)} \|\mathcal{M}^{\Lambda-D}(Gf)\|_{L^p(\mathbb{R}^d)},\end{aligned} \tag{6.5}$$

where

$$\widehat{\mathfrak{T}^{A,B}_{j,K}}(\xi) = \widehat{T^{A,B}_{j,K}}(\xi) \prod_{\ell} \prod_{\nu \neq \mu} \bigl(1 - \Omega_{k_{d(\mu)} - k_{d(\nu)}}(\xi(\ell,\nu), \xi(\ell,\mu))\bigr)$$
$$\times \prod_{(\ell,\nu)\in A} \psi(2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu)) \prod_{(\ell,\nu)\in B} \psi^c(2^{k_{d(\nu)}+j\ell}\xi(\ell,\nu))$$

and Gf is the composition of directional maximal functions in Lemma 3.2. The second term in the right-hand side of (6.5) is bounded by

$$\sum_{D\in\mathcal{P}(A)} \|\mathcal{M}^{\Lambda-D}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{6.6}$$

Thus in proving (6.2), it suffices to show that

$$\Bigl\| \sup_{(j,K)\in\mathbb{Z}\times\mathbb{Z}(d(1),\dots,d(m))} |\mathfrak{T}^{A,B}_{j,K} * f| \Bigr\|_{L^p(\mathbb{R}^d)} \lesssim B(p,m-1)\|f\|_{L^p(\mathbb{R}^d)}. \tag{6.7}$$

To establish (6.7) we proceed to a dyadic decomposition based on the size of the phase function. We set

$$\widehat{L^{\ell,\nu}_n}(\xi) = \chi(2^n\xi(\ell,\nu)).$$

Denote card(Λ) by the cardinality of the set Λ. For each $n = (n_{\ell,\nu})_{(\ell,\nu)\in\Lambda} \in \mathbb{Z}^{\mathrm{card}(\Lambda)}$, we define $\mathcal{L}_{j,K,n} = \circledast_{(\ell,\nu)\in\Lambda} L^{\ell,\nu}_{k_{d(\nu)}+j\cdot\ell-n_{\ell,\nu}}$ where $\circledast_{i=1}^N A_i = A_1 * \cdots * A_N$. Then we can decompose

$$\mathfrak{T}^{A,B}_{j,K} * f = \sum_{n\in\mathbb{Z}^{\mathrm{card}(\Lambda)}} \mathfrak{T}^{A,B}_{j,K} * \mathcal{L}_{j,K,n} * f.$$

It is easy to see that the support of $\widehat{\mathcal{L}_{j,K,n}}$ is contained in

$$2^{-n_{\ell,\nu}-2} \le 2^{k_{d(\nu)}+j\ell}|\xi(\ell,\nu)| \le 2^{-n_{\ell,\nu}+1}$$

for each $(\ell,\nu)\in\Lambda$.

In order to show (6.7), we prove that for each fixed $n = (n_{\ell,\nu})$,

$$\begin{aligned} \Bigl\| \sup_{(j,K)\in\mathbb{Z}\times[\mathbb{Z}(d(1),\dots,d(m))]} |\mathfrak{T}^{A,B}_{j,K} * \mathcal{L}_{j,K,n} * f| \Bigr\|_{L^p(\mathbb{R}^d)} \\ \lesssim 2^{-\epsilon\sum_{(\ell,\nu)\in\Lambda}|n_{\ell,\nu}|} B(p,m-1)\|f\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{6.9}$$

We shall prove (6.9) in the rest of our paper. To do this we first claim that in the region $\{\xi : \widehat{\mathfrak{T}^{A,B}_{j,K} * \mathcal{L}_{j,K,n}}(\xi) \neq 0\}$

$$|\widehat{T^{A,B}_{j,K}}(\xi)| \lesssim 2^{-\epsilon\sum_{(\ell,\nu)\in B}|n_{\ell,\nu}|} \tag{6.10}$$

$$|\widehat{T^{A,B}_{j,K}}(\xi)| \lesssim 2^{-\epsilon\sum_{(\ell,\nu)\in A}|n_{\ell,\nu}|}. \tag{6.11}$$

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Proof of (6.10) **and** (6.11) Recall that

$$\widehat{T^{A,B}_{j,K}}(\xi) = \int e^{i\sum_{(\ell,\nu)\in B}\xi(\ell,\nu)2^{k_{d(\nu)}+j\ell}t^\ell} \prod_{(\ell,\nu)\in A}(e^{i\xi(\ell,\nu)2^{k_{d(\nu)}+j\ell}t^\ell}-1)\varphi(t)\,dt,$$

where

$$|\xi(\ell,\nu)|2^{k_{d(\nu)}+j\ell} \approx 2^{n_{\ell,\nu}} \gtrsim 1 \text{ for } (\ell,\nu)\in B,$$
$$|\xi(\ell,\nu)|2^{k_{d(\nu)}+j\ell} \approx 2^{n_{\ell,\nu}} \lesssim 1 \text{ for } (\ell,\nu)\in A.$$

Let $\mathfrak{n}(\ell) = \max\{n_{\ell,\nu} : (\ell,\nu)\in B_\ell\}$ where $B_\ell = \{(\ell,\nu)\in B :\ \ell \text{ is fixed}\}$. Then from the conditions (4.2) and (6.8), we can observe

$$\Big|\sum_{(\ell,\nu)\in B_\ell}\xi(\ell,\nu)2^{k_{d(\nu)}+j\ell}\Big| \sim\ 2^{\mathfrak{n}(\ell)}.$$

Thus by applying Van der Corput's lemma, there exists a constant $c>0$ such that

$$|\widehat{T^{A,B}_{j,K}}(\xi)| \lesssim 2^{-c\max\{\mathfrak{n}(1),\dots,\mathfrak{n}(q)\}}.$$

By using the Mean Value Theorem,

$$|\widehat{T^{A,B}_{j,K}}(\xi)| \lesssim \prod_{(\ell,\nu)\in A}|2^{k_{d(\nu)}+j\cdot\ell}\xi(\ell,\nu)| \approx 2^{-\epsilon\sum_{(\ell,\nu)\in A}|n_{\ell,\nu}|}.$$

By putting things together, we complete the proof of the claim. ∎

For the proof of (6.9) we replace the supremum by a square summation. In doing this, we need to be careful for the number of parameters $k_{d(1)},\dots,k_{d(m)}$, and j. We rewrite $\mathfrak{T}^{A,B}_{j,K} * \mathfrak{L}_{j,K,n}$ as

$$T^{A,B}_{j,K} * \Big(\circledast_{(\ell,\nu),(\ell,\mu)\in\Lambda}(\delta - A^{\mathfrak{a}^\nu_\ell,\mathfrak{a}^\mu_\ell}_{(k_{d(\nu)}+j\ell)-(k_{d(\mu)}+j\ell)})\Big)$$
$$* \Big(\circledast_{(\ell,\nu)\in A}P^{\mathfrak{a}^\nu_\ell}_{k_{d(\nu)}+j\ell}\Big) * \Big(\circledast_{(\ell,\nu)\in B}(\delta - P^{\mathfrak{a}^\nu_\ell}_{k_{d(\nu)}+j\ell})\Big) * \circledast_{(\ell,\nu)\in\Lambda}L^{\ell,\nu}_{k_{d(\nu)}+j\cdot\ell-n_{\ell,\nu}} \tag{6.12}$$

where $\mathfrak{a}^\nu_\ell$ is defined as in (5.4) and δ is a dirac measure at 0 in $\mathbb{R}^d$. Note that all measures in (6.12) depend only on the parameters

$$\{k_{d(\nu)} + j\ell : (\ell,\nu)\in\Lambda\}. \tag{6.13}$$

We can check that the measure $T^{A,B}_{j,K}$ is determined by $k_{d(\nu)}+j\ell$ for each $(\ell,\nu)\in\Lambda$ in view of (6.3). All other measures

$$A^{\mathfrak{a}^\nu_\ell,\mathfrak{a}^\mu_\ell}_{(k_{d(\nu)}+j\ell)-(k_{d(\mu)}+j\ell)},\quad P^{\mathfrak{a}^\nu_\ell}_{k_{d(\nu)}+j\ell},\quad L^{\ell,\nu}_{k_{d(\nu)}+j\cdot\ell-n_{\ell,\nu}}$$

are determined by $k_{d(\nu)}+j\ell$ for each $(\ell,\nu)\in\Lambda$, as we have seen from their definitions. Let us investigate Λ. We can assume that for each $\nu=1,\dots,m$, there exists at least one $\ell\in\{1,\dots,q\}$ such that $(\ell,\nu)\in\Lambda$. If not, our maximal operator belongs to the class $\mathcal{A}_r$ with $r\le m-1$. Now we consider the following two cases.

Case I. Suppose that for each $\nu=1,\dots,m$, there exists a unique ℓ such that $(\ell,\nu)\in\Lambda$. Let us denote such ℓ by $\ell(\nu)$. By combining this with (6.13), we see that the measures in (6.12) actually depend on only m parameters,

$$\mathfrak{j}_1=k_{d(1)}+j\ell(1),\dots,\mathfrak{j}_m=k_{d(m)}+j\ell(m),$$

which allow us to simplify the notation by renaming

$$\begin{aligned}
\mathcal{T}_{\mathfrak{j}_1,\dots,\mathfrak{j}_m}&=T^{A,B}_{j,K},\\
\mathcal{A}_{\mathfrak{j}_1,\dots,\mathfrak{j}_m}&=\underset{(\ell,\nu),(\ell,\mu)\in\Lambda}{\circledast}\big(\delta-A^{\mathfrak{a}^\nu_\ell,\mathfrak{a}^\mu_\ell}_{(k_{d(\nu)}+j\ell)-(k_{d(\mu)}+j\ell)}\big),\\
\mathcal{P}_{\mathfrak{j}_1,\dots,\mathfrak{j}_m}&=\Big(\underset{(\ell,\nu)\in A}{\circledast}P^{(\ell,\nu)}_{k_{d(\nu)}+j\ell}\Big)*\Big(\underset{(\ell,\nu)\in B}{\circledast}\big(\delta-P^{\mathfrak{a}^\nu_\ell}_{k_{d(\nu)}+j\ell}\big)\Big),\\
\mathcal{L}_{\mathfrak{j}_1,\dots,\mathfrak{j}_m}&=\underset{(\ell,\nu)\in\Lambda}{\circledast}L^{\ell,\nu}_{k_{d(\nu)}+j\cdot\ell-n_{\ell,\nu}}.
\end{aligned}$$

Let $J=(\mathfrak{j}_1,\dots,\mathfrak{j}_m)$. Then the left-hand side of (6.9) is bounded by the square sum

$$\Big\|\Big(\sum_{J\in\mathbb{Z}^m}|\mathcal{T}_J*\mathcal{A}_J*\mathcal{P}_J*\mathcal{L}_J*f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^d)}. \tag{6.14}$$

Case II. Suppose that there exists a ν among $1,\dots,m$ such that $(\ell,\nu)\in\Lambda$ for more than one ℓ. For example, for $\nu=m$, there exist two $\ell(m)$ and $\ell'(m)$ where $\ell(m)\ne\ell'(m)$ such that both $(\ell(m),m)$ and $(\ell'(m),m)$ are in Λ. Thus we see that $\mathfrak{T}^{A,B}_{j,K}(s)*\mathcal{L}_{j,K,n}$ actually depends on $m+1$ parameters:

$$\mathfrak{j}_1=k_{d(1)}+j\ell(1),\dots,\mathfrak{j}_m=k_{d(m)}+j\ell(m)\quad\text{and}\quad\mathfrak{j}_{m+1}=k_{d(m)}+j\ell'(m).$$

This follows from the fact that there exists an invertible $(m+1)\times(m+1)$ matrix G

$$G=\begin{pmatrix}1&0&\cdots&0&\ell(1)\\0&1&\cdots&0&\ell(2)\\0&0&\ddots&0&\vdots\\0&0&\cdots&1&\ell(m)\\0&0&\cdots&1&\ell'(m)\end{pmatrix}$$

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satisfying $G(k_{d(1)}, \dots, k_{d(m)}, j) = (\mathfrak{j}_1, \dots, \mathfrak{j}_m, \mathfrak{j}_{m+1})$. We then rename

$$\mathcal{T}_{\mathfrak{j}_1,\dots,\mathfrak{j}_{m+1}} = T^{A,B}_{j,K},$$

$$\mathcal{A}_{\mathfrak{j}_1,\dots,\mathfrak{j}_{m+1}} = \Big(\mathop{\circledast}_{(\ell,\nu),(\ell,\mu)\in\Lambda} (\delta - A^{\mathfrak{a}^\nu_\ell,\mathfrak{a}^\mu_\ell}_{(k_{d(\nu)}+j\ell)-(k_{d(\mu)}+j\ell)}) \Big),$$

$$\mathcal{P}_{\mathfrak{j}_1,\dots,\mathfrak{j}_{m+1}} = \Big(\mathop{\circledast}_{(\ell,\nu)\in A} P^{\mathfrak{a}^\nu_\ell}_{k_{d(\nu)}+j\ell} \Big) * \Big(\mathop{\circledast}_{(\ell,\nu)\in B} (\delta - P^{\mathfrak{a}^\nu_\ell}_{k_{d(\nu)}+j\ell}) \Big),$$

$$\mathcal{L}_{\mathfrak{j}_1,\dots,\mathfrak{j}_{m+1}} = \mathop{\circledast}_{(\ell,\nu)\in\Lambda} L^{\ell,\nu}_{k_{d(\nu)}+j\cdot\ell-n_{\ell,\nu}}.$$

Let $J = (\mathfrak{j}_1, \dots, \mathfrak{j}_{m+1})$. Then the left-hand side of (6.9) is bounded by the square sum

$$\Big\| \Big(\sum_{J\in\mathbb{Z}^{m+1}} |\mathcal{T}_J * \mathcal{A}_J * \mathcal{P}_J * \mathcal{L}_J * f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)}. \tag{6.15}$$

Now we turn to the estimate of (6.9). To prove this we shall use (6.10), (6.11), (6.14) and (6.15).

Proof of (6.9) **for** $p = 2$ We use the Plancherel theorem and the orthogonality of $\mathcal{L}_J$ combined with (6.10) and (6.11) to obtain the estimates for (6.14) and (6.15) of the form

$$\Big\| \Big(\sum_{J\in Z} |\mathcal{T}_J * \mathcal{A}_J * \mathcal{P}_J * \mathcal{L}_J * f|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-\epsilon \sum_{(\ell,\nu)\in\Lambda} |n_{\ell,\nu}|} \|f\|_{L^2(\mathbb{R}^d)}, \tag{6.16}$$

where Z is taken as $\mathbb{Z}^m$ and $\mathbb{Z}^{m+1}$ for Case I and Case II, respectively. Therefore (6.9) for $p = 2$ has been proved with the bound $B(p, m-1) = 1$. ■

Hence (6.9) combined with (6.7) and (6.2) completes the proof of (6.1) for $p = 2$, which was the goal of this section.

7 Bootstrap Argument

By applying Lemmas 3.2 through 3.4 combined with the $L^p(\ell^2) \to L^p(\ell^2)$ boundedness of the strong maximal function majorizing $\mathcal{P}_J * f$ or $\mathcal{L}_J * f$, we obtain that

$$\Big\| \Big(\sum_{J\in Z} |\mathcal{A}_J * \mathcal{P}_J * \mathcal{L}_J * f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{7.1}$$

More direct proof of (7.1) is applying the multi-parameter Marcinkiewicz multiplier theorem after changing the square sum into a linear sum in the same way as Lemmas 3.1–3.4. We shall frequently use (7.1) for the L^p estimate of (7.2) in this section. By using (5.1) and (6.1), we obtain Lemma 4.1 for $p = 2$. By iterated application of Lemma 4.1, we obtain Lemma 4.2 when $p = 2$. In this section we use a bootstrap argument to prove Lemmas 4.1 and 4.2 for $1 < p < 2$.

We first deal with the range $4/3 < p < 2$.

Proof of (5.1) By using Lemma 4.2 with $p = 2$, we see that the maximal operator defined by

$$f \mapsto \sup_k \mathcal{M}^{(d(1),\dots,d(m))}_{\nu,\mu,k} f$$

is bounded in $L^2(\mathbb{R}^d)$ with the L^2-operator norm $B(2, m-1)$. Thus we are able to apply Lemma 3.6 to obtain (5.7), which yields (5.1) for $4/3 < p < 2$. ■

Proof of (6.1) Since Lemma 4.2 holds for $p = 2$, we can apply Lemma 3.5 with $q = 2$ and (7.1) to obtain that for $4/3 < p < 2$

$$\Big\| \Big(\sum_{J\in Z} |\mathcal{T}_J * \mathcal{A}_J * \mathcal{P}_J * \mathcal{L}_J * f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \lesssim B(p, m-1) \|f\|_{L^p(\mathbb{R}^d)}. \tag{7.2}$$

Now we interpolate (6.16) with (7.2) to obtain that for $4/3 < p < 2$,

$$\Big\| \Big(\sum_{J\in Z} |\mathcal{T}_J * \mathcal{A}_J * \mathcal{P}_J * \mathcal{L}_J * f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-\epsilon_1 \sum_{(\ell,\nu)\in\Lambda} |n_{\ell,\nu}|} \|f\|_{L^p(\mathbb{R}^d)}, \tag{7.3}$$

which yields (6.9) for $4/3 < p < 2$. We combine this with (6.6) to obtain (6.1) for $4/3 < p < 2$. ■

By (5.1) and (6.1) we now obtain Lemma 4.1 for $4/3 < p < 2$. We repeat the argument in Lemma 4.1 finitely many times to obtain Lemma 4.2 for $4/3 < p < 2$.

For the second step we consider the range $8/7 < p \le 4/3$. We use the result of Lemma 4.2 for $4/3 < p \le 2$ and (7.1) in applying Lemmas 3.5 and 3.6 to obtain (5.7) and (7.2). We see that (5.7) yields (5.1). We also see that (7.2) combined with (7.3) and (6.6) gives (6.1). We repeat this process so that p is moving backward, $4/3 \mapsto 8/7 \mapsto 16/15 \mapsto \cdots$, satisfying the range restriction of Lemma 3.5 such that $1/p < (1 + 1/q)/2$. ■

Remark 3 For the case $m = 1$, that is, $\mathcal{M}^\Lambda \in \mathcal{A}_1$, the oscillatory term is given by

$$P_{j,K}(\xi, t) = \sum_{\ell=1}^{q} \xi(\ell, 1) 2^{k+j\ell} t^\ell,$$

where $K = (k, \dots, k)$. Then we have already a good decay property

$$\begin{aligned} |\widehat{\mu_{j,K}}(\xi_1, \dots, \xi_d)| &\lesssim \min\{|2^{k+j\ell}\xi(\ell, 1)|^{-\epsilon} : \ell = 1, \dots, q\} \\ &\approx \min\{2^{-\epsilon|n_\ell|}, \ell = 1, \dots, q\}, \end{aligned}$$

where $2^{n_\ell - 1} \le 2^{k+j\ell}|\xi(\ell, 1)| \le 2^{n_\ell+1}$ and $n_\ell > 0$. This combined with (5.6) leads us to (6.16). Thus we use Lemma 3.5 to obtain Lemmas 4.1 and 4.2 by showing only (6.1) with $B(p, 0) = 0$.

8 Proof of Theorem 1.2

We shall obtain Theorem 1.2 by using Theorem 1.1 and the lifting lemma used in [5, p. 484]. We first deal with the localized version of the maximal operator in (1.2) defined by

$$\mathcal{S}_\rho f(x) = \sup_{K\in\mathbb{Z}^{qd},0<r<\rho} \frac{1}{2r}\int_{-r}^{r} |f(x-P_K(t))|\,dt.$$

Let $N = qd$ and $u = (u^1,\dots,u^d) \in \mathbb{R}^N$ where $u^\tau = (u^\tau_1,\dots,u^\tau_q) \in \mathbb{R}^q$. We define a linear transform $L\colon \mathbb{R}^N \to \mathbb{R}^d$ by $L(u) = (\sum_{\ell=1}^q a^1_\ell u^1_\ell,\dots,\sum_{\ell=1}^q a^d_\ell u^d_\ell)$, and define a maximal operator

$$\mathcal{M}^{\mathrm{lac}}_N f(u) = \sup_{K\in\mathbb{Z}^N,r>0} \frac{1}{2r}\int_{-r}^{r} |f(u-Q_K(t))|\,dt,$$

where $Q_K(t) = ((2^{k_{1,1}}t_1,\dots,2^{k_{1,q}}t^q),\dots,(2^{k_{q,1}}t_1,\dots,2^{k_{q,q}}t^q))$. We know that $\mathcal{M}^{\mathrm{lac}}_N$ is bounded in $L^p(\mathbb{R}^N)$ from Theorem 1.1. By using this we obtain that for any $R>0$,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathcal{S}_\rho f(x)|^p\,dx &\lesssim \frac{1}{R^N}\int_{|u|<R}\int_{\mathbb{R}^d} |\mathcal{S}_\rho f(x)|^p\,dxdu\\
&\lesssim \frac{1}{R^N}\int_{\mathbb{R}^d}\int_{\mathbb{R}^N} |\mathcal{M}^{\mathrm{lac}}_N F_x(u)|^p\,dudx\\
&\lesssim \frac{1}{R^N}\int_{\mathbb{R}^d}\int_{\mathbb{R}^N} |F_x(u)|^p\,dudx\\
&\lesssim \frac{(R+\rho)^N}{R^N}\int_{\mathbb{R}^d} |f(x)|^p\,dx,
\end{aligned}$$

where $F_x(u) = f(x+L(u))\chi_{B_{R+\rho}}(u)$ with $\chi_{B_{R+\rho}}$ a characteristic function supported in the ball centered zero with radius $R+\rho$. Take $R\to\infty$; then we obtain that $\|\mathcal{S}_\rho\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \le C$ with C independent of ρ. By using the Monotone Convergence Theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathcal{S}f(x)|^p\,dx &= \int_{\mathbb{R}^d} \lim_{\rho\to\infty} |\mathcal{S}_\rho f(x)|^p\,dx\\
&= \lim_{\rho\to\infty}\int_{\mathbb{R}^d} |\mathcal{S}_\rho f(x)|^p\,dx\\
&\le C^p\int_{\mathbb{R}^d} |f(x)|^p\,dx.
\end{aligned}$$

This completes the proof of Theorem 1.2.

Remark 4 One may work with the maximal operator associated with $Q_K(t)$ in order to prove Theorem 1.1. However, one can check that things are not simplified by just applying a lifting argument.

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