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ON BOUNDS FOR THE LIPSCHITZ CONSTANT OF THE REMAINDER IN POLYNOMIAL APPROXIMATION

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Suppose f is a function possessing a kth order derivative, the derivative being Lipschitz continuous of order α, $0 < \alpha \le 1$, on $[-1,1]$. Let p_n be a polynomial of degree $\le n$ approximating to f on $[-1,1]$ such that if $r_n = f - p_n$ then $||r_n||_\infty \le An^{-k-\alpha}$. Define

$$M_n(\beta) = \sup_{x_1,x_2\in[-1,1]} |r_n(x_2) - r_n(x_1)|/|x_2-x_1|^\beta ,$$

where $0 < \beta \le 1$. Upper bounds are obtained for $M_n(\beta)$ when $k \ge 1$ thereby generalizing results previously given for functions which are only Lipschitz continuous on $[-1,1]$.

1. Introduction

The recent interest in quadrature rules for the evaluation of Cauchy principal value integrals over finite intervals has required a knowledge of the behaviour of the remainder when a function is approximated by a polynomial. A useful result was a lemma given in 1957 by Kalandiya [2]. More recently, Ioakimidis [1] has given what he calls an "improvement" of Kalandiya's lemma. We shall now summarise these

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results.

Suppose that a given function f is Lipschitz continuous of order α, where $0 < \alpha \leq 1$, on the interval $[-1,1]$. That is, for every pair of points $x_1, x_2 \in [-1,1]$ we have

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1|^{\alpha}, \tag{1.1}$$

where L and α are independent of x_1 and x_2. We write $f \in \text{Lip}\,\alpha$. The Lipschitz constant is the smallest number L for which (1.1) is satisfied and if we know this we sometimes write $f \in \text{Lip}_L\alpha$. For each positive integer n, let p_n be a polynomial of degree $\leq n$ which approximates to f on $[-1,1]$ in some way, and denote by r_n the remainder in this approximation so that

$$r_n = f - p_n. \tag{1.2}$$

Obviously for $f \in \text{Lip}\,\alpha$, $r_n \in \text{Lip}\,\alpha$. Furthermore, we know that $r_n \in \text{Lip}\,\gamma$ for all $\gamma \leq \alpha$. We shall be interested in obtaining bounds for the quantity $M_n(\beta)$, say, defined by

$$M_n(\beta) = \sup_{x_1, x_2 \in [-1,1]} \frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^{\beta}}, \tag{1.3}$$

where β is a number whose value may be limited by α (see Theorem 1.1, below). Thus $M_n(\beta)$ is the Lipschitz constant for the function r_n considered as an element of the space Lip β.

The uniform norm for any continuous function g defined on $[-1,1]$ will be denoted and defined in the usual way by

$$||g||_{\infty} = \max_{-1 \leq x \leq 1} |g(x)|. \tag{1.4}$$

It is well known, see for example Meinardus [3, Theorem 43], that if p_n is the polynomial of best uniform approximation of degree $\leq n$ to $f \in \text{Lip}\,\alpha$, then $||r_n||_{\infty} \leq An^{-\alpha}$. The results of Kalandiya and Ioakimidis can be given together in the following theorem. (Throughout, $A_1, A_2, A_3, \ldots$ will denote positive constants independent of n).

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THEOREM 1.1. *Suppose* $f \in Lip\ \alpha$ *and* p_n *is a polynomial of degree* $\leq n$ *such that*

$$(1.5) \qquad ||r_n||_\infty \leq A_1 n^{-\alpha} .$$

Then

(i) *for every polynomial of degree* $\leq n$ *satisfying* (1.5) *and for* $0 < \beta \leq \alpha/2$,

$$(1.6) \qquad M_n(\beta) \leq A_2 n^{-\alpha+2\beta} ;$$

(ii) *there exists a polynomial of degree* $\leq n$ *satisfying* (1.5) *and such that for* $0 < \beta \leq \alpha$,

$$(1.7) \qquad M_n(\beta) \leq A_3 n^{-\alpha+\beta} .$$

Theorem 1.1 (i) was first stated and proved by Kalandiya [2]. Theorem 1.1 (ii) was stated and proved recently by Ioakimidis [1]. The latter proof makes use of a result due to Stečkin [4] which we shall state as Theorem 1.2. As a result of it, Ioakimidis' proof is considerably simpler than that of Kalandiya's and indeed can be used to give a simpler proof of Kalandiya's lemma.

For any continuous function f defined on $[-1,1]$, its modulus of continuity is denoted and defined by

$$(1.8) \qquad \omega(f;\ \delta) = \max_{\substack{|x_2-x_1|\leq\delta \\ x_1,x_2\in[-1,1]}} |f(x_2) - f(x_1)| .$$

Two simple properties of ω which follow immediately from this definition and which we shall use subsequently are

$$(1.9) \qquad \begin{cases} \omega(f;\ \delta) \leq 2||f||_\infty & \text{and} \\ \omega(f;\ \delta) \leq \delta.||f'||_\infty , \end{cases}$$

the latter result following from the mean value theorem where we assume that f' exists and is continuous on $[-1,1]$.

THEOREM 1.2. (Stečkin [4]). *Suppose* p_n *is a polynomial of degree* $\leq n$. *Then*

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$$(1.10) \qquad |p_n'(x)| \le \begin{cases} (n/2)(1-x^2)^{-\frac{1}{2}}\,\omega(p_n;\ \pi/n), & |x| < 1, \\ (n^2/2)\omega(p_n;\ \pi/n), & |x| \le 1\ . \end{cases}$$

If we choose $p_n = T_n$, the Chebyshev polynomial of the first kind, then $\omega(T_n;\ \pi/n) = 2$. Since $|T_n'(x)| \le n(1-x^2)^{-\frac{1}{2}}$ for $|x| < 1$ and since $|T_n'(x)| \le n^2$ for $|x| \le 1$, it follows that the coefficients of ω on the right hand side of (1.10) cannot be improved.

Returning to Theorem 1.1 (i) we observe that the exponent of n in (1.6) cannot be improved either, in that there exists $f \in \text{Lip}\ \alpha$ and a polynomial satisfying (1.5) such that $M_n(\beta) \ge A_4 n^{-\alpha+2\beta}$. To see this suppose we choose $f \equiv 0$ and $p_n(x) = A_1 n^{-\alpha}\, T_n(x)$. Then (1.5) is satisfied. If we choose $x_1 = 1$, $x_2 = \cos(\pi/n)$ then, from (1.3),

$$\begin{aligned} M_n(\beta) &\ge \frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} \\ &= \frac{A_1|T_n(x_2) - T_n(x_1)|}{n^\alpha|x_2 - x_1|^\beta} \\ &= \frac{2^{1-\beta}A_1}{n^\alpha(\sin(\pi/2n))^{2\beta}} \\ &\ge \frac{2^{1+\beta}A_1}{\pi^{2\beta}} \cdot n^{-\alpha+2\beta}\ , \end{aligned}$$

which is the desired result.

In this paper we wish to extend the result of Theorem 1.1 to the case when, for some integer $k \ge 1$, the kth order derivative of f is in Lip α. We then modify the bound for $||r_n||_\infty$ as given in (1.5) but again we require bounds for the Lipschitz constant $M_n(\beta)$. The results are given in Theorem 2.1.

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2. Functions whose derivatives are Lipschitz continuous.

Without any further ado we shall now state and prove the principal result of this paper.

THEOREM 2.1. *Suppose that for some* $k \geq 1$, $f^{(k)} \in Lip\ \alpha$ *and let* p_n *be a polynomial of degree* $\leq n$ *such that*

$$||r_n||_\infty \leq A_5 n^{-k-\alpha}. \tag{2.1}$$

Then

(i) for every polynomial of degree $\leq n$ *satisfying (2.1) and for* $0 < \beta \leq 1$

$$M_n(\beta) \leq A_6 n^{-k-\alpha+2\beta}\ ; \tag{2.2}$$

(ii) there exists a polynomial of degree $\leq n$ *satisfying (2.1) and such that for* $0 < \beta \leq 1$,

$$M_n(\beta) \leq A_7 n^{-k-\alpha+\beta}\ . \tag{2.3}$$

Proof. We observe, see Meinardus [3, Theorem 45] that if p_n is the polynomial of best uniform approximation to f on $[-1,1]$ then it satisfies (2.1).

(i) First, we shall show that for every polynomial satisfying (2.1),

$$||r_n'||_\infty \leq A_8 n^{-k-\alpha+2}\ . \tag{2.4}$$

Let q^*_{n-1} denote the polynomial, of degree $\leq (n-1)$, of best uniform approximation to f' on $[-1,1]$. Since the $(k-1)$st derivative of f' is in Lip α we have [3, Theorem 45] that

$$||f' - q^*_{n-1}||_\infty \leq A_9 (n-1)^{-(k-1)-\alpha} \leq A_{10} n^{-k-\alpha+1}\ , \tag{2.5}$$

for $n \geq 2$. For any $x \in [-1,1]$,

$$|r_n'(x)| = |(f'(x) - q^*_{n-1}(x)) + (q^*_{n-1}(x) - p_n'(x))|$$

$$\leq A_{10} n^{-k-\alpha+1} + |(s_n - p_n)'(x)|, \tag{2.6}$$

say, where s_n is any polynomial of degree $\leq n$ such that $s_n' = q^*_{n-1}$. By Theorem 1.2, for any $x \in [-1,1]$,

$$\begin{aligned}|(s_n-p_n)'(x)| &\le (n^2/2)\omega(s_n-p_n;\ \pi/n)\\ &= (n^2/2)\omega((s_n-f)+(f-p_n);\ \pi/n)\\ &\le (n^2/2)\{\omega(s_n-f;\ \pi/n)+\omega(r_n;\ \pi/n)\}\\ &\le (n^2/2)\{(\pi/n)||s_n'-f'||_\infty + 2||r_n||_\infty\}\ ,\end{aligned}$$

on using (1.9). From (2.1) and (2.5) we find

$$|(s_n-p_n)'(x)| \le \{A_5 + (\pi/2)A_{10}\}n^{-k-\alpha+2}\ .$$

Substituting this back into (2.6) then, since x is arbitrary, we establish (2.4) where $A_8 = A_5 + (1+\pi/2)A_{10}$.

We can now complete the proof of (i). If x_1, x_2 are any two points of $[-1, 1]$ such that $|x_2-x_1| \ge 1/n^2$ then

(2.7) $$\frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} \le 2A_5 n^{-k-\alpha+2\beta}\ .$$

On the other hand if $|x_2 - x_1| \le 1/n^2$ then, for $0 < \beta \le 1$, we have

$$\begin{aligned}\frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} &\le |x_2 - x_1|^{1-\beta}\ ||r_n'||_\infty\\ &\le A_8 n^{-k-\alpha+2\beta}\ ,\end{aligned}$$ (2.8)

by (2.4). Inequality (2.2) now follows immediately from (2.7) and (2.8) where $A_6 = \max\{2A_5, A_8\}$.

(ii) To prove (2.3) we shall use an imbedding argument similar to that given by Ioakimidis in [1] for the proof of Theorem 1.1 (ii). First we shall show the existence of a polynomial p_n which not only satisfies (2.1) but also

(2.9) $$||r_n'||_\infty \le A_{11}n^{-k-\alpha+1}\ .$$

To do this, choose $c > 1$ and write $x = ct$. On the interval $-1 \le t \le 1$, define a function $F(t)$, say, such that (i) $F^{(k)}(t) \in \text{Lip}\ \alpha$ and (ii) $F(t) = f(ct)$ on $-1/c \le t \le 1/c$. In the intervals $[-1, -1/c]$

and $[1/c, 1]$ we need to continue the definition of f so that F possesses a continuous kth order derivative which, on $-1 \le t \le 1$, is in Lip α. One way of doing this is to define $F^{(k)}$ as

$$F^{(k)}(t) = \begin{cases} f^{(k)}(-1) & , \ -1 \le t \le -1/c , \\ f^{(k)}(ct) & , \ -1/c \le t \le 1/c , \\ f^{(k)}(1) & , \ 1/c \le t \le 1 , \end{cases}$$

and then to recover F by integration choosing constants of integration appropriately on $[-1,-1/c]$ and $[1/c, 1]$. Let P_n^* denote the polynomial of best uniform approximation of degree $\le n$ to F on $-1 \le t \le 1$. Then, since $F^{(k)} \in \text{Lip}\ \alpha$, $||F - P_n^*||_\infty \le A_{12}n^{-k-\alpha}$. For $-1 \le x \le 1$, choose

$$p_n(x) = p_n(ct) = P_n^*(t) , \ -1/c \le t \le 1/c.$$

Then certainly (2.1) is satisfied for this polynomial. Now, for $-1 \le x \le 1$,

$$r_n'(x) = \frac{1}{c}\frac{d}{dt}(F(t) - P_n^*(t))$$

(2.10)
$$= \frac{1}{c}\left\{(\frac{dF}{dt} - Q_{n-1}^*(t)) + (Q_{n-1}^*(t) - \frac{dP_n^*}{dt})\right\} ,$$

where $Q_{n-1}^*(t)$ is the polynomial of best uniform approximation of degree $\le (n-1)$ to dF/dt on $-1 \le t \le 1$. Again, for $n \ge 2$ and any $t \in [-1, 1]$ we know that

(2.11)
$$\left|\frac{dF}{dt} - Q_{n-1}^*(t)\right| \le A_{13}n^{-k-\alpha+1} .$$

If we define $S_n(t)$ to be a polynomial of degree $\le n$ such that $dS_n/dt = Q_{n-1}^*(t)$ then from (2.10) and (2.11) we have

(2.12)
$$c|r_n'(x)| \le A_{13}n^{-k-\alpha+1} + |\frac{d}{dt}(S_n(t) - P_n^*(t))|.$$

For points in $[-1/c, 1/c]$ we have, by Theorem 1.2,

$$\left|\frac{d}{dt}(S_n(t) - P_n^*(t))\right| \le \frac{n}{2(1-t^2)^{\frac{1}{2}}}\,\omega(S_n - P_n^*; \tfrac{\pi}{n})$$

$$\le \frac{nc}{2(c^2-1)^{\frac{1}{2}}}\left\{\omega(S_n - F; \tfrac{\pi}{n}) + \omega(F - P_n^*; \tfrac{\pi}{n})\right\}$$

$$\le \frac{nc}{2(c^2-1)^{\frac{1}{2}}}\left\{\frac{\pi}{n}\max_{-1\le t\le 1}\left|\frac{dF}{dt} - Q_{n-1}^*(t)\right| + \right.$$

$$\left. + 2A_{12}n^{-k-\alpha}\right\}$$

$$\le \frac{c}{2(c^2-1)^{\frac{1}{2}}}(\pi A_{13} + 2A_{12})n^{-k-\alpha+1} , \tag{2.13}$$

on using (2.11). Combining (2.13) with (2.12) gives (2.9) where $A_{11} = A_{12}(c^2-1)^{-\frac{1}{2}} + A_{13}\{(\pi/2)(c^2-1)^{-\frac{1}{2}} + c^{-1}\}$.

We can now complete the proof of (ii). If x_1, x_2 are any two points of $[-1,1]$ such that $|x_2 - x_1| \ge 1/n$ then

$$\frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} \le 2A_5 n^{-k-\alpha+\beta} . \tag{2.14}$$

On the other hand if $|x_2 - x_1| \le 1/n$ then, from (2.9),

$$\frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} \le |x_2 - x_1|^{1-\beta}.\,||r_n'||_\infty \tag{2.15}$$

$$\le A_{11}n^{-k-\alpha+\beta} ,$$

provided that $\beta \le 1$. From (2.14) and (2.15), (2.3) follows where we choose $A_7 = \max\{2A_5, A_{11}\}$. This completes the proof of the theorem.

As a final comment we note, by choosing $f \equiv 0$, $p_n = A_5 n^{-k-\alpha} T_n$ and taking points $x_1 = 1$, $x_2 = \cos(\pi/n)$, that the exponent of n in (2.2) cannot be improved.

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