ON LATTICES OF VARIETIES OF METABELIAN GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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This paper presents an example to show that the lattice of subvarieties of $\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_{11}$ is non-distributive. The example is used further to show that a certain 'canonic'' description for non-nilpotent subvarieties of $\mathfrak{A}_p\mathfrak{A}_{p^2}$, p prime, is generally not unique.

1. Introduction

The notation and terminology used follows Hanna Neumann [4] with the addition of lat $\mathfrak{B}$ and lat G to denote respectively the lattice of subvarieties of a variety $\mathfrak{B}$ and the lattice of verbal subgroups of a group G.

Recently, Kovács and Newman [3] showed that lat $(\mathfrak{A}_{p^\alpha}\mathfrak{A}_p)$ is distributive for all primes p and all positive integers α. In contrast to this however, in some unpublished work the same authors demonstrated non-distributivity in lat $(\mathfrak{A}_2\mathfrak{A}_8 \wedge \mathfrak{N}_6)$, thereby showing that lat $(\mathfrak{A}_p\mathfrak{A}_{p^\alpha})$ is generally not distributive. In § 2 of this paper another example of non-distributivity in lat $(\mathfrak{A}_p\mathfrak{A}_{p^\alpha})$ is given, in this case with α as small as it can be, namely $\alpha = 2$, and with $p = 3$. The result is:

THEOREM 1. *The lattice of subvarieties of* $\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_{11}$ *is not distributive.*

Note that since lat $\mathfrak{A}^2$ has minimum condition (Cohen [2]) every metabelian variety $\mathfrak{B}$ can be expressed as the irredundant join of finitely many join-irreducible subvarieties, and in this context non-distributivity means precisely that not every $\mathfrak{B}$ has a unique expression of this kind. However, in lat $(\mathfrak{A}_p\mathfrak{A}_{p^2})$, p prime, a weaker form of uniqueness persists, namely that described in the second part of Theorem 2 below. This theorem, the proof of which occupies the bulk of the author's Ph.D. thesis (Australian National University, 1968), is stated here without proof; it is hoped that a proof will be published at a later date.

THEOREM 2. *The varieties* $\mathfrak{Z}_k$, $k = 1, 2, \cdots$, *defined by*

$$\mathfrak{Z}_k = \begin{cases} \mathfrak{A}_p\mathfrak{A}_{p^2} \wedge \mathfrak{N}_k\mathfrak{A}_p \wedge \mathfrak{B}_{p^2}, & \text{if } 1 \leqq k \leqq p-1 \\ \mathfrak{A}_p\mathfrak{A}_{p^2} \wedge \mathfrak{N}_k\mathfrak{A}_p, & \text{if } p \leqq k \end{cases}$$

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form a properly ascending chain of subvarieties of $\mathfrak{A}_p\mathfrak{A}_{p^2}$, *and this chain, with* $\mathfrak{A}_p\mathfrak{A}_{p^2}$ *itself adjoined, makes up a complete list of the non-nilpotent join-irreducible subvarieties of* $\mathfrak{A}_p\mathfrak{A}_{p^2}$. *Moreover, to every non-nilpotent proper subvariety* $\mathfrak{V}$ *of* $\mathfrak{A}_p\mathfrak{A}_{p^2}$ *there exists a nilpotent variety* $\mathfrak{L}$ *and a unique* $\mathfrak{J}_k$ *such that* $\mathfrak{V} = \mathfrak{J}_k \vee \mathfrak{L}$.

In § 3 a closer examination of the example used to establish Theorem 1 will yield the following demonstration of the non-uniqueness, in a strong sense, of the nilpotent component $\mathfrak{L}$ mentioned in Theorem 2.

THEOREM 3. *There exists a subvariety* $\mathfrak{V}$ *of* $\mathfrak{A}_3\mathfrak{A}_9$ *such that* $\mathfrak{V} = \mathfrak{J}_3 \vee \mathfrak{L} = \mathfrak{J}_3 \vee \mathfrak{L}'$, *where* $\mathfrak{J}_3$ *is the non-nilpotent join-irreducible subvariety of* $\mathfrak{A}_3\mathfrak{A}_9$ *defined in Theorem* 2 *and* $\mathfrak{L}$, $\mathfrak{L}'$ *are distinct nilpotent varieties both minimal with respect to the property that their join with* $\mathfrak{J}_3$ *is* $\mathfrak{V}$.

It is natural to ask whether Theorems like 1 and 3 hold for all primes p, and, in relation to Theorem 1, whether the class can be reduced, and if so, how far. Towards an answer to these questions, I have obtained the following information (the proofs will be omited): An example very similar to that in § 2 can be constructed to show that lat $(\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_9)$ is non-distributive, but this smaller class example does not yield the additional result of Theorem 3. Further, essentially the same constructions work for $p = 5$, giving that lat $(\mathfrak{A}_5\mathfrak{A}_{25} \wedge \mathfrak{N}_{25})$ is not distributive and that there exists $\mathfrak{V} \in$ lat $(\mathfrak{A}_5\mathfrak{A}_{25})$ such that $\mathfrak{V} = \mathfrak{J}_5 \vee \mathfrak{L} = \mathfrak{J}_5 \vee \mathfrak{L}'$ with $\mathfrak{L}$, $\mathfrak{L}'$ both nilpotent and minimal but distinct. Almost certainly these examples generalise to cover all primes $p \geqq 3$ but the length of the calculations seems to increase with the prime. For $p = 2$ the construction definitely fails, so that whether or not lat $(\mathfrak{A}_2\mathfrak{A}_4)$ is distributive remains very much an open question. Note however that neither lat $(\mathfrak{A}_2\mathfrak{A}_8)$ nor lat $(\mathfrak{A}_4\mathfrak{A}_4)$ is distributive, the former on account of the Kovács and Newman example previously mentioned, and the latter on account of a result of Bryce [1], who shows that lat $(\mathfrak{A}_{p^2}\mathfrak{A}_{p^2} \wedge \mathfrak{N}_{p+2})$ is not distributive for any prime p.

2. Proof of theorem 1

There is a more-or-less standard method of proving results like Theorem 1; it consists of demonstrating bad behaviour among the verbal subgroups of some suitably chosen relatively free group G and then drawing conclusions about var G. Part of the reason for requiring that G should be relatively free is to ensure that lat G is a sublattice of the lattice of normal subgroups of G, so that in lat G the join and meet of any pair of verbal subgroups of G is respectively their product and set-theoretic intersection. The method is summed up in the following:

LEMMA 4. *Let* G *be a relatively free group. If* lat G *is not distributive then neither is* lat (var G). *In fact, if for some* $C, D_1, D_2 \in$ lat G

$$C \cap D_1D_2 \neq (C \cap D_1)(C \cap D_2), \tag{1}$$

then

(2) $$\mathfrak{U} \vee (\mathfrak{W}_1 \wedge \mathfrak{W}_2) \neq (\mathfrak{U} \vee \mathfrak{W}_1) \wedge (\mathfrak{U} \vee \mathfrak{W}_2),$$

where $\mathfrak{W}_i = \text{var}\,(G/D_i)$ *for* $i = 1, 2$ *and* $\mathfrak{U}$ *is any variety for which* $U(G) = C$.

Proof. The proof is by contradiction. Let F be an absolutely free group of the same rank as G and let $\gamma : F \to G$ be the natural epimorphism. As is easily checked, the map $\mu : \text{lat}\, X_\infty \to \text{lat}\, F$, given by $V\mu = V(F)$ for all $V \in \text{lat}\, X_\infty$, is a lattice epimorphism, and consequently the negation of (2) implies that

$$U(F) \cap W_1(F)W_2(F) = (U(F) \cap W_1(F))(U(F) \cap W_2(F)).$$

Since $W_i(F) \supseteq \ker \gamma$, $i = 1, 2$, the modular law in lat F implies further that

(3) $$U(F)(\ker \gamma) \cap W_1(F)W_2(F) = (U(F)(\ker \gamma) \cap W_1(F))(U(F)(\ker \gamma) \cap W_2(F)).$$

Now if Λ denotes the lattice of verbal subgroups of F which contain $\ker \gamma$ then the map $\bar{\gamma} : \Lambda \to \text{lat}\, G$ induced by γ is a lattice isomorphism (cf. 13.32 in [4]) and therefore an application of $\bar{\gamma}$ to (3) yields

$$U(G) \cap W_1(G)W_2(G) = (U(G) \cap W_1(G))(U(G) \cap W_2(G))$$

which contradicts (1). This completes the proof.

Remark. The assumption in Lemma 4 that G is relatively free cannot in general be dispensed with. For if $\{a, b, c\}$ is a free generating set for $H = F_3(\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_3)$ and $G = H/K$, where K is the (central) cyclic subgroup of H generated by $a^9[a, b, c]$, then lat (var G) is distributive whereas lat G is not even modular.

In consequence of Lemma 4, it is sufficient for the proof of Theorem 1 to demonstrate non-distributivity in lat G, where $G = F_2(\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_{11})$. The example to be exhibited occurs among the verbal subgroups of G contained in $G_{(11)}$, where $G_{(11)}$ is the last non-trivial term of the lower central series of G and is clearly an elementary abelian 3-group. With $\{a, b\}$ a free generating set for G, set $c_i = [b, ia, (10-i)b]$ for $i = 2, \cdots, 9$. Then:

(4) *The set* $\{c_2, \cdots, c_9\}$ $\mathfrak{A}_3$*-freely generates* $G_{(11)}$.

This may be proved as follows: Let $\{a^*, b^*\}$ be a free generating set for $G^* = F_2(\mathfrak{A}_3\mathfrak{A} \wedge \mathfrak{N}_{11})$, let $c_i^* = [b^*, ia^*, (10-i)b^*]$ for $i = 1, \cdots, 10$, and let K be the subgroup of G^* generated by $\{(a^*)^{27}, (b^*)^{27}, c_1^*, c_{10}^*\}$. It may be shown by routine commutator calculations that $[x, y^{27}] = 1$ and $[x, y, z^9] = [x, y, 9z]$ are laws in G^*, so that K is contained in both the centre and the $\mathfrak{A}_3\mathfrak{A}_9$-subgroup of G^*. Moreover it is a straightforward matter to check that G^*/K satisfies the laws $x^{27} = 1$, $[x^9, y^9] = 1$ and $[x, y, z^9] = 1$, and since these laws define $\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_{11}$ within $\mathfrak{A}_3\mathfrak{A} \wedge \mathfrak{N}_{11}$ this means that $G^*/K \in \mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_{11}$. Thus K contains, and therefore is, the $\mathfrak{A}_3\mathfrak{A}_9$-subgroup of G^*, and so it is the kernel of the natural epimorphism $\phi : G^* \to G$ given by $a^* \mapsto a$, $b^* \mapsto b$. Now it follows from Theorem

36.32 in [4] that the set $\{c_1^*, \cdots, c_{10}^*\}$ is an $\mathfrak{A}_3$-free generating set for $G_{(11)}^*$, and since $G_{(11)} = G_{(11)}^*\phi$ it only remains for the proof of (1) to show that $G_{(11)}^* \cap K$ is generated by $\{c_1^*, c_{10}^*\}$. But, modulo the derived group $G_{(2)}^*$ of G^*, $\{a^{*27}, b^{*27}\}$ freely generates a free abelian group and consequently $G_{(2)}^*$, and, a fortiori, $G_{(11)}^*$ does not contain any element of the form $(a^{*27})^m(b^{*27})^n$. Since K is abelian, and trivially $c_1^*, c_{10}^* \in G_{(11)}^* \cap K$, this completes the proof of (4).

The knowledge of this $\mathfrak{A}_3$-free generating set for $G_{(11)}$ enables the subgroups of $G_{(11)}$ to be easily described and distinguished; the next task is to obtain a usable criterion for determining which of them are verbal, or equivalently fully invariant, in G.

Let α, β, γ be the automorphisms of G given by

$$\alpha : a \mapsto ab, \quad b \mapsto b;$$
$$\beta : a \mapsto b, \quad b \mapsto a;$$
$$\lambda : a \mapsto a^{-1}, b \mapsto b.$$

Let M denote the $\mathfrak{A}_3$-subgroup of G and for any endomorphism η of G denote by η/M the endomorphism of G/M induced by η. Then, as is readily checked, $\{\alpha/M, \beta/M, \gamma/M\}$ is a generating set for the automorphism group of G/M. (Use the fact that Aut $(G/M) \cong GL(2, 3)$.) To make use of this information the following two facts are required:

(i) if η_1, η_2 are endomorphisms of G such that $\eta_1/M = \eta_2/M$ then η_1 and η_2 agree on $G_{(11)}$;

(ii) if η is an endomorphism of G such that ker $(\eta/M) \neq \{1\}$ then ker $\eta \supseteq G_{(11)}$.

Both (i) and (ii) follow easily from the fact that $G_{(12)} = \{1\}$. Now suppose that S is a subgroup of $G_{(11)}$ which admits the automorphisms α, β, γ and let η be an arbitrary endomorphism of G. Either ker $\eta \supseteq G_{(11)}$ in which case S certainly admits η, or, by (ii), $\eta/M \in$ Aut (G/M). In the latter case $\eta/M = \nu/M$ for some $\nu \in gp(\alpha, \beta, \gamma)$ and since S admits ν it follows fom (i) that S admits η. Thus a subgroup S of $G_{(11)}$ is fully invariant in G if (and trivially only if) it admits α, β, γ.

The action of these automorphisms on $c_2, \cdots, c_9$ is easily calculated, and is tabulated below.

c_i	c_i^{α}	c_i^{β}	c_i^{γ}
c_2	c_2	c_9^{-1}	c_2
c_3	$c_2^{-1}c_3$	c_8^{-1}	c_3^{-1}
c_4	c_4	c_7^{-1}	c_4
c_5	$c_2c_4c_5$	c_6^{-1}	c_5^{-1}
c_6	$c_2^{-1}c_3c_4c_5^{-1}c_6$	c_5^{-1}	c_6
c_7	$c_4^{-1}c_7$	c_4^{-1}	c_7^{-1}
c_8	$c_2c_4^{-1}c_5^{-1}c_7c_8$	c_3^{-1}	c_8
c_9	$c_2^{-1}c_3c_4^{-1}c_5c_6^{-1}c_7c_8^{-1}c_9$	c_2^{-1}	c_9^{-1}

From this table it is a purely routine matter to verify that the subgroups

$$D_1 = gp(c_2, c_3c_5c_7, c_4c_6c_8, c_9),$$
$$D_2 = gp(c_2c_4, c_3c_5c_7, c_4c_6c_8, c_7c_9),$$
$$C = gp(c_4, c_7)$$

each admit α, β, γ and are therefore fully invariant, so verbal, in G. However, $C \cap D_1 = \{1\} = C \cap D_2$ and $C < D_1D_2$, and hence

$$\{1\} = (C \cap D_1)(C \cap D_2) \neq C \cap D_1D_2 = C, \tag{5}$$

which gives the required non-distributivity.

3. Proof of theorem 3

Continuing with the example of non-distributivity in lat G discussed in § 2, it should now be observed that $C = M_{(4)} = \{[x_1, x_2, x_3, x_4]\}(M)$. This can be checked by routine commutator expansion calculations making appropriate use of the laws of $\mathfrak{A}_3\mathfrak{A}_9 \wedge \mathfrak{N}_{11}$ and the fact that M is generated by all commutators and cubes in G. Thus $C = I_3(G)$, where $\mathfrak{I}_3$ is the non-nilpotent join-irreducible subvariety of $\mathfrak{A}_3\mathfrak{A}_9$ defined in Theorem 2. Consequently, if $\mathfrak{W}_i = \text{var}\,(G/D_i)$ for $i = 1, 2$, then by (5) and Lemma 4

$$\mathfrak{I}_3 \vee (\mathfrak{W}_1 \wedge \mathfrak{W}_2) \neq (\mathfrak{I}_3 \vee \mathfrak{W}_1) \wedge (\mathfrak{I}_3 \vee \mathfrak{W}_2),$$

and since the $\mathfrak{W}_i$ are both nilpotent subvarieties of $\mathfrak{A}_3\mathfrak{A}_9$ Theorem 3 is an immediate corollary to the following more general, and presumably well-known, result:

LEMMA 5. *If* $\mathfrak{U}, \mathfrak{W}_1, \mathfrak{W}_2$ *are varieties of groups, and*

$$\mathfrak{U} \vee (\mathfrak{W}_1 \wedge \mathfrak{W}_2) \neq (\mathfrak{U} \vee \mathfrak{W}_1) \wedge (\mathfrak{U} \wedge \mathfrak{W}_2), \tag{6}$$

then there exist varieties of groups $\mathfrak{V}, \mathfrak{L}_1, \mathfrak{L}_2$, *with* $\mathfrak{L}_1 \neq \mathfrak{L}_2$ *and* $\mathfrak{L}_i \in \mathfrak{W}_i$ *for* $i = 1, 2$, *such that each* $\mathfrak{L}_i$ *is minimal with respect to the property that its join with* $\mathfrak{U}$ *is* $\mathfrak{V}$.

PROOF. If $\mathfrak{V}, \mathfrak{X}_1, \mathfrak{X}_2$ are defined by

$$\mathfrak{V} = (\mathfrak{U} \vee \mathfrak{W}_1) \wedge (\mathfrak{U} \vee \mathfrak{W}_2)$$
$$\mathfrak{X}_i = \mathfrak{W}_i \wedge (\mathfrak{U} \vee \mathfrak{W}_j) \qquad i, j = 1, 2,\ i \neq j,$$

then it follows from (6) by modularity that

$$\mathfrak{V} = \mathfrak{U} \vee \mathfrak{X}_1 = \mathfrak{U} \vee \mathfrak{X}_2 \neq \mathfrak{U} \vee (\mathfrak{X}_1 \wedge \mathfrak{X}_2).$$

For $i = 1, 2$, let $\mathscr{L}_i = \{\mathfrak{Y} \in \text{lat}\,\mathfrak{X}_i | \mathfrak{U} \vee \mathfrak{Y} = \mathfrak{V}\}$. If $\{\mathfrak{Y}_\delta | \delta \in \Delta\}$ is any descending chain in $\mathscr{L}_i$ then since $\mathfrak{U} \vee (\bigwedge_{\delta \in \Delta} \mathfrak{Y}_\delta) = \bigwedge_{\delta \in \Delta}(\mathfrak{U} \vee \mathfrak{Y}_\delta)$ (21.26 in [4]) it follows that $\bigwedge_{\delta \in \Delta} \mathfrak{Y}_\delta \in \mathscr{L}_i$. Thus every totally ordered subset of $\mathscr{L}_i$ has a lower bound in $\mathscr{L}_i$ and hence, by the minimum principle, $\mathscr{L}_i$ contains a minimal element $\mathfrak{L}_i$. Moreover, $\mathfrak{L}_1 \neq \mathfrak{L}_2$ for otherwise

$$\mathfrak{V} = \mathfrak{U} \vee \mathfrak{X}_1 \supseteq \mathfrak{U} \vee (\mathfrak{X}_1 \wedge \mathfrak{X}_2) \supseteq \mathfrak{U} \vee (\mathfrak{L}_1 \wedge \mathfrak{L}_2) = \mathfrak{U} \vee \mathfrak{L}_1 = \mathfrak{V}$$

contradicting $\mathfrak{V} \neq \mathfrak{U} \vee (\mathfrak{X}_1 \wedge \mathfrak{X}_2)$. This completes the proof.

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