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THE OSCULATORY PACKING OF A THREE DIMENSIONAL SPHERE

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1. Introduction. Packings by unequal spheres in three dimensional space have interested many authors. This is to some extent due to the practical applications of such investigations to engineering and physical problems (see, for example, [**16**; **17**; **31**]). There are a few general results known concerning complete packings by spheres in N-dimensional Euclidean space, due mainly to Larman [**20**; **21**]. For osculatory packings, although there is a great deal of specific knowledge about the two-dimensional situation, the results for higher dimensions, such as [**4**], rely on general methods which do not give particularly precise information. For example there has not been, up to this time, even an heuristic estimate for the exponent of any packing in a space of dimension higher than two, because the packing process is not well enough understood to generate large numbers of spheres in such a packing.

In this paper we shall give a precise description of osculatory packings of the three dimensional unit sphere. That is, we describe a process, quite analogous to the well-known two-dimensional process, which generates all the spheres in the osculatory packing of a unit sphere. The analogue to the two and three-dimensional processes can be described in dimensions higher than three but in higher dimensions it does not lead to a packing since the generated spheres, in general, intersect one another. Infinite packings of N-dimensional spheres can, by inversion, be related to packings of $(N - 1)$-dimensional space by equal spheres, and since, for $N - 1 > 2$, there are many unsolved problems in this area of study it is not surprising that the higher dimensional packings should be more difficult to understand.

We shall be making much use of the notion of the "separation" between two spheres. The separation between two spheres X, Y with radii r, s and whose centres are at distance d apart, is defined by the formula

$$\Delta(X, Y) = (d^2 - r^2 - s^2)/2rs$$

This inversive invariant seems to have been first systematically used by Darboux [**11**] and Clifford [**8**]. It is simply related to the "inversive distance" of Coxeter [**10**, p. 116], and is the negative of the "inclination" used by Mauldon [**22**].

The key to our proof that, for $N = 3$, the generated spheres do not inter-

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sect, is the observation that the separation between any two generated spheres is an odd integer (Corollary 4). A special case of this result, for a subset of the circles in the two-dimensional packing, was proved by Coxeter in [**10**, p. 117]. His proof used rather specific knowledge concerning the sequence of circles in question. Our original proof of Corollary 4 used inversion quite extensively. However the proof we present here uses "polyspherical coordinates" and seems to be more transparent. In § 3 we present a brief but complete introduction to this coordinate system, basing our account on a fundamental formula due to Darboux [**11**] and Frobenius [**14**]. In this system a sphere is represented as a point on a hyperboloid of one sheet in $(N + 2)$-dimensional space (formula (9)), showing that there are non-Euclidean aspects to the packing problem. We do not pursue these further here.

In § 4, we describe the sphere generating procedure in all dimensions and give a formula for the separation of any two generated spheres. This leads, for $N = 3$, to the important result, Theorem 5, which proves that the spheres form a packing.

The next section, § 5, contains the proof that our procedure gives osculatory packings of the unit sphere. We present this in two parts, Theorem 10 and Theorem 11. The proofs of these results are more geometrical than the others in this paper. We also describe how to produce complete packings of all of Euclidean three-space from the packings of the unit sphere.

Although the spheres generated by our process do not intersect, the same sphere will be generated more than once (in fact, infinitely often). It is this occurrence that distinguishes the two and three-dimensional situations. For practical and aesthetic reasons, one would like an algorithm which gives each sphere exactly once. We have developed such an algorithm but, because a complete description of it here would unduly lengthen this paper, we shall give this elsewhere [**7**]. The algorithm is well-adapted to practical computation because of its "tree-like" structure. We have used it to generate the penta-spherical coordinates of the 305594 spheres whose curvatures are at most 300 in a packing of the unit sphere which we call the "Soddy" packing, since it contains all the spheres in Soddy's "bowl of integers" [**28**]. Using the method suggested by Melzak [**24**], we have obtained the heuristic result that the exponent of this packing is approximately 2.42. This is consistent with the known result for the two-dimensional packing exponent S, that $1.300197 < S < 1.314534$, see [**6**], since one suspects that these exponents are the minimal exponents t_N, and it can be shown by an analysis similar to that of [**19**] that $t_N \geqq t_{N-1} + 1$.

Although we are principally concerned with the three-dimensional case, we have proved most of the results for general N, in the hope that these will be useful in investigating packings in higher dimensions.

I would like to thank the referee J. B. Wilker for pointing out that I had failed to treat all possibilities in the original proof of Theorem 5 and for numerous other helpful remarks.

2. Preliminary definitions. By *sphere* we shall mean an N-sphere (or N-ball). We write, for $a \in E_N$ and $r \neq 0$, if $\xi = (\xi_1, \ldots, \xi_N)$,

$$S(a, r) \begin{cases} = \{\xi : |\xi - a| < r\} & \text{if } r > 0, \\ = \{\xi : |\xi - a| > -r\} & \text{if } r < 0. \end{cases}$$

The *curvature* of a sphere is the reciprocal of its radius. We shall also consider a half-space to be a sphere with curvature zero.

If we let U be an open set in E_N, then a *complete packing* of U is a sequence of disjoint open spheres of positive radius each contained in U and such that the set $U \backslash \bigcup S_n$ has Lebesque measure zero. An *osculatory packing* of an open set U of finite measure is a packing in which, for some integer m, S_n has the largest radius of spheres contained in $U \backslash (S_1 \cup \ldots \cup S_{n-1})$ for $n = m, m + 1, \ldots$. An osculatory packing is known to be complete [**2**].

The *exponent* of a complete packing of an open set of finite measure is defined by

$$e(C, U) = \sup\{t : \textstyle\sum r_n{}^t = \infty\} = \inf\{t : \textstyle\sum r_n{}^t < \infty\}, \tag{1}$$

where r_n is the radius of S_n. This exponent was first introduced in these terms by Melzak [**23**], but had also been used by Gilbert [**15**].

Given two spheres $X = S(a, r)$ and $Y = S(b, s)$, we define the *separation* between X and Y to be

$$\Delta(X, Y) = (|a - b|^2 - r^2 - s^2)/2rs. \tag{2}$$

If $X = S(a, r)$ and Y is a half-space, let d be the distance from a to the bounding hyperplane of Y, measured so that $d \geqq 0$ if $a \notin Y$ and $d < 0$ if $a \in Y$. Then, we define

$$\Delta(X, Y) = d/r. \tag{3}$$

Note that if X and Y intersect, then $\Delta(X, Y) = -\cos\theta$ where θ is the dihedral angle between the outward normals at a point of intersection. This allows one to define the separation between two half-spaces consistently. Observe that if X and Y have positive radii, then $\Delta(X, Y) = 1$ if and only if X and Y are externally tangent, and that $\Delta(X, Y) = -1$ if and only if X and Y are internally tangent. If $|\Delta(X, Y)| \geqq 1$, Coxeter [**10**] defines δ, given by $\cosh \delta = |\Delta(X, Y)|$ to be the *inversive distance* between X and Y.

By *inversion in the sphere* $S(a, r)$, we mean inversion in its boundary. Note that under inversion in $S(a, r)$, the sphere $S(a, r)$ becomes $S(a, -r)$. The separation $\Delta(X, Y)$ is an inversive invariant. One can show that if X has finite radius r, if Y has radius s and if Y', the image of Y under inversion in X has radius s', then

$$\Delta(X, Y) = \frac{r}{2}\left(\frac{1}{s'} - \frac{1}{s}\right). \tag{4}$$

3. Polyspherical coordinates. The most natural description of the sphere generating process to be described in the next section is in terms of polyspherical coordinates. These seem to have been used first by Darboux [**11**] and Clifford [**8**] and are described (for 2 and 3 dimensions) by Lachlan [**18**]. They are also used in the treatises of Coolidge [**9**, pp. 254–261] and Woods [**32**, p. 138, p. 282, p. 418]. We shall give a brief but complete development of the results we need, beginning with a fundamental result due to Darboux [**11**] and, independently, Frobenius [**14**].

LEMMA 1. (Darboux-Frobenius Formula). *Let* $X_1, \ldots, X_{N+3}$, $Y_1 \ldots Y_{N+3}$ *be* $2N + 6$ *spheres in* E_N. *Then* $\det(\Delta(X_i, Y_j)) = 0$.

Proof. We assume, by a preliminary inversion, if necessary, that all the spheres have finite radii. If X has centre $c = (c_1, \ldots, c_N)$ and radius r, let $u(X)$ be the following column vector (T denoting transpose)

$$u(X) = (1/r)(\tfrac{1}{2}, |c|^2 - r^2, -c_1, \ldots, -c_N)^T.$$

And, if Y has centre d and radius s, let

$$v(Y) = (1/s)(|d|^2 - s^2, \tfrac{1}{2}, d_1, \ldots, d_N)^T.$$

Then, $\Delta(X, Y) = v(Y)^T u(X)$. Since $u(X_1), \ldots, u(X_{N+3})$ are $N + 3$ vectors in $\mathbf{R}^{N+2}$, their linear dependence implies

$$\det(\Delta(X_i, Y_j)) = \det(v(Y_j)^T u(X_i)) = 0.$$

We shall be interested in the special case of this formula in which we choose $N + 2$ spheres $X_1, \ldots, X_{N+2}$ common to both sets of spheres. Lemma 1 was apparently discovered by Clifford [**8**, p. 335] for this special case in 1868 but was not published until after his death. Suppose that Y and Z are two spheres. Let $c(Y)$ denote the $(N + 2)$-vector

$$c(Y) = (\Delta(Y, X_1), \ldots, \Delta(Y, X_{N+2}))^T. \tag{5}$$

Let Δ denote the matrix $(\Delta(X_i, X_j))$. Then, Lemma 1 gives

$$\det\begin{pmatrix} \Delta(Y, Z) & c(Y)^T \\ c(Z) & \Delta \end{pmatrix} = 0. \tag{6}$$

Expanding by the first row and column, and letting adjΔ be the matrix of cofactors of Δ (Δ is symmetric), we have

$$\Delta(Y, Z) \det \Delta - c(Y)^T(\text{adj } \Delta)c(Z) = 0. \tag{7}$$

Hence, if $\det \Delta \neq 0$, we have

$$\Delta(Y, Z) = c(Y)^T \Delta^{-1} c(Z), \tag{8}$$

and in particular,

$$c(Y)^T \Delta^{-1} c(Y) = -1. \tag{9}$$

Now, in (8), suppose $X_1, \ldots, X_{N+2}, Y$ are finite spheres and let Z be a plane at distance d from the centre of Y and at distance d_i from the centre of X_i, so that, if $\epsilon_1, \ldots, \epsilon_{N+2}, \eta$ are the curvatures of $X_1, \ldots, X_{N+2}, Y$, we have

$$\Delta(Y, Z) = d\eta \quad \text{and} \quad \Delta(Z, X_i) = d_i\epsilon_i.$$

Letting Z recede to infinity, $d_i/d \to 1$ for each i, so (8) implies that

$$\eta = c(Y)^T\Delta^{-1}\epsilon, \tag{10}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_{N+2})^T$. By continuity (10) holds if some of the spheres have curvature zero (i.e. are half spaces). Now in (10), letting Y be a plane which recedes to infinity we have

$$\epsilon^T\Delta^{-1}\epsilon = 0. \tag{11}$$

We define the *polyspherical coordinates* of Y with respect to $X_1, \ldots, X_{N+2}$ by

$$a(Y) = \Delta^{-1}c(Y). \tag{12}$$

Then (10) takes the form

$$\eta = a(Y)^T\epsilon, \tag{13}$$

and (8) becomes

$$\Delta(Y, Z) = a(Y)^T\Delta a(Z) = a(Y)^Tc(Z). \tag{14}$$

One can obtain the Cartesian equations of Y quite simply from $a(Y)$. First, we define the *canonical equation* of a sphere as follows: if X has finite radius r, and centre c, and if $\xi = (\xi_1, \ldots, \xi_N)$, let

$$x(\xi) = (|\xi - c|^2 - r^2)/2r \tag{15}$$

If X has infinite radius, so is a half-space with boundary passing through the point b, say, and with outward unit normal n, then

$$x(\xi) = n \cdot (\xi - b). \tag{16}$$

Now, if $x_i(\xi) < 0$, $y(\xi) < 0$ are the canonical equations of X_i and Y, then

$$y(\xi) = \sum \{a_i(Y)x_i(\xi) : i = 1, \ldots, N + 2\}. \tag{17}$$

To see this, let Z be a sphere with centre ξ and radius 1. Then, it is easy to see that

$$\Delta(Y, Z) = y(\xi) - \eta/2. \tag{18}$$

But, from (14), and then (13),

$$\begin{aligned} y(\xi) &= \eta/2 + \Delta(Y, Z) \\ &= \eta/2 + \textstyle\sum_i a_i(Y)\Delta(Z, X_i) \\ &= \eta/2 + \textstyle\sum_i a_i(Y)(x_i(\xi) - \epsilon_i/2) \\ &= \textstyle\sum_i a_i(Y)x_i(\xi). \end{aligned} \tag{19}$$

We will be interested in the special case that $X_1, \ldots, X_{N+2}$ are mutually tangent so that $\Delta(X_i, X_j) = 1$, if $i \neq j$, and $\Delta(X_i, X_i) = -1$. In this case $\Delta = J - 2I$, where J is the matrix all of whose entries are 1, and I is the identity matrix. Since $J^2 = (N+2)J$, one sees by inspection that

$$\Delta^{-1} = (2N)^{-1}(J - NI).$$

Then, (11) becomes

$$(\epsilon_1 + \ldots + \epsilon_{N+2})^2 = N(\epsilon_1^2 + \ldots + \epsilon_{N+2}^2). \tag{20}$$

Formula (20) is quite often called "Soddy's formula", after the popular poems [**26**; **27**] for the cases $N = 2$ and 3. There is an extensive literature on this formula, it having been rediscovered many times. Pedoe [**25**] is a good reference. There, he proposes the name, the "generalized Descartes formula", since Aeppli has traced (20) back to Descartes for $N = 2$. I have not seen it mentioned in any of these papers that the result for $N = 3$ appears in the 1886 paper of Lachlan [**19**, p. 498], and is reproduced in Coolidge [**9**, p. 258]. Coxeter [**10**] gives a non-computational proof of (20). Observe that the convention concerning the sign of the ϵ_i has been handled by our assumption $\Delta(X_i, X_j) = 1$ if $i \neq j$. Notice that if $(\epsilon_1, \ldots, \epsilon_{N+2})$ is a solution of (20), then so is $(-\epsilon_1, \ldots, -\epsilon_{N+2})$; exactly one of these solutions corresponds to a set $(X_1, \ldots, X_{N+2})$ of *disjoint* spheres with $(\Delta(X_i, X_j)) = J - 2I$ and having curvatures $\epsilon_1, \ldots, \epsilon_{N+2}$. This solution will have either all components non-negative or else one negative component. At most two of the components can be zero as can be seen geometrically or else by the Schwarz inequality applied to (20). Henceforth, we shall consider only those solutions of (20) which correspond to disjoint spheres.

One can also derive Mauldon's formula given in [**22**], for the case $\Delta(X_i, X_j) = -\gamma$ for $i \neq j$, from formula (11). By an imitation of the analysis given there, we can show that if D is any symmetric matrix with all diagonal elements equal to -1, which has $(N + 1)$ negative and one positive eigenvalue, then there are spheres $X_1, \ldots, X_{N+2}$ with $(\Delta(X_i, X_j)) = D$. In this case $\epsilon^T D^{-1} \epsilon = 0$ has real solutions and one can choose X_i to have curvature ϵ_i for $i = 1, \ldots, N + 2$. The set of such D exhausts the set of non-singular Δ. We shall not pursue this line of investigation here as we shall not need these results.

We should perhaps note that Coolidge [**9**] and Woods [**32**] generally choose their spheres to be orthogonal so that $(\Delta(X_i, X_j)) = -I$. As one can see from (11), this necessitates choosing one sphere with an imaginary radius, which we do not allow here.

4. The sphere generating process. We now describe a process for generating a collection of spheres $\mathscr{G}$ in E_N. As motivation for the process, the reader should consider the packing, in E_2, of a curvilinear triangle bounded by mutually tangent circular sides which is described for example in [**3**; **13**; **15**].

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We shall be using the results of § 3, and throughout this section we define

$$\Delta = J - 2I.$$

Given $(N + 1)$ mutually tangent spheres in E_N there are exactly two spheres which are tangent to all $N + 1$. We shall begin with an $(N + 2)$-tuple of disjoint spheres $M = (X_1, \ldots, X_{N+2})$ such that $\Delta(X_i, X_j) = 1$ for $i \neq j$. We shall apply $N + 2$ operations to M, denoted $\theta_1, \ldots. \theta_{N+2}$. The operation θ_i applied to M produces the $(N + 2)$-tuple

$$M(i) = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{N+2}, Y),$$

where Y is the unique sphere which is tangent to all the spheres in M except X_i, and is not the sphere X_i. We shall write

$$M(i) = (X_1(i), \ldots, X_{N+2}(i)),$$

so that

$$\begin{cases} X_j(i) = X_j, & j \leqq i - 1 \\ X_j(i) = X_{j+1}, & i \leqq j \leqq N + 1 \\ X_{N+2}(i) = Y. \end{cases} \tag{21}$$

We thus obtain $(N + 2)$ "new" $(N + 2)$-tuples $M(1), \ldots, M(N + 2)$. We can repeat this procedure with these new $(N + 2)$-tuples obtaining $(N + 2)^2$ new $(N + 2)$-tuples $M(i, j)$, $(i = 1, \ldots, N + 2)$, $(j = 1, \ldots, N + 2)$. Proceeding in this way, at the mth stage, we have $(N + 2)^m$ new $(N + 2)$-tuples which we shall index by a parameter $\alpha = (i_1, \ldots, i_m)$, where each i_k runs independently over the integers $1, 2, \ldots, N + 2$. We shall let G_m denote the set of such α, and G_0 will denote a set consisting of a single vector with no components. Then, define $G = \bigcup \{G_m : m = 0, 1, \ldots\}$. By the above process, we can thus produce, for each $\alpha \in G$, an $(N + 2)$-tuple of spheres

$$M(\alpha) = (X_1(\alpha), \ldots, X_{N+2}(\alpha)).$$

We should point out that we are interested in generating the spheres $X_{N+2}(\alpha)$. Thus, the ordering of the spheres in $M(\alpha)$ implied by (21) is purely a conventional device, which seems appropriate since the "new" sphere $X_{N+2}(\alpha)$ occupies a special position. Another attractive choice would be to order $M(i)$ as $(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_{N+2})$. The same set of spheres will be generated but with different labels. These remarks should make it clear that if we begin with any $(N + 2)$-tuple $(X_1(\alpha), \ldots, X_{N+2}(\alpha))$ and apply the above procedure, we generate the same set of spheres as if we begin with $(X_1, \ldots, X_{N+2})$.

We shall denote the collection of all the spheres $X_i(\alpha)$, $(i = 1, \ldots, N + 2)$, $\alpha \in G$ by $\mathscr{G}$.

Our next object is to obtain an expression for the curvature $\epsilon_i(\alpha)$, and the polyspherical coordinates of $X_i(\alpha)$, in terms of $X_1, \ldots, X_{N+2}$. We refer the reader to Coxeter [**10**] who considered the curvatures of the sequence $X_{N+2}(1^m)$, where $1^m = (1, 1, \ldots, 1) \in G_m$. We first treat the curvatures. Let us find the curvature $\epsilon_{N+2}(i)$ of $X_{N+2}(i)$, given that the curvatures $X_1, \ldots, X_{N+2}$ are $\epsilon_1, \ldots, \epsilon_{N+2}$. Since (20) is quadratic in each curvature and since the sets $\{\epsilon_1(i), \ldots, \epsilon_{N+2}(i)\}$, $\{\epsilon_1, \ldots, \epsilon_{N+2}\}$ differ in exactly one element, we must have that the two numbers ϵ_i, $\epsilon_{N+2}(i)$ are the two roots of (20) considered as an equation for ϵ_i, so that

$$(22) \qquad \epsilon_{N+2}(i) = -\epsilon_i + \frac{2}{N-1}(\epsilon_1 + \ldots + \epsilon_{i-1} + \epsilon_{i+1} + \ldots + \epsilon_{N+2}).$$

(see [**10**, p. 111]). That is, the relation between the curvatures of the spheres in $M(\alpha)$, and those in $M(\alpha, i)$ is linear. (Here, if $\alpha = (i_1, \ldots, i_m)$, then $(\alpha, i) = (i_1, \ldots, i_m, i)$). Thus, there are matrices A_i such that

$$(23) \qquad (\epsilon_1(\alpha, i), \ldots, \epsilon_{N+2}(\alpha, i)) = (\epsilon_1(\alpha), \ldots, \epsilon_{N+2}(\alpha))A_i.$$

Thus, with $\alpha = (i_1, \ldots, i_m)$, and $A(\alpha) = A_{i_1} \ldots A_{i_m}$, we have

$$(24) \qquad (\epsilon_1(\alpha), \ldots, \epsilon_{N+2}(\alpha)) = (\epsilon_1, \ldots, \epsilon_{N+2})A(\alpha).$$

The matrix A_i can be described as follows: Let $e_1, \ldots, e_{N+2}, e$ denote the column vectors for which e_i has all components zero except for a 1 in the ith position, and e has all components 1. Then

$$(25) \qquad A_i = \left(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{N+2}, \frac{2}{N-1}e - \frac{N+1}{N-1}e_i\right).$$

Note that A_i has integer entries only in case $N = 2$ or 3. This fact is of considerable significance as we shall see.

LEMMA 2. *For any* $\alpha \in G$, $\alpha = (i_1, \ldots, i_m)$, *let*

$$A(\alpha) = A_{i_1} \ldots A_{i_m},$$

where A_i *is the matrix of* (25). *Let* $a(X_i(\alpha))$ *denote the column vector of pentaspherical coordinates of* $X_i(\alpha)$ *with respect to* $X_1, \ldots, X_{N+2}$. *Then,* $a(X_i(\alpha))$ *is the* ith *column of the matrix* $A(\alpha)$.

Proof. Let Y be any sphere. Let $\alpha \in G$ and $k \in \{1, \ldots, N+2\}$. The vectors $c'(Y)$ with components $\Delta(Y, X_i(\alpha))$ and $c''(Y)$ with components $\Delta(Y, X_i(\alpha, k))$ satisfy the same equation (9), and have $(N+1)$ components in common (although the order is different). Hence, by the same reasoning as used to obtain (22), we have

$$(26) \qquad c''(Y)^T = c'(Y)^T A_k.$$

Letting Y be successively $X_1, \ldots, X_{N+2}$ we see that (26) implies that

$$(27) \qquad (\Delta(X_i, X_j(\alpha, k))) = (\Delta(X_i, X_j(\alpha)))A_k.$$

Hence, by induction,

$$(28) \qquad (\Delta(X_i, X_j(\alpha))) = \Delta A(\alpha).$$

Finally, using the definition (12) of $a(X_j(\alpha))$, we see that

$$(29) \qquad (a_1(X_1(\alpha)), \ldots, a(X_{N+2}(\alpha))) = \Delta^{-1}(\Delta(X_i, X_j(\alpha))) = A(\alpha).$$

COROLLARY 3. *Let* $\alpha, \beta \in G$ *and* $i, j \in \{1, \ldots N + 2\}$. *Then* $\Delta(X_i(\alpha), X_j(\beta))$ *is the* (i, j)th *entry of the matrix* $A(\alpha)^T \Delta A(\beta)$.

Proof. This follows immediately from Lemma 2 and equation (14).

COROLLARY 4. *Suppose that* $N = 2$ *or* 3, *that* $\alpha, \beta \in G$, *and*

$$i, j \in \{1, 2, \ldots, N + 2\}.$$

Then $\Delta(X_i(\alpha), X_j(\beta))$ *is an odd integer.*

Proof. Since A_i and Δ have integer entries, it follows from Corollary 3 that $\Delta(X_i(\alpha), X_j(\beta))$ is an integer. Computing modulo 2, we have $\Delta \equiv J$, and since the column sums of each A_i are odd integers, we have, by induction on the number of components in α and β,

$$A(\alpha)^T \Delta A(\beta) \equiv J (\text{mod } 2).$$

Remark. Coxeter [**10**, p. 117] proved a special case of Corollary 4, when $N = 2$ and α, β have all components equal to 1. His proof is quite different from the above, using more specific knowledge concerning the sequence of disks in question. This result was what suggested to us that Corollary 4 might be valid.

THEOREM 5. *Let* $\mathscr{G}$ *be the collection of all spheres* $X_i(\alpha)$, $\alpha \in G$, $i \in \{1, \ldots, N + 2\}$. *If* $N = 2$ *or* 3, *then* $\mathscr{G}$ *is a packing of* E_N. *That is,* $\mathscr{G}$ *is a collection of disjoint spheres.*

This is false for all $N > 3$.

Proof. We first consider $N = 2$ or 3. We must show that if $\alpha, \beta \in \mathscr{G}$, $i, j \in \{1, \ldots, N + 2\}$, and if $X = X_i(\alpha)$, $Y = Y_j(\beta)$ then either $X = Y$ or else X and Y are disjoint. Let us suppose then that $X \neq Y$ but that $X \cap Y$ is non-empty. We may assume in addition that the total number of components in α and β is minimal under the condition that the preceding sentence holds since our initial configuration $X_1, \ldots, X_{N+2}$ consists of disjoint spheres. Then we must have $X = X_{N+2}(\alpha)$ and $Y = X_{N+2}(\beta)$, while $X_1(\alpha), \ldots, X_{N+1}(\alpha)$ are disjoint from Y and $X_1(\beta), \ldots, X_{N+1}(\beta)$ are disjoint from X. By Corollary 4, $|\Delta(X, Y)| \geqq 1$ so the boundaries of X and Y can intersect in at most one point; otherwise the boundaries would coincide and then the fact that X and Y are not disjoint would imply that $X = Y$. Also, it is clear that $X \cup Y$ is properly contained in E_N or else $X_{1(\alpha)}$ would not be disjoint from both X and Y. The only remaining possibility is that one of X, Y is properly contained in the other, say $X \subsetneqq Y$. But $X_1(\alpha), \ldots, X_{N+1}(\alpha)$ do not intersect Y and yet they have $N + 1$ *distinct* points of contact with X. This is clearly impossible.

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For $N > 3$ we need only produce an example. Note that the spheres produced by the iteration of A_{N+1} form a sequence of spheres each mutually tangent to the previous sphere and to N fixed spheres. This sequence has been studied by Wilker [**30**] who showed that the points of successive contact lie on a circle and that the sequence is eventually self-intersecting.

We can also prove this independently by computing the separations $d_n = \Delta(X_{N+1}, X_{N+2}((N+2)^n))$. These satisfy the following recurrence in which $b = 2/(N-1)$,

$$(30) \qquad \begin{cases} d_{-1} = -1 \\ d_0 \;\;= 1 \\ d_n \;\;= 2 + b - d_{n-2} + bd_{n-1} \end{cases}$$

since

$$(31) \qquad (1, \ldots, 1, c, d)A_{N+1} = (1, \ldots, 1, d, 2 + b - c + bd).$$

By a simple computation using (30),

$$(32) \qquad d_3 = -1 + 4b^2 + 2b^3 = -1 + 16N(N-1)^{-3},$$

so, if $N > 4$ it follows that $0 < |d_3| < 1$, and (32) shows that the two spheres X_{N+1} and $X_{N+2}((N+1)^3)$ intersect. For $N = 4$, one finds that $0 < |d_4| < 1$, which completes the proof.

Remarks. 1. It is possible to use the algorithm of [**7**] to give a completely computational proof of Theorem 5 for $N = 3$. One shows first that if $Y \in \mathscr{G}\backslash\{X_1, \ldots, X_5\} = \mathscr{H}$ then the vector $c(Y)$ has components which are positive integers. Choosing curvatures for $X_1, \ldots, X_5$ as $-1, 2, 2, 3, 3$, one can then show that $\epsilon(Y) \geqq 1$ for all $Y \in \mathscr{H}$. Using these facts Y is shown to be disjoint from $X_1, \ldots, X_5$. By invariance under inversion, this is now true if $X_1, \ldots, X_5$ have any curvatures $\epsilon_1, \ldots, \epsilon_5$ satisfying (20). Since any quintuple $X_1(\alpha), \ldots, X_5(\alpha)$ can be used as the initial quintuple, this proves Theorem 5.

2. The paper of Wilker [**30**] mentioned above showed that the sequence of spheres generated by the iterates of A_{N+1} is self-intersecting. This can be shown to be true for iterates of A_i for any $i > 3$ and $N > 3$, by investigating the spectra of the various A_i which is rather easy since the characteristic polynomial can be explicitly computed. The only matrix of finite order (other than A_{N+2} for all N) is the 3-dimensional A_4 for which $A_4^6 = I$. This fact is the basis of Soddy's beautiful "hexlet" described by him in [**27**] and [**28**] and also investigated in more detail by Wilker [**30**].

5. Osculatory packings in three dimensions. Suppose that $N = 2$ or 3, that X_1 is a sphere of curvature -1, say $S(a, -1)$, and that $U = S(a, 1)$. Let $X_2, \ldots, X_{N+2}$ be spheres such that $X_1, \ldots, X_{N+2}$ are mutually tangent. By Theorem 5, the collection $\mathscr{G}' = \mathscr{G}\backslash\{X_1\}$ forms a packing of U. It is

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well-known, and easily proved that for $N = 2$, $\mathscr{G}'$ is an osculatory packing of U, and hence a complete packing. In this section, we prove the analogous result for $N = 3$. There is a natural division into two cases depending on whether or not the centre of U lies in the interior of the convex hull of the centres of $X_2, \ldots, X_5$. In the first case the packing $\mathscr{G}'$ is the only osculatory packing of U which begins with $X_2, \ldots, X_5$ whereas in the second case there may be many osculatory packing with this property. These cases correspond to Theorems 9 and 10 respectively.

For the proof of these theorems we need a number of lemmas. Since the proofs of two of these are by induction we have proved these for general N although they are needed only for $N = 3$.

LEMMA 6. *Let $X_1, \ldots, X_N$, $(N \geqq 3)$ be mutually tangent N-spheres with curvatures $\epsilon_1, \ldots, \epsilon_N$. Let η denote the maximum curvature for a sphere Y tangent to all X_i, $(i = 1, \ldots, N)$. Then η is the larger root of*

$$(33) \qquad (\epsilon_1 + \ldots + \epsilon_N + \eta)^2 = (N - 1)(\epsilon_1^2 + \ldots + \epsilon_N^2 + \eta^2),$$

so the centre of Y is in the hyperplane which contains the centres of $X_1, \ldots, X_N$

Proof. Let Z, with curvature ζ, touch $X_1, \ldots, X_N$ and Y. Then, by (20)

$$(34) \qquad (\epsilon_1 + \ldots + \epsilon_N + \eta + \zeta)^2 = N(\epsilon_1^2 + \ldots + \epsilon_N^2 + \eta^2 + \zeta^2).$$

Since (34) has a real root for ζ, the discriminant of (34) considered as a polynomial in ζ, must be non-negative, so

$$(35) \qquad (\epsilon_1 + \ldots + \epsilon_N + \eta)^2 \geqq (N - 1)(\epsilon_1^2 + \ldots + \epsilon_N^2 + \eta^2).$$

The largest η satisfying (35) is the largest solution of (33).

LEMMA 7. *Let $X_i = S(a_i, r_i)$, $(i = 1, \ldots, N + 1)$ be mutually tangent N-spheres with positive radii. Let L denote the convex hull of their centres. Let r be the radius of the smaller sphere tangent to all X_i. Then*

$$(36) \qquad L \subset \bigcup \{S^-(a_i, r_i + r) : i = 1, \ldots, N + 1\} = T$$

Proof. We use induction on N. The case $N = 1$ is trivial. Note that

$$(37) \qquad \bigcap \{S^-(a_i, r_i + r) : i = 1, \ldots, N + 1\} = \{p\},$$

where p is the centre of the tangent sphere of radius r. Since each $S^-(a_i, r_i + r)$ is convex, (37) implies that the set T is starlike with respect to the point p. We claim that the boundary of L is covered by T. Once this has been shown, it will follow that $L \subset T$, for otherwise there would be an open set $O \subset L$ which is excluded by T. But T also excludes the complement of a large sphere, so the complement of T would be disconnected, contradicting the fact that T is starlike.

To see that T does contain the boundary of L, consider a face L' of L, the hull of $a_1, \ldots, a_N$, say, and let Π be the hyperplane containing L'. Let r' be the radius of the smaller $(N-1)$-sphere tangent to $\Pi \cap X_1, \ldots, \Pi \cap X_N$. Then, by Lemma 6, $r' \leqq r$. Hence

$$T = \bigcup \{S^-(a_i, r_i + r) : i = 1, \ldots, N+1\}$$
$$\supset \bigcup \{S^-(a_i, r_i + r') \cap \Pi : i = 1, \ldots, N\} \supset L',$$

where the last step uses the induction hypothesis.

We need an analogue to Lemma 7 in case one sphere X_1 has negative radius. In this case, the analogue of the convex hull of the centres of the spheres is the following set L : L is the closure of the set difference $K \backslash H$, where K is the polyhedral cone with vertex at a_1 generated by the convex hull of $a_2, \ldots, a_{N+1}$, and H is the convex hull of $a_1, \ldots, a_{N+1}$.

LEMMA 8. *Let X_i $(i = 1, \ldots, N+1)$ be spheres as in Lemma 7, except that $r_1 < 0$. Suppose that a_1 is not in the convex hull of $a_2, \ldots, a_{N+1}$. Let r be as in Lemma 7, and let L be the set described in the previous paragraph. Then*

$$L \subset \bigcup \{S^-(a_i, r_i + r) : i = 1, \ldots, N+1\}. \tag{38}$$

Proof. This is similar to the proof of Lemma 7 except that $S^-(a_1, r_1 + r)$ is not starlike. We note that a proof of (38) amounts to proving

$$M = L \cap S^-(a_1, -r_1 - r) \subset T_N = \bigcup \{S^-(a_i, r_i + r) : i = 2, \ldots, N+1\} \tag{39}$$

(N.B. the index $i \geqq 2$ in the right member of (39)). Note that T_N is starlike with respect to p, the centre of the smaller sphere tangent to all X_i. As in Lemma 7, we show that T_N contains the boundary of M, and for this we use induction on N and Lemma 7. Note that the boundary of M consists of some planar faces (from L) and a spherical face (from $S^-(a_1, -r_1 - r)$). The planar face containing $a_2, \ldots, a_{N+1}$ is covered by T_N by Lemma 7, and the faces containing a_1 and $(N-1)$ of $a_2, \ldots, a_{N+1}$ are covered by T_N, by induction. As for the spherical face F, we see that $S^-(a_i, r_i + r) \cap S^-(a_1, -r_1 - r) = C_i$ is a spherical cap for $i = 2, \ldots, N+1$, and

$$\bigcap \{C_i : i = 2, \ldots, N+1\} = \{p\}.$$

Each C_i is starlike as a subset of $S^-(a_1, -r_1 - r)$, (considering great circles as lines); hence to show C_i covers F, we need only show it covers the boundary of F. But this follows by induction since the boundary of F is a union of intersections of $S^-(a_1, -r_1 - r)$ with the plane faces of L.

Remark. The reader should not get the impression, from Lemma 7, that if $X_1, \ldots, X_{N+2}$ are spheres with curvatures $\epsilon_1 \leqq \ldots \leqq \epsilon_{N+2}$, then the centre of X_{N+2} lies in the convex hull of the centres of $X_1, \ldots, X_{N+1}$ since this is, in general, *false* if $N \neq 2$. The next lemma gives the true state of affairs when $N = 3$. These results will be used in the proof of Theorem 10.

LEMMA 9. *Let $X_1, \ldots, X_5$ be mutually tangent spheres with centres $a_1, \ldots, a_5$ and curvatures $0 < \epsilon_1 \leqq \ldots \leqq \epsilon_5$. Suppose that ϵ_5 is the larger curvature of the two spheres tangent to $X_1, \ldots, X_4$. Then a_5 lies in the convex cone with vertex at a_4 and generated by the convex hull of a_1, a_2 and a_3. Furthermore a_5 lies in the convex hull of $a_1, \ldots, a_4$ if and only if*

$$\epsilon_1 + \ldots + \epsilon_5 \geqq 3\epsilon_4. \tag{40}$$

The condition $\epsilon_1 + \ldots + \epsilon_5 < 3\epsilon_4$ is equivalent to

$$\zeta < \epsilon_4 \leqq \epsilon_5 < \eta \tag{41}$$

where

$$\eta = \epsilon_1 + \epsilon_2 + \epsilon_3 + 2(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)^{\frac{1}{2}} \tag{42}$$

and

$$\zeta = \epsilon_1 + \epsilon_2 + \epsilon_3 + (\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)^{\frac{1}{2}}. \tag{43}$$

Proof. Let $a = \epsilon_1 + \ldots + \epsilon_4$ and $b = \epsilon_1^2 + \ldots + \epsilon_4^2$ so that

$$\epsilon_5 = (a + (3a^2 - 6b)^{\frac{1}{2}})/2. \tag{44}$$

Our proof will use $X_1, \ldots, X_5$ as a basis for pentaspherical coordinates. Thus if Y is a sphere with $c(Y) = (\Delta(Y, X_i))$, then its curvature γ is given by (10) as

$$\gamma = c(Y)^T\Delta^{-1}\epsilon. \tag{45}$$

Let us introduce

$$(\kappa_1, \ldots, \kappa_5)^T = \Delta^{-1}(\epsilon_1, \ldots, \epsilon_5)^T \tag{46}$$

so that

$$\kappa_i = (\epsilon_1 + \ldots + \epsilon_5 - 3\epsilon_i)/6. \tag{47}$$

Let Y_j be the plane orthogonal to $\{X_1, \ldots, X_4\}\backslash\{X_j\}$ for $j = 1, \ldots, 4$. Then by (45) and (46) since the curvature of Y_j is 0,

$$0 = \Delta(Y_j, X_j)\kappa_j + \Delta(Y_j, X_5)\kappa_5. \tag{48}$$

Now observe that, since $\epsilon_4 \geqq \epsilon_3$ and $\epsilon_5 \geqq \epsilon_3$, we have

$$\begin{aligned} \kappa_1 \geqq \kappa_2 \geqq \kappa_3 &= (\epsilon_1 + \epsilon_2 - 2\epsilon_3 + \epsilon_4 + \epsilon_5)/6 \\ &\geqq (\epsilon_1 + \epsilon_2)/6 > 0. \end{aligned} \tag{49}$$

Also, $\epsilon_4 \leqq \epsilon_5$ implies that $a^2 > 2b$ so (44) implies

$$\kappa_5 = (a - 2\epsilon_5)/6 < 0. \tag{50}$$

The equation (48) with (49) and (50) shows that $\Delta(Y_j, X_j)$ and $\Delta(Y_j, X_5)$ have the same sign for $j = 1, 2, 3$ and the same or opposite sign for $j = 4$

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according to whether $\kappa_4 \geqq 0$ or $\kappa_4 < 0$. Since (40) is just the condition $\kappa_4 \geqq 0$ and $\kappa_4 < 0$ is easily seen to be equivalent to (41), this completes the proof.

THEOREM 9. *Let $U = S(a_0, 1)$ be a unit sphere, and let $X_1 = S(a_0, -1)$. Let $X_2, \ldots, X_5$ be spheres contained in U such that $X_1, \ldots, X_5$ are mutually tangent. Suppose that a_0 lies in the interior of the convex hull of the centres of $X_2, \ldots, X_5$. Let $\mathcal{G}$ be as in Theorem 5. Then $\mathcal{G}' = \mathcal{G}\backslash\{X_1\}$ is an osculatory packing of U, and, moreover, is the unique osculatory packing which begins with $S_1 = X_2, \ldots, S_4 = X_5$, (unique, apart from the order in which spheres of equal radii are listed).*

Proof. Let $C = \{S_n\}$ be an osculatory packing of U with $S_n = X_{n+1}$ for $n = 1, \ldots, 4$. We must show that $S_n \in \mathcal{G}'$ for all n. This will prove the theorem since if C were a proper subcollection of the packing $\mathcal{G}'$, then C could not be complete. The proof will use induction, and since it is rather complicated we shall explain the strategy first. By definition, S_{n+1} (for $n \geqq 4$) has the largest radius of spheres contained in $R_{n+1} = U\backslash(S_1 \cup \ldots \cup S_n)$; that is, given any $x \in R_{n+1}$, if we let $d_{n+1}(x) = \text{dist}(x, \partial R_{n+1})$, then r_{n+1}, the radius of S_{n+1} satisfies $r_{n+1} = \max\{d_{n+1}(x) : x \in R_{n+1}\}$. We shall inductively introduce a subdivision of E_N into polyhedra $L_1, \ldots, L_k$ so that

$$\max\{d_{n+1}(x) : x \in R_{n+1} \cap L_i\} = \rho_i \tag{51}$$

is attained at a unique point p_i in L_i and the sphere $S(p_i, \rho_i)$ is one of the spheres in $\mathcal{G}'$.

Formally, our induction assumption contains the following assertions at the nth stage:

(i) $S_1, \ldots, S_n$ are in $\mathcal{G}'$.

(ii) For $n \geqq 5$, if S is in $\mathcal{G}'$ and the curvature of S is strictly less than the curvature of S_n, then S is one of $S_1, \ldots, S_{n-1}$.

(iii) E_3 can be partitioned into polyhedra $L_1, \ldots, L_k$ each of which has vertices at the centres $a_1, \ldots, a_n$ of $S_1, \ldots, S_n$. Certain of the L_i are frustra of polyhedral cones with vertex at a_0 the centre of U and in this case we consider a_0 to be one of the vertices of L_i.

(iv) Each L_i is starlike with respect to a point p_i which is the centre of a sphere Y_i in $\mathcal{G}'$ tangent to all S_m with $m \leqq n$ whose centres are the vertices of L_i. We say that these spheres *determine* L_i.

(v) The vertices of L_i are the centres of *all* spheres U and S_m with $m \leqq n$ which touch Y_i.

(vi) The faces of L_i are triangles. The vertices of these triangles are the centres of spheres which are mutually tangent. If W_1, W_2, W_3 are three such spheres then there is a sphere W_4 such that the centre of W_4 is a vertex of L_i, and $W_1, \ldots, W_4$ are in some order $X_1(\alpha), \ldots, X_4(\alpha)$ for some $\alpha \in G$, and $Y_i = X_5(\alpha)$.

(vii) The radius of Y_i is ρ_i, given by (51).

We begin the induction with $n = 4$. Then (i) and (ii) are trivial. For (iii), there are five sets $L_1, \ldots, L_5$ which are respectively the convex hull of $a_1, \ldots, a_4$ and the frustra of the polyhedral cones with vertex a_0 generated by three of $a_1, \ldots, a_4$ (as in the paragraph preceding Lemma 8). These form a partition of E_3 since a_0 is in the interior of L_1 by assumption. For (iv), let Y_i denote the sphere of smaller radius tangent to the spheres which determine L_i. Then $Y_i \in \mathscr{G}'$. The centre p_i of Y_i can be shown to be in L_i using arguments like those in the proof of Lemma 9 and the fact that $a_0 \in L_1$. Parts (v) and (vi) are clear, and (vii) follows from Lemmas 7 and 8.

Now we assume (i)–(vii) for $n - 1$ and proceed to n. By (iv), (v) and (vii) (for $n - 1$), we have

$$\max\{d_n(x) : x \in R_n\} = \max(\rho_1, \ldots, \rho_k) = \rho_i \quad \text{say.}$$

Hence $S_n = Y_i \in \mathscr{G}'$ proving (i). To prove (ii), denote the curvature of a sphere S by $\epsilon(S)$, and we see that if $\epsilon(S) < \epsilon(S_{n-1})$, the result is true by the induction assumption, while if $\epsilon(S_{n-1}) \leqq \epsilon(S) < \epsilon(S_n)$, and S is not one of $S_1, \ldots, S_{n-1}$ then S_n does not have the minimal curvature of spheres contained in R_n, which contradicts its definition.

We now proceed to the construction (iii) which requires some care. We shall let $L_1, \ldots, L_k$ denote the partition at stage $(n - 1)$, and temporarily denote the new partition by $L_1', \ldots, L_{k'}'$. By (iii), (iv) and (vi) for $n - 1$, we may subdivide L_i (where $Y_i = S_n$), into a number of tetrahedra, by joining $a_n = p_i$ to the vertices of L_i. Let these tetrahedra be $T_1, \ldots, T_s$, and let the smaller sphere tangent to the spheres which determine T_j be Z_j. By using (vi) we see that the four spheres determining T_j together with Z_j are $X_1(\beta), \ldots, X_5(\beta)$ for some $\beta \in G$, with $Z_j = X_5(\beta)$. We examine each T_j in turn and ask whether or not the centre of Z_j is in the interior of T_j. If so then T_j becomes one of the L_m'. If not, then by Lemma 9, with an appropriate numbering, the curvature of Z_j, say ϵ_5, and the curvatures of the four spheres determining T_j, say $\epsilon_1, \ldots, \epsilon_4$, must satisfy

$$\epsilon_1 \leqq \epsilon_2 \leqq \epsilon_3 \leqq \epsilon_4 \leqq \epsilon_5$$

$$\zeta \leqq \epsilon_4 \leqq \epsilon_5 \leqq \eta$$

where η and ζ are given by (42) and (43). Note that ϵ_4 is the curvature of S_n. Let W_1, W_2, W_3, be the spheres of curvatures ϵ_1, ϵ_2, ϵ_3 respectively. Let W be the sphere which touches all of W_1, W_2, W_3 and Z_j, but is not S_n. Then $W \in \mathscr{G}'$ using the fact established above that W_1, W_2, W_3, Z_j, S_n are in some order $X_1(\beta), \ldots, X_5(\beta)$ for some $\beta \in G$. Now, if γ is the curvature of W then by (22),

$$\begin{aligned} \gamma &= \epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5 \\ &\leqq \epsilon_1 + \epsilon_2 + \epsilon_3 - \zeta + \eta \\ &= \zeta \leqq \epsilon_4. \end{aligned}$$

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There are three cases. If $\zeta < \epsilon_4$, then by (ii), W is one of $S_1, \ldots, S_{n-1}$. By Lemma 9, $\zeta < \epsilon_4$ implies that the centre of Z_j is in the convex hull of the centres of W, W_1, W_2 and W_3, which is a subset of some L_r. In this case, we define $L_r \cup T_j$ to be one of the sets L_m'. Note that since Z_j is tangent to W, W_1, W_2 and W_3 it follows from (iv) (for $n - 1$), that $Z_j = Y_r$. We make this same construction if $\zeta = \epsilon_4 = \gamma$, in case W is one of $S_1, \ldots, S_{n-1}$. In this case the centre of Z_j is on a boundary face of T_j. Finally, if $\zeta = \epsilon_4 = \gamma$, but W is not one of $S_1, \ldots, S_{n-1}$, we let T_j be one of the L_m'. Having done this for $T_1, \ldots, T_s$, we now let $L_1', \ldots, L'_{k'}$ consist of all L_m' just constructed, together with the unaltered sets L_p from the previous stage.

Having now completed the construction required by (iii), we revert to the notation $L_1, \ldots, L_k$ for the partition of E_3. The sphere Y_i is the sphere of smaller radius tangent to any four of the spheres determining L_i. It is clear by the construction that (iv) and (v) are valid with this choice of Y_i, and (vi) was proved during the proof of (iii).

It is in the proof of (vii) that we use Theorem 5 in a crucial way. Let r_i be the radius of Y_i. By (iv), Y_i, touches all the spheres which determine L_i so, by Lemma 7 or 8 applied to appropriate quadruples of these spheres, we see that

$$\rho_i = \max\{\operatorname{dist}(x, \partial R_n) : x \in R_n \cap L_i\} \leqq r_i. \tag{52}$$

However, $Y_i \in \mathscr{G}'$ and is not one of the spheres $S_1, \ldots, S_n$, or X_1, so, by Theorem 5, Y_i does not intersect any of these spheres. Thus equality holds in (52), and is attained only for $x = p_i$, the centre of Y_i, (see equation (37)).

This completes the induction and the proof of the theorem.

Remarks. 1. By inversion, given any five spheres $X_1, \ldots, X_5$ which are mutually tangent, we can invert them into spheres $X_1', \ldots, X_5'$ which satisfy the conditions of Theorem 9, (see [**10**, p. 109]). Since osculatory packings are complete, and since inversion preserves sets of measure zero, it is thus clear that $\mathscr{G}'$ is a complete packing of U for any choice of $X_2, \ldots, X_5$. However, it is not clear that this packing is osculatory. Theorem 11 shows that $\mathscr{G}'$ is osculatory, but the uniqueness aspect of Theorem 10 may not be true.

2. We can use a construction similar to that in Remark 1 to generate complete packings of E_3 by spheres with positive radii: Invert the configuration of Theorem 9 with respect to a sphere centred at the point of contact of X_1 and X_2. Then X_1 and X_2 invert into non-intersecting half-spaces, and the remaining spheres in $\mathscr{G}'$ form a packing of the region between the (parallel) boundaries of X_1 and X_2. This is easily seen to be a complete packing. By stacking together a countable number of copies of this packing, we produce a complete packing of all of E_3.

3. There are other ways one could imagine for packing E_3 completely, some of which would undoubtedly be more efficient (or at least as efficient), than the one just proposed. Gilbert [**15**], and Hudson [**17**] suggest (implicitly)

filling the "interstices" of a close packing of E_3 by equal spheres, by first placing the largest spheres possible, then the next largest, and so on. This is certainly a process resembling what we have done. However, on closer examination, in terms of the separations of the spheres involved, this packing more closely resembles our four-dimensional process.

4. The idea used in Remark 2 is equally valid in higher dimensions. That is, suppose we have a packing of the region between two mutually tangent spheres X_1 and X_2 in E_N, where X_1 has negative curvature and X_2 has positive curvature. If we invert in the point of contact, then, considering only the spheres which touch both X_1 and X_2 (if there are any), these form a packing by equal N-spheres of the region between two parallel hyperplanes. The cross section of these spheres, by a hyperplane midway between these two, is a packing by equal spheres of E_{N-1}. In the case of our packing $\mathscr{G}'$, this is the well-known closest packing of E_2 by equal circles.

It would be extremely interesting to investigate the packings produced in this way from the osculatory packings of, say, a four-dimensional sphere, since one would intuitively expect these to have fairly high densities. Indeed, it seems clear that such a packing will contain configurations such as those suggested by Boerdijk [**1**; **12**, p. 297 and 306], which have local densities greater than the presumed best packings, with density $\pi/\sqrt{18}$.

THEOREM 11. *Let $U, X_1, \ldots, X_5$ be as in Theorem* 10, *except that the centre of U need not be in the interior of the convex hull of the centres of $X_2, \ldots, X_5$. Then $\mathscr{G}'$ is an osculatory packing of U.*

Proof. We show that there is a choice of $Y_2, \ldots, Y_5$ mutually tangent spheres in $\mathscr{G}'$, all touching X_1, for which the centre of U lies in the convex hull of the centres of $Y_2, \ldots, Y_5$ (although possibly on the boundary of this set). We select $Y_2, \ldots, Y_5$ as follows: let Y_2 be a sphere of minimal curvature ϵ_2 in $\mathscr{G}'$ which touches X_1, and let Y_i, $(i = 3, 4, 5)$, be a sphere of minimal curvature ϵ_i in $\mathscr{G}'$ which touches $Y_2, \ldots, Y_{i-1}$ and X_1. We shall show that if Π_i is the half-space orthogonal to $\{Y_2, \ldots, Y_5\}\backslash\{Y_i\}$, then the centres of X_1 and Y_i lie on the same side of the boundary of Π_i. Let $c = \Delta(X_1, \Pi_i)$ and $d = \Delta(X_1, \Pi_i)$. We assume $d > 0$ and we wish to prove $c \leqq 0$ (since X_1 has negative curvature). As in equation (48), we have

$$(53)\quad 0 = c((-1 + \epsilon_2 + \ldots + \epsilon_5) + 3) + d((-1 + \epsilon_2 + \ldots + \epsilon_5) - 3\epsilon_i).$$

The coefficient of c in (53) is clearly positive. Also ϵ_i is the smaller root of the quadratic (20) (with $N = 3$ and $\epsilon_1 = -1$), and hence $2\epsilon_i$ is less than the sum of the two roots which is $-1 + \epsilon_2 + \ldots + \epsilon_5 - \epsilon_i$. This shows that the coefficient of d in (53) is non-negative, and equals zero if and only if the two roots of (20) for ϵ_i are equal. Thus $c \leqq 0$.

Since the above holds for $i = 2, \ldots, 5$, the centre a_0 of X_1 lies in the convex hull H of the centres of $Y_2, \ldots, Y_5$. If a_0 is in the interior of H, Theorem 10 applies. If a_0 lies in the interior of a two-dimensional face of H,

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say the face opposite Y_i, then the two spheres Y_i, Y_i' tangent to all of $\{X_1, Y_2, \ldots, Y_5\}\backslash\{Y_i\}$ have the same curvature, and we can repeat the proof of Theorem 10 beginning the induction at $n = 6$. Finally, if a_0 is in the interior of an edge of H, then the spheres Y_2, Y_3 are tangent along a diameter of U and there are six spheres of equal curvature tangent to X_1, Y_2 and Y_3 and forming a closed ring (the "hexlet" [**27**]). We may again repeat the proof of Theorem 10 starting with these eight spheres.

6. Concluding remarks. It is quite clear that the packings of U given by $\mathscr{G}'$ have the same exponent M, independent of the choice of $X_2, \ldots, X_5$, since inversion in a suitable sphere, with centre outside U will map any of these packings into any other. Such mappings are Lipschitz and have Lipschitz inverses. An interesting choice for $(\epsilon_1, \ldots, \epsilon_5)$ is $(-1, 2, 2, 3, 3)$ since, according to (24), the curvatures of all spheres in the packing are integers. Soddy noted this fact in [**28**] for certain subcollections of the packing $\mathscr{G}'$, so we shall call the packing, beginning with spheres of these curvatures, the Soddy packing. Observe that, in this case, $\mathscr{G}'$ is not the only osculatory packing which begins with the four spheres $X_2, \ldots, X_5$ since there are many spheres of curvature 3 not in $\mathscr{G}'$ which will fit in $U\backslash(X_2 \cup \ldots \cup X_5)$.

Using an algorithm described in [**7**], the IBM 360/65 computer at the University of British Columbia quickly (135 seconds) counted the number $W(C)$ of spheres in the Soddy packing, with curvature C at most 300. It is interesting to note (and easily proved from (24)), that $W(C) = 0$ for $C \equiv 1$ (modulo 3). The total number of spheres with curvatures at most 300 is 305594 and these occupy .94727 of the volume of U. Using a method suggested by Melzak [**24**] for $N = 2$, we obtain

$$\sum \{W(N) : N \leqq C\} \approx (.2988455)C^{-M_1},$$

where

$$M_1 = 2.42009, \tag{54}$$

suggesting that $M \approx 2.42$.

As an additional numerical experiment, we used the initial curvatures $(-1, a, a, a, a)$, where $a = 1 + \frac{1}{2}\sqrt{6}$, corresponding to the centres of $X_2, \ldots, X_5$ being at the vertices of a regular tetrahedron. In this case, the computer counted the number of spheres for which the integer part of the curvature is C, for each C less than 600. There were 1693595 such spheres, and the result corresponding to (54) was

$$M_2 = 2.41748 \tag{55}$$

again suggesting $M \approx 2.42$.

It should be possible to give rigorous upper and lower bounds on M analogous to the bounds obtained for the two-dimensional S in [**5**; **6**]. However, the

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methods developed by various authors [**3**; **5**; **6**; **13**; **23**; **29**] for the two-dimensional problem depend very much on the fact that, if the largest disk is removed from a curvilinear triangle, then three new triangles are formed. No such result is true in three dimensions, where the interior of the set

$$R_n = U \backslash (S_1 \cup \ldots \cup S_{n-1})$$

is connected, for all n. The best bounds we have are thus

$$2.03 < M < 2.8228 \ldots = (3 + \sqrt{7})/2. \tag{56}$$

The lower bound is due to Larman [**20**] and the upper bound due to this author [**4**].

The construction used in Theorem 10 of this paper is reminiscent of the construction used in the proof of [**4**, Theorem 2], and could possibly be used to improve the upper bound in (56), but not by much.

In the two-dimensional case, it is easy to see how to choose $\alpha \in G$ in order that each disk in $\mathscr{G} \backslash \{X_1, \ldots, X_4\}$ shall have a unique representation as $X_4(\alpha)$. One simply uses only those α which have components in the set $\{1, 2, 3\}$, so A_4 is unnecessary. The situation for three dimensions is more complicated. It is easy to see that A_5 is unnecessary since the columns of $A_4{}^5$ are a permutation of those of A_5. It can also be shown that if α and β have components only in $\{1, 2, 3\}$ then $X_5(\alpha) \neq X_5(\beta)$ if $\alpha \neq \beta$. However, there are many relations of the form $A(\alpha)e_5 = A(\beta)e_5$, if α or β has some components equal to 4. The algorithm developed in [**7**] gets around this difficulty by replacing the A_i by operations which are not linear, but it is still an interesting question as to whether all relations of the form $A(\alpha)e_5 = A(\beta)e_5$ can be discovered.

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