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A REGULAR SUMMABILITY METHOD WHICH SUMS THE GEOMETRIC SERIES TO ITS PROPER VALUE IN THE WHOLE COMPLEX PLANE

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ABSTRACT. In this paper an explicit regular sequence-to-sequence summability method is presented which sums the geometric series to the value $1/(1-z)$ in all of $\mathbb{C}\setminus\{1\}$ and to infinity at the point 1. The method also provides compact convergence in $\mathbb{C}\setminus[1,\infty)$ and therefore improves well-known results by Le Roy, Lindelöf and Mittag-Leffler.

Several authors (Le Roy [1], Lindelöf [2], Mittag-Leffler [3]) have given explicit regular summability methods which sum the geometric series to the function $1/(1-z)$ in its Mittag–Leffler star $\mathbb{C}\setminus[1,\infty)$.

In this paper we present a regular method which sums the geometric series to the value $1/(1-z)$ in all of $\mathbb{C}\setminus\{1\}$ and to infinity at the point 1. The method described in the following theorem provides compact convergence in $\mathbb{C}\setminus[1,\infty)$—so do the methods in [1], [2], [3]—, and pointwise convergence in all of $\mathbb{C}\setminus\{1\}$. Moreover we get uniform convergence on every compact subset of $H=\{x+iy\colon x>1,\, y\geq 0\}$.

THEOREM. *The continuous method defined by*[1]

$$c_k(x)=\frac{\log x}{x}e^{-(k/x)(\log k-i\pi)} \qquad (x>1,\, k=0,1,\ldots) \tag{1}$$

is regular, and the transform

$$\sigma_x(z)=\sum_{k=0}^{\infty} c_k(x)\cdot(1+z+\cdots+z^k) \qquad (z\in\mathbb{C},\, x>1) \tag{2}$$

of the geometric series has the following properties.

$$\lim_{x\to\infty}\sigma_x(z)=\frac{1}{1-z} \tag{3}$$
uniformly on every compact subset of $\mathbb{C}\setminus[1,\infty)$ *resp.* $H=\{x+iy \mid x>1,\, y\geq 0\}$.

$$\lim_{x\to\infty}\sigma_x(1)=\infty. \tag{4}$$

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(1) We define $0\log 0=0$.

REMARKS

(a) In (1) we might replace π by any function $f:(1,\infty)\to[0,\infty)$ which satisfies the conditions (i) $f(x)=o(\log x)$ as $x\to\infty$, and (ii) $\liminf_{x\to\infty} f(x)>\pi/2$, and still $\sigma_x(z)$ would have the properties (3) and (4). A proof to this more general version of the above theorem—without (4)—is given in [4].

(b) From (1) we can also obtain a discrete row-finite method $A=(a_{n,k})_{n,k=0}^{\infty}$ with the same summation properties, e.g. by defining

$$a_{n,k}=\begin{cases}\dfrac{\log n}{n}e^{-(k/n)(\log k-i\pi)} & \text{if } n=2,3,\ldots \text{ and } k\le n^n,\\ 0 & \text{else.}\end{cases}$$

The A-transform $\sigma_n(z)$ of the geometric series also satisfies (3). The simple proof to this can be found in [4].

Proof to the Theorem

1. At first we show that $c_k(x)$ is regular by checking the Toeplitz conditions. Clearly $\lim_{x\to\infty} c_k(x)=0$ for $k=0,1,\ldots$. It remains to prove that

$$\lim_{x\to\infty}\sum_{k=0}^{\infty} c_k(x)=1 \tag{5}$$

and that the series $\sum_{k=0}^{\infty}|c_k(x)|$ are uniformly bounded for $x>1$. We will also show that

$$\lim_{x\to\infty}\sum_{k=0}^{\infty}|c_k(x)|=1. \tag{6}$$

(It is easily seen that the series in (6) converge for $x>1$.) We define for $x>1$

$$H(x)=\sum_{0\le k\le x/\sqrt{(\log x)}}|c_k(x)|,$$

$$R(x)=\sum_{k>x/\sqrt{(\log x)}}|c_k(x)|.$$

Then we obtain that

$$\sum_{k=0}^{\infty}|c_k(x)|=H(x)+R(x),$$

and—since

$$\max_{0\le k\le x/\sqrt{(\log x)}}\left|\exp\left(\frac{ik\pi}{x}\right)-1\right|=o(1)$$

as $x \to \infty$—

$$\sum_{k=0}^{\infty} c_k(x) = H(x) + \frac{\log x}{x} \cdot \sum_{0 \le k \le x/\surd(\log x)} \exp\left(-\frac{k \cdot \log k}{x}\right)$$

$$\times \left(\exp\left(\frac{ik\pi}{x}\right) - 1\right) + O(R(x))$$

$$= H(x)(1+o(1)) + O(R(x)) \quad \text{as} \quad x \to \infty.$$

In order to complete the proof for regularity we need to show that

(7) $\qquad H(x) \to 1 \quad \text{as} \quad x \to \infty,$

(8) $\qquad R(x) \to 0 \quad \text{as} \quad x \to \infty,$ and that

(9) $\qquad H(x), R(x)$ are uniformly bounded for $x > 1$.

As to (9) we observe that the expressions $\exp[-(k/x)\log(k/x)]$ are uniformly bounded by a constant K for $x > 1$, $k \ge 0$. Therefore both $H(x)$ and $R(x)$ are bounded by

$$\frac{\log x}{x} \sum_{k=0}^{\infty} \exp\left(-\frac{k}{x} \log x\right) \exp\left(-\frac{k}{x} \log \frac{k}{x}\right) \le K \frac{\log x}{x}$$

$$\times \sum_{k=0}^{\infty} \left(\exp\left(-\frac{\log x}{x}\right)\right)^k = \frac{\log x/x}{1-\exp(-\log x/x)}$$

which is bounded for $x > 1$. Thus (9) is proved.

To show (8) we use the estimate

$$|R(x)| \le \frac{\log x}{x} \cdot \sum_{k > x/\surd(\log x)} \exp\left(-\frac{k}{x} \log x\right) \exp\left(-\frac{k}{x} \log \frac{k}{x}\right) \le K \frac{\log x}{x}$$

$$\times \sum_{k > x/\surd(\log x)} \left(\exp\left(-\frac{\log x}{x}\right)\right)^k \le K \frac{\log x/x}{1-\exp(-\log x/x)} \exp(-\surd(\log x))$$

which converges to 0 as $x \to \infty$.

For (7) we can use the relation

$$\max_{0 \le k \le x/(\log x)} \left|\exp\left(-\frac{k}{x} \log \frac{k}{x}\right) - 1\right| = o(1) \quad \text{as} \quad x \to \infty.$$

We get

$$H(x) = \frac{\log x}{x} \sum_{0 \le k \le x/\surd(\log x)} \exp\left(-\frac{k}{x} \log x\right)\left(1 + \left(\exp\left(-\frac{k}{x} \log \frac{k}{x}\right) - 1\right)\right)$$

$$= \frac{\log x}{x} \sum_{0 \le k \le x/\surd(\log x)} \left(\exp\left(-\frac{\log x}{x}\right)\right)^k (1+o(1))$$

$$= \frac{\log x/x}{1-\exp(-\log x/x)} (1 + O(e^{-\surd(\log x)}))$$

$$= 1 + o(1) \quad \text{as} \quad x \to \infty.$$

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2. Our next aim is to show (4), which will be done by proving that

$$\lim_{x\to\infty}\sum_{k=1}^{\infty} k\cdot c_k(x)=\infty. \tag{10}$$

(Hence (4) is obtained by adding up the limits in (5) and (10).)

Like in 1. we write

$$c_k(x)=\frac{\log x}{x}\exp\left(-\frac{k}{x}\log x\right)\cdot\exp\left(-\frac{k}{x}\log\frac{k}{x}+i\pi\frac{k}{x}\right)$$

and use the fact that

$$\exp\left(-\frac{k}{x}\log\frac{k}{x}+i\pi\frac{k}{x}\right)=1+o(1)\quad\text{as}\quad x\to\infty$$

uniformly for $k\le x/\sqrt{(\log x)}$ $(x>1)$, and that the same term is uniformly bounded for $x>1$, $k\in\mathbb{N}_0$.

From this it follows that

$$\sum_{k=1}^{\infty} kc_k(x)=\frac{\log x}{x}\sum_{k=1}^{\infty} k\exp\left(-\frac{k}{x}\log x\right)(1+o(1)) \tag{11}$$
$$+O\left(\frac{\log x}{x}\sum_{k>x/\sqrt{(\log x)}} k\exp\left(-\frac{k}{x}\log x\right)\right)$$

as $x\to\infty$.

The first term on the right hand side of (11) is equal to

$$\frac{\log x}{x}\cdot e^{-(\log x)/x}(1-e^{-(\log x)/x})^{-2}\cdot(1+o(1))=\frac{x}{\log x}(1+o(1))$$

and the O-term is

$$O\left(\frac{\log x}{x}\sum_{\mu=1}^{\infty}\left(\mu+\frac{x}{\sqrt{(\log x)}}\right)\cdot e^{-\sqrt{(\log x)}}e^{-(\mu/x)\log x}\right)$$
$$=O\left(e^{-\sqrt{(\log x)}}\left(\frac{\log x}{x}(1-e^{-(\log x)/x})^{-2}+\sqrt{(\log x)}(1-e^{-(\log x)/x})^{-1}\right)\right)$$

which is $o(x/\log x)$. Thus we have

$$\sum_{k=1}^{\infty} kc_k(x)=\frac{x}{\log x}\cdot(1+o(1))$$

which implies (10).

3. In order to prove (3) we now derive an "integral representation" for the transform $\tau_x(z)=\sum_{k=0}^{\infty} c_k(x)z^k$ of the sequence $(z^n)_{n\in\mathbb{N}_0}$. Namely if

$$z=\rho e^{i\theta},\qquad \rho>0,\qquad 0\le\theta<2\pi\quad\text{and}\quad x>\frac{4\pi}{2\pi-\theta}$$

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then we have

$$(12)\qquad \tau_x(z)=i\frac{\log x}{x}\int_{1/2}^{\infty} e^{-t(\theta+\pi/2x)}\exp\Big(-i\frac{t}{x}\log(t/\rho^x)\Big)\,dt+\frac{\log x}{x}Q(z)$$

—where

$$|Q(z)|\le A\Big(\rho+\frac{1}{2\pi-\theta}\Big)$$

and A is a uniform constant not depending on ρ, θ or x.

To show (12) we consider the curves $\gamma_n=\gamma^{(1)}+\gamma_n^{(2)}+\gamma_n^{(3)}+\gamma_n^{(4)}$ $(n=0,1,\ldots)$ with the parametrisations

$$\gamma^{(1)}(t)=\frac{1}{2}e^{-it},\qquad -\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

$$\gamma_n^{(2)}(t)=-it,\qquad \frac{1}{2}\le t\le n+\frac{1}{2}$$

$$\gamma_n^{(3)}(t)=\Big(n+\frac{1}{2}\Big)e^{it},\qquad -\frac{\pi}{2}\le t\le\frac{\pi}{2}$$

$$\gamma_n^{(4)}(t)=-it,\qquad -\Big(n+\frac{1}{2}\Big)\le t\le-\frac{1}{2}.$$

Also we define

$$(13)\qquad F(u)=\frac{\exp\Big(u(\log\rho+i\theta)-\dfrac{u}{x}(\operatorname{Log}u-i\pi)\Big)}{e^{2\pi iu}-1}\quad\text{for}\quad u\notin(-\infty,0]\cup\mathbb{N}.$$

With the residue theorem we obtain

$$(14)\qquad \sum_{k=1}^{n}c_k(x)z^k=\frac{\log x}{x}\int_{\gamma_n}F(u)\,du\quad\text{for}\quad n=1,2,\ldots.$$

There is some positive constant K such that

$$(15)\qquad \left|\frac{1}{e^{2\pi iu}-1}\right|\le K\quad\text{and}\quad\left|\frac{1}{1-e^{-2\pi iu}}\right|\le K\quad\text{for}\quad u\in\gamma_n^{(\nu)},\qquad \nu=2,3,4,\quad n=0,1,\ldots.$$

For $n=0,1,\ldots$ we have

$$(16)\qquad \int_{\gamma_n^{(3)}}F(u)\,du=i\Big(n+\frac{1}{2}\Big)\cdot\int_{-\pi/2}^{\pi/2}F\Big(\Big(n+\frac{1}{2}\Big)e^{it}\Big)e^{it}\,dt.$$

If $u=(n+\frac{1}{2})e^{it}$, $0\le t\le\pi/2$, then the modulus of the integrand of the last integral is equal to

$$\left|\frac{1}{e^{2\pi iu}-1}\right|\exp\Big(\Big(n+\frac{1}{2}\Big)\Big(\cos t\log\frac{\rho}{(n+\frac{1}{2})^{1/x}}-\Big(\theta+\frac{\pi-t}{x}\Big)\sin t\Big)\Big)$$

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which is not greater than

$$K\exp\left(\left(n+\frac{1}{2}\right)\cdot\cos t\log\frac{\rho}{(n+\frac{1}{2})^{1/x}}\right);$$

and for

$$u=\left(n+\frac{1}{2}\right)e^{it},\qquad -\frac{\pi}{2}\le t\le 0$$

it is equal to

$$\left|\frac{1}{1-e^{-2\pi iu}}\right|\exp\left(\left(n+\frac{1}{2}\right)\left(\cos t\log\frac{\rho}{(n+\frac{1}{2})^{1/x}}\right)-\left(\theta+\frac{\pi-t}{x}-2\pi\right)\sin t\right)$$

which also doesn't exceed

$$K\exp\left(\left(n+\frac{1}{2}\right)\cos t\log\frac{\rho}{(n+\frac{1}{2})^{1/x}}\right).$$

Therefore we get the following estimate for the integral in (16).

$$\left|\int_{\gamma_n^{(3)}} F(u)\,du\right|\le(2n+1)K\int_0^{\pi/2} e^{-\alpha\cos t}\,dt \tag{17}$$

where

$$\alpha=\left(n+\frac{1}{2}\right)\log\frac{(n+1/2)^{1/x}}{\rho}.$$

If $n>\rho^x$, then α is positive and we can write

$$\int_0^{\pi/2} e^{-\alpha\cos t}\,dt=\int_0^{\pi/2} e^{-\alpha\sin t}\,dt\le\int_0^{\infty} e^{-2\alpha t/\pi}\,dt=\frac{\pi}{2\alpha}.$$

Hence for $n>\rho^x$ we have

$$\left|\int_{\gamma_n^{(3)}} F(u)\,du\right|\le\frac{(2n+1)K\pi}{2\alpha}=K\pi/\log\frac{(n+1/2)^{1/x}}{\rho}$$

from which it follows that

$$\lim_{n\to\infty}\int_{\gamma^{(3)}} F(u)\,du=0. \tag{18}$$

Substituting $n=0$ in (17) we obtain for $\gamma^{(1)}=-\gamma_0^{(3)}$

$$\left|\int_{\gamma^{(1)}} F(u)\,du\right|\le K\int_0^{\pi/2}(2^{1/x}\rho)^{(\cos t)/2}\,dt$$

$$\le K\int_0^{\pi/2}\left(2\pi\left(\rho+\frac{1}{2\pi-\theta}\right)\right)^{(\cos t)/2} dt.$$

After omitting the exponent $(\cos t)/2$ and evaluating the last integral we obtain

$$\left|\int_{\gamma^{(1)}} F(u)\,du\right|\le K\pi^2\left(\rho+\frac{1}{2\pi-\theta}\right). \tag{19}$$

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In order to show (12) we still have to consider the integrals $\int_{\gamma_n^{(2)}}$ and $\int_{\gamma_n^{(4)}}$. We shall now give an estimate for the integrals $\int_{\gamma_n^{(2)}} F(u)\,du = -i\int_{1/2}^{n+1/2} F(-it)\,dt$. For $\frac{1}{2} \le t \le n+\frac{1}{2}$ we have

$$|F(-it)| = \frac{1}{1-e^{-2\pi t}} \exp\left(-t\left(2\pi - \theta - \frac{3\pi}{2x}\right)\right)$$

which doesn't exceed

$$\frac{1}{1-e^{-\pi}} \exp\left(-\frac{5}{8}\, t(2\pi-\theta)\right) \quad \text{for} \quad x > \frac{4\pi}{2\pi-\theta}.$$

Therefore, as $n \to \infty$, $\int_{\gamma_n^{(2)}} F(u)\,du$ approaches a number the modulus of which is less than

$$\frac{1}{1-e^{-\pi}} \int_0^\infty \exp\left(-\frac{5}{8}\, t(2\pi-\theta)\right) dt = \frac{8}{5(1-e^{-\pi})} \cdot \frac{1}{2\pi-\theta}.$$

Hence

$$\left|\lim_{n\to\infty} \int_{\gamma_n^{(2)}} F(u)\,du\right| \le \frac{2}{1-e^{-\pi}} \cdot \frac{1}{2\pi-\theta}. \tag{20}$$

We complete the proof for (12) by considering the integrals

$$\int_{\gamma_n^{(4)}} F(u)\,du = -i\int_{1/2}^{n+1/2} F(it)\,dt = i(I_1(n)+I_2(n))$$

—where

$$I_1(n) = \int_{1/2}^{n+1/2} e^{-t(\theta+\pi/2x)} \exp\left(-i\frac{t}{x}\log(t/\rho^x)\right) dt,$$

$$I_2(n) = \int_{1/2}^{n+1/2} \frac{1}{1-e^{-2\pi t}} \exp\left(-t\left(2\pi+\theta+\frac{\pi}{2x}\right) - i\frac{t}{x}\log(t/\rho^x)\right) dt.$$

The Weierstraß M-test shows that the limits $\lim_{n\to\infty} I_1(n)$, $\lim_{n\to\infty} I_2(n)$ both exist. The integral $I_1(n)$ approaches the integral in (12) as $n \to \infty$. And for $I_2(n)$ we can give the estimate

$$|I_2(n)| \le \int_0^\infty \frac{1}{1-e^{-\pi}}\, e^{-2\pi t}\,dt = \frac{1}{2\pi(1-e^{-\pi})}.$$

These two results together with (18), (19) and (20) show that we may take the limit as $n \to \infty$ in (14) to obtain (12).

4. With the help of (12) we are now able to prove (3). It is easy to verify the identity

$$\sigma_x(z) = \frac{1}{1-z}(\tau_x(1) - z\tau_x(z)) \quad \text{for} \quad z \ne 1,$$

and the Toeplitz condition (5) implies that $\lim_{x\to\infty} \tau_x(1) = 1$. Therefore it suffices to show that

$$\lim_{x\to\infty} \tau_x(z) = 0 \tag{21}$$

uniformly on all compact sets described in (3). In fact it even suffices to show that (21) holds

(a) uniformly on every disc $D_r := \{z : |z| \le r\}$ with $0 < r < 1$,

(b) uniformly on every sector $\Delta(\theta_0, R) := \{\rho e^{i\theta} : \frac{1}{2} \le \rho \le R,\ \theta_0 \le 2\pi - \theta_0\}$ with $R > 1$, $0 < \theta_0 < \pi$, and

(c) uniformly on every sector $D(r, R) := \{\rho e^{i\theta} : r \le \rho \le R, 0 \le \theta \le \pi/2\}$ with $1 < r < R$.

ad (*a*): If $0 < r < 1$ and $x > 1$, then we have uniformly on D_r

$$|\tau_x(z)| \le \sum_{k=0}^{\infty} |c_k(x)|\, r^k \le \sum_{0 \le k \le \sqrt{x}} |c_k(x)| + \sum_{k > \sqrt{x}} |c_k(x)| r^{\sqrt{x}}$$
$$\le \frac{\log x}{x}(1 + \sqrt{x}) + r^{\sqrt{x}} \sum_{k=0}^{\infty} |c_k(x)|.$$

Because of (6) and $0 < r < 1$ the last expression approaches zero as $x \to \infty$ which proves (21) for the case (a).

ad (*b*): Let $R > 1$ and $0 < \theta_0 < \pi$. For $x > 4\pi/\theta_0$ we may use the "integral representation" (12) for all elements $z = \rho e^{i\theta}$ of $\Delta(\theta_0, R)$, and by taking absolute values in (12) we obtain the inequality

$$|\tau_x(z)| \le \frac{\log x}{x} \int_{1/2}^{\infty} e^{-t(\theta + \pi/2x)}\, dt + \frac{\log x}{x} A\left(\rho + \frac{1}{2\pi - \theta}\right) \le \frac{\log x}{x} \int_0^{\infty} e^{-t\theta_0}\, dt$$
$$+ \frac{\log x}{x} A\left(R + \frac{1}{\theta_0}\right) = \frac{\log x}{x}\left(AR + \frac{A+1}{\theta_0}\right).$$

This estimate for $|\tau_x(z)|$ implies (21) for the case (b).

ad (*c*): Let $1 < r < R$. Again, if

$$x > \frac{4\pi}{2\pi - \pi/2} = \frac{8}{3},$$

we may use the "integral representation" (12) for all elements $z = \rho e^{i\theta}$ of $D(r, R)$. After cutting the integral in (12) into three parts we obtain the inequality

$$(22) \qquad |\tau_x(z)| \le \frac{\log x}{x}(|I_1| + |I_2| + |I_3| + |Q(z)|)$$

where

$$I_1 = \int_{1/2}^{\sqrt{x}} e^{-t(\theta + \pi/2x)} \exp\left(-i\frac{t}{x}\log(t/\rho^x)\right) dt, \qquad I_2 = \int_{\sqrt{x}}^{x^2} \ldots,\ I_3 = \int_{x^2}^{\infty} \cdots.$$

Now we can use the inequality in (12) for $|Q(z)|$ and give trivial estimates for I_1

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and I_3 to obtain

$$(23)\qquad \begin{cases} |Q(z)| \le A\left(R + \dfrac{1}{2\pi - \pi/2}\right) \\ |I_1| \le \sqrt{x} \\ |I_3| \le \displaystyle\int_{x^2}^{\infty} e^{-t(\pi/2x)}\, dt = \frac{2x}{\pi} e^{-(\pi/2)x}. \end{cases}$$

Also we have

$$iI_2 = x\int_{\sqrt{x}}^{x^2} \frac{1}{G(t)}\, d\exp\left(-i\frac{t}{x}\log(t/\rho^x)\right)$$

where

$$G(t) = e^{t(\theta+\pi/2x)}(\log(\rho^x/t) - 1).$$

As $r > 1$, there exists a constant $x_1 > \frac{8}{3}$ (depending on r) such that $x^2 < r^x/e$ for $x \ge x_1$, and hence $G(t)$ is positive throughout the interval $[\sqrt{x}, x^2]$ if $x \ge x_1$. The derivative $G'(t)$ in this interval is given by

$$G'(t) = e^{t(\theta+\pi/2x)}\left(\theta + \frac{\pi}{2x}\right)g(t)$$

where

$$g(t) = \log\frac{\rho^x}{te} - \frac{1}{t\left(\theta + \dfrac{\pi}{2x}\right)} \ge \log\frac{r^x}{x^2 e} - \frac{2\sqrt{x}}{\pi}$$

Since this lower bound for $g(t)$ in $[\sqrt{x}, x^2]$ tends to infinity as $x \to \infty$, there exists a constant $x_2 \ge x_1$ such that for $x \ge x_2$ $g(t)$—and therefore also $G'(t)$—is positive in this interval. Thus, if $x \geqq x_2$, the function $1/G(t)$ is positive and decreasing in $[\sqrt{x}, x^2]$ and we may apply the second mean value theorem to the real and imaginary part of I_2, which yields

$$|I_2| \le \frac{4x}{G(\sqrt{x})} \le \frac{4x}{\exp\left(\dfrac{\pi}{2\sqrt{x}}\right)(\log(r^x/\sqrt{x}) - 1)} \xrightarrow[\text{as } x\to\infty]{} \frac{4}{\log r}.$$

Therefore there exists a constant $x_3 \ge x_2$ such that $|I_2| \le 5/log\, r$ for $x \ge x_3$. Inserting this inequality and (23) in (22) we obtain the estimate

$$|\tau_x(z)| \le \frac{\log x}{x}\left(\sqrt{x} + \frac{5}{\log r} + \frac{2x}{\pi} e^{-\pi x/2} + 2AR\right)$$

which holds uniformly for $x \ge x_3$ and $z \in D(r, R)$. This implies (21) for the case (c) which completes the proof to our theorem.

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