Canad. Math. Bull. Vol. **30** (3), 1987

MORE ON FATOU'S LEMMA IN SEVERAL DIMENSIONS

BY

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ABSTRACT. Recently, Balder proved a version of Fatou's lemma in several dimensions which, *inter alia*, generalizes a version of this lemma due to Artstein. Here we show how the latter result can be used to derive the former, by using Chacon's biting lemma.

1. **Introduction**. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $\mathcal{L}_1^m \equiv (\mathcal{L}_1(\Omega))^m$ stand for the space of all μ-integrable functions from Ω into $\mathbb{R}^m$. For any $f \equiv (f^1, \ldots, f^m)$ in $\mathcal{L}_1^m$ we denote by $f^- \in \mathcal{L}_1^m$ the function consisting of the negative parts (coordinatewise) of f, i.e., $(f^-)^i \equiv (f^i)^- \equiv \max(-f^i, 0)$, $1 \leqslant i \leqslant m$; similarly, we define $f^+ \equiv (-f)^-$. Unless mentioned otherwise, all operations and relations in $\mathbb{R}^m$ (such as limits, integration, inequalities, etc.) are understood to take place or hold coordinatewise.

Recently, the present author gave the following result [3], [4, Cor. 3.9]:

THEOREM 1. *Suppose that* $\{f_k\}$ *is a sequence in* $\mathcal{L}_1^m$ *such that*

$$\lim_k \int_\Omega f_k \, d\mu \text{ exists (in } \mathbb{R}^m\text{),} \tag{1}$$

$$\{f_k^-\} \text{ is uniformly integrable.} \tag{2}$$

Then there exists a function f_* *in* $\mathcal{L}_1^n$ *such that*

$$\int_\Omega f_* \, d\mu \leqslant \lim_k \int_\Omega f_k \, d\mu, \tag{3}$$

$$f_*(\omega) \text{ is a limit point of } \{f_k(\omega)\} \text{ a.e. in } \Omega. \tag{4}$$

This version of Fatou's lemma in several dimensions, as it is called, generalizes the original result due to Schmeidler [12], as well as later versions of Hildenbrand-Mertens [9], [10], Artstein [1], and Cesari-Suryanarayana [8]. As can be learned from the case $m = 1$, when Theorem 1 is equivalent to Fatou's classical lemma [2, Thm. 7.5.2], Theorem 1 is the sharpest possible result of its kind. We refer the interested reader to [9] and [3], [5] for some applications of this result to existence problems in mathematical economics and optimal control theory.

Received by the editors January 3, 1986.

AMS Subject Classification (1980): 28A20.

The method of proof introduced in [3], [4] constitutes a departure from the earlier lines of approach. Thus it is a very natural question (posed to the author by Zvi Artstein) to ask how the earlier methods can be strengthened so as to yield the same result. The purpose of this note is to provide an answer to this question: we show that Artstein's version of the multidimensional Fatou lemma (cf. Theorem 2 below) implies Theorem 1, its more general counterpart, by using Chacon's biting lemma [7]:

BITING LEMMA. *Suppose that $\{g_k\}$ is a sequence in $\mathscr{L}_1(\Omega)$ such that*

$$\sup_k \int_\Omega |g_k|\,d\mu < +\infty.$$

Then there exist a function g_* in $\mathscr{L}_1(\Omega)$, a subsequence $\{k_j\}$ of $\{k\}$, and a nonincreasing sequence $\{B_p\}$ of sets in $\mathscr{F}$, $\mu(\cap_{p=1}^\infty B_p) = 0$, such that for every p

$$\lim_j \int_{\Omega\setminus B_p} g_{k_j} h\,d\mu = \int_{\Omega\setminus B_p} g_* h\,d\mu \qquad \text{for all } h \in \mathscr{L}_\infty(\Omega). \tag{5}$$

In the Appendix to this note we present an elegant proof of Chacon's biting lemma, due to W. Thomsen and D. Plachky [11, pp. 201–202], which is entirely based on the Yosida-Hewitt decomposition theorem. It is reproduced here, in the English language, with their kind permission.

As we have mentioned above, our starting point for proving Theorem 1 in this note is the following multidimensional Fatou lemma, due to Artstein [1], which generalizes a similar result of Hildenbrand-Mertens [10].

THEOREM 2. *Suppose that $\{\phi_k\}$ is a sequence in $\mathscr{L}_1^m$ such that*

$$\lim_k \int_\Omega \phi_k\,d\mu \text{ exists,}$$

$$\{\phi_k\} \text{ is uniformly integrable.}$$

Then there exists a function $\phi_* \in \mathscr{L}_1^m$ such that

$$\int_\Omega \phi_*\,d\mu = \lim_k \int_\Omega \phi_k\,d\mu,$$

$$\phi_*(\omega) \text{ is a limit point of } \{\phi_k(\omega)\} \text{ a.e. in } \Omega.$$

As we noted in [3] for the converse direction of proof, Theorem 1 immediately implies Theorem 2 by taking $f_k \equiv (\phi_k, -\phi_k)$.

2. **Proof of Theorem 1 by Theorem 2**. First we note that by (1) for every i, $1 \leqslant i \leqslant m$,

$$\sup_k \int_\Omega (f_k^+)^i\,d\mu < +\infty.$$

By m-fold application of the biting lemma, there exist a function g_* in $\mathscr{L}_1^m$, a subsequence $\{k_j\}$ of $\{k\}$, and a nonincreasing sequence $\{B_p\}$ of sets in $\mathscr{F}$, $\mu(\cap_p B_p) = 0$,

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such that for every i, $1 \leqslant i \leqslant m$, and p

$$\lim_j \int_{C_p \cap A} (f_{k_j}^+)^i \, d\mu = \int_{C_p \cap A} g_*^i \, d\mu \quad \text{for all } A \in \mathcal{F}. \tag{6}$$

Here $C_p \equiv \Omega \backslash B_p$; we shall also write $D_p \equiv C_{p+1} \backslash C_p$. In particular, it follows from (6) that for every p

$$\lim_j \int_{D_p} f_{k_j}^+ \, d\mu = \int_{D_p} g_* \, d\mu. \tag{7}$$

Also, (6) implies by the Dunford-Pettis criterion that for every p

$$\{f_{k_j}^+\} \text{ is uniformly integrable over } D_p. \tag{8}$$

By (2) the sequence $\{f_{k_j}^-\}$ is uniformly integrable (over Ω). Hence, by the Dunford-Pettis criterion there exist a function g_{**} in $\mathcal{L}_1^m$ and a subsequence $\{k_j\}$—which we may take to be $\{k_j\}$ itself without any loss of generality—such that $\{f_{k_j}^-\}$ converges weakly in the topology $\sigma(\mathcal{L}_1^m(\Omega), \mathcal{L}_\infty^m(\Omega))$ to g_{**}. *A fortiori*, for every p

$$\lim_j \int_{D_p} f_{k_j}^- \, d\mu = \int_{D_p} g_{**} \, d\mu, \tag{9}$$

$$\{f_{k_j}\} \text{ is uniformly integrable over } D_p \tag{10}$$

By (7)−(10) we can apply Theorem 2 for every p to the domain D_p and the sequence $\{\phi_j\}$, with $\phi_j \equiv (f_{k_j}^+, f_{k_j}^-)$. It thus follows that for every p there exists a function $(\bar{f}_p, \bar{\bar{f}}_p)$ in $\mathcal{L}_1^{2m}(D_p)$ such that

$$\int_{D_p} \bar{f}_p \, d\mu = \int_{D_p} g_* \, d\mu, \int_{D_p} \bar{\bar{f}}_p \, d\mu = \int_{D_p} g_{**} \, d\mu, \tag{11}$$

$$(\bar{f}_p(\omega), \bar{\bar{f}}_p(\omega)) \text{ is a limit point if } \{(f_{k_j}^+(\omega), f_{k_j}^-(\omega))\} \text{ a.e. in } D_p. \tag{12}$$

By (12) it follows that for every p

$$\bar{f}_p(\omega) \geqslant 0, \bar{\bar{f}}_p(\omega) \geqslant \text{ a.e. in } D_p. \tag{13}$$

Clearly, we thus have, by the monotone convergence theorem, for the functions $\bar{f}, \bar{\bar{f}}$, defined by setting $\bar{f}(\omega) \equiv \bar{f}_p(\omega)$ and $\bar{\bar{f}}(\omega) \equiv \bar{\bar{f}}_p(\omega)$ if $\omega \in D_p$, and $\bar{f}(\omega) = \bar{\bar{f}}(\omega) \equiv 0$ if $\omega \in \cap_p B_p = \Omega \backslash (\cap_p D_p)$, the following:

$$\int_\Omega \bar{f} \, d\mu = \sum_p \int_{D_p} \bar{f}_p \, d\mu = \int_\Omega g_* \, d\mu, \tag{14}$$

$$\int_\Omega \bar{\bar{f}} \, d\mu = \sum_p \int_{D_p} \bar{\bar{f}}_p \, d\mu = \int_\Omega g_{**} \, d\mu. \tag{15}$$

Therefore it follows that $\bar{f}, \bar{\bar{f}}$, we well as f_*, defined by $f_* \equiv \bar{f} - \bar{\bar{f}}$, belong to $\mathcal{L}_1^m$. By (12) we obviously have now

$$f_*(\omega) \text{ is a limit point of } \{f_{k_j}(\omega)\} \text{ a.e. in } \Omega,$$

and this implies (4). Also, by (5) and weak convergence of $\{f_{k_j}^-\}$ to g_{**}, established above, we have for any i, $1 \leqslant i \leqslant m$, for every p

$$\lim_k \int_\Omega f_k^i d\mu = \lim_j \left(\int_{B_p} f_{k_j}^i d\mu + \int_{C_p} f_{k_j}^i d\mu \right) \geqslant \alpha_p^i + \int_{C_p} (g_*^i - g_{**}^i) d\mu,$$

where

$$\alpha_p^i = \liminf_j \int_{B_p} f_{k_j}^i d\mu \geqslant -\limsup_j \int_{B_p} (f_{k_j}^-)^i d\mu.$$

Since we know that $\lim_p \mu(B_p) = 0$, it follows from (2) that for every $\epsilon > 0$

$$\lim_k \int_\Omega f_k^i d\mu \geqslant -\epsilon + \int_{C_p} (g_*^i - g_{**}^i) d\mu \text{ for } p \text{ large enough.}$$

Consequently, by the dominated convergence theorem it follows that

$$\lim_k \int_\Omega f_k d\mu \geqslant \int_\Omega (g_* - g_{**}) d\mu.$$

By the definition of f_* and (14)−(15) this implies (3). QED

3. **Conclusions**. Let us briefly review the relations existing between the various versions of Fatou's lemma in several dimensions that have appeared in the literature ([1], [3], [8], [10], [12]). On the one hand, Artstein's result generalizes that of Hildenbrandt-Mertens; on the other, Balder's result generalizes those of Schmeidler and Cesari-Suryanarayana. Further, as indicated in the introduction, Balder's result immediately implies Artstein's result. Conversely, as we have seen in this note, Artstein's result plus the biting lemma imply Balder's (and *a fortiori* Schmeidler's) result. To this author it is an open question how to derive his or Artstein's result from the original result given by Schmeidler.

ACKNOWLEDGEMENTS. I am indebted to Zvi Artstein for asking me the question that led to this note, and I thank D. Plachky and W. Thomsen for giving me permission to reproduce their proof of the biting lemma.

Appendix. Here we present a simple, elegant proof of Chacon's biting lemma. This proof appears in [11, pp. 201−202], and is due to W. Thomsen and D. Plachky.

Without loss of generality we may suppose that all functions g_k are nonnegative (otherwise, consider positive and negative parts separately). Let $\mathscr{F}_0$ be the σ-algebra generated by all functions of the sequence $\{g_k\}$; then $\mathscr{F}_0$ is countably generated. Hence, $\mathscr{F}_0$ is also generated by a countable algebra, say $\mathscr{A}_0 \equiv \{A_n : n \in \mathbb{N}\}$. By a standard diagonal argument there is a subsequence $\{k_j\}$ of $\{k\}$ such that for every n

$$\nu(A_n) = \lim_j \int_{A_n} g_{k_j} d\mu \text{ exists.} \tag{a1}$$

Obviously, we also know that for every μ-null set N in $\mathscr{F}_0$

$$\nu(N) \equiv \lim_j \int_N g_{k_j} d\mu = 0.$$

Let $\mathcal{A}$ be the algebra generated by $\mathcal{A}_0$ and the μ-null sets in $\mathcal{F}_0$. Clearly, the extension of ν to $\mathcal{A}$ is finitely additive. By the Yosida-Hewitt decomposition theorem [6, Thm. 10.2.1] ν can be decomposed as

$$\nu = \nu_f + \nu_{c'}$$

where ν_f is a purely finitely additive set function and ν_c a σ-additive measure on $\mathcal{A}$. Let ρ_c be the uniquely determined measure on $\mathcal{F}_0$ extending ν_c (apply Carathéodory's extension theorem [2]). Then ρ_c is evidently absolutely continuous on $\mathcal{F}_0$ with respect to μ. Let g_* be the associated Radon-Nikodym derivative; then

(a2) $$\rho_c(A) = \int_A g_* \quad \text{for all } A \in \mathcal{F}_0.$$

Since ν_f is purely finitely additive and μ is σ-additive on $\mathcal{A}$, there must exist for every p a set B'_p in $\mathcal{A}$ with

$$\nu_f(\Omega \backslash B'_p) < 2^{-p}, \ \mu(B'_p) < 2^{-p}.$$

For $B_p \equiv \bigcup_{n \geqslant p} B'_n$ it follows elementarily that the sequence $\{B_p\}$ in $\mathcal{F}_0$ is nonincreasing, with $\mu(\bigcap_p B_p) = 0$. Let ρ_f denote any finitely additive positive extension of ν_f from $\mathcal{A}$ to $\mathcal{F}_0$ (apply [6, Thm. 3.2.10]). For $C_p \equiv \Omega \backslash B_p$ we have

$$0 \leqslant \rho_f(C_p) \leqslant \rho_f(\Omega \backslash B'_n) = \nu_f(\Omega \backslash B'_n) < 2^{-n} \quad \text{for all } n \geqslant p;$$

hence $\rho_f(C_p) \equiv 0$ for every p.

Next, we define for every j the measure ρ_j on $\mathcal{F}_0$ by

$$\rho_j(A) \equiv \int_A g_{k_j} d\mu, \quad A \in \mathcal{F}_0.$$

By hypothesis, $\sup_j \rho_j(\Omega) < +\infty$; thus, by Tychonov's theorem, there exists a sub*net* $\{\rho_\gamma\}$ of $\{\rho_j\}$ such that

$$\tau(A) \equiv \lim_\gamma \rho_\gamma(A) \text{ exists for all } \mathcal{A} \in \mathcal{F}_0.$$

Clearly, the pointwise limit τ is finitely additive on $\mathcal{F}_0$, and by (a1)

(a3) $$\tau(A) = \nu(A) \quad \text{for all } A \in \mathcal{A}.$$

Applying the Yosida-Hewitt decomposition theorem once more, we find that τ can be decomposed as

$$\tau = \tau_f + \tau_c$$

into a purely finitely additive part τ_f and a σ-additive part τ_c on $\mathcal{F}_0$. When restricted to $\mathcal{A}$, τ_c must coincide with ν_c, as a consequence of (a3) and the uniqueness of the Yosida-Hewitt decomposition. By Carathéodory's theorem it follows that $\tau_c = \rho_c$ on $\mathcal{F}_0$. For the same reasons the restrictions of τ_f to $\mathcal{A}$ coincides with ν_f; hence, for every

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p we have $\tau_f(C_p) = 0$, since $0 \leqslant \tau_f(C_p) \leqslant \tau_f(\Omega \backslash B_n') = \nu_f(\Omega \backslash B_n') < 2^{-n}$ for every $n \geqslant p$. This means that for every p

$$\text{(a4)} \qquad \lim_\gamma \rho_\gamma(A \cap C_p) \equiv \tau(A \cap C_p) = \tau_c(A \cap C_p) + 0 = \rho_c(A \cap C_p) \quad \text{for all } A \in \mathcal{F}_0.$$

independently from the choice of the subnet $\{\rho_\gamma\}$. Thus, we conclude that every subnet of $\{\rho_j\}$ will have a sub-subnet $\{\rho_\gamma\}$ satisfying (a4), which means that for every p

$$\lim_j \rho_j(A \cap C_p) = \rho_c(A \cap C_p) \quad \text{for all } A \in \mathcal{F}_0.$$

By (a2) this amounts to saying that for every p

$$\lim_j \int_{A \cap C_p} g_{k_j} d\mu = \int_{A \cap C_p} g_* d\mu \quad \text{for all } A \in \mathcal{F}_0,$$

and this obviously implies that (5) holds for all $\mathcal{F}_0$-measurable h in $\mathcal{L}_\infty(\Omega)$. By taking the conditional expectation with respect to $\mathcal{F}_0$ for al h in $\mathcal{L}_\infty(\Omega)$, the proof is finished. Q.E.D.

Note added in proof:
The open question in section 3 has been answered affirmatively by Professor Czesław Olech in his paper "On n-dimensional extensions of Fatou's lemma", Preprint CRM-1355, Centre de Recherches Mathématiques, Université de Montréal, Montréal, 1986.

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