Canad. Math. Bull. Vol. **50** (4), 2007 pp. 486–503

Higher-Dimensional Modular Calabi–Yau Manifolds

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Abstract. We construct several examples of higher-dimensional Calabi–Yau manifolds and prove their modularity.

1 Introduction

As a consequence of Wiles' proof of the Taniyama–Shimura–Weil conjecture [20] there has been considerable interest in the modularity of Calabi–Yau manifolds in recent years.

The case of dimension 2 was first considered by Shioda and Inose [17] who studied $K3$ surfaces with maximal Picard number, the so-called singular $K3$ surfaces. They showed that these surfaces can be defined over number fields and computed their Hasse–Weil zeta functions. In the case of a singular $K3$ surface the transcendental lattice is 2-dimensional. If the surface is defined over $\mathbb{Q}$, then Livné [11] showed that the corresponding 2-dimensional Galois representation is related to a weight 3 modular form.

In dimension 3 rigid Calabi–Yau manifolds are simplest in the sense that they have 2-dimensional middle cohomology. By a variant of the Fontaine–Mazur conjecture [7], also asked by Yui (see [21] for a recent account), one expects that the middle cohomology of a rigid Calabi–Yau threefold defined over $\mathbb{Q}$ gives rise to an L-series which is that of a weight 4 modular form. After numerous examples by various authors were exhibited, Dieulefait and Manoharmayum [6] proved the modularity conjecture for rigid Calabi–Yau threefolds under mild conditions on the primes of bad reduction. Examples and results about non-rigid modular Calabi–Yau threefolds can be found, *e.g.*, in [8, 9]. For a very recent survey, including lists of practically all known examples, we refer the reader to the book by Meyer [13].

However, practically no examples seem to be known in higher dimension, and it is the aim of this paper to fill this gap. The first type of examples we give, arises inductively from the Kummer construction described in Proposition 2.1. The manifolds obtained in this way are resolutions of quotients of products of Calabi–Yau manifolds by a group of the form $\mathbb{Z}_2^n$. With this method one can construct several examples of modular Calabi–Yau manifolds (in any dimension). The middle cohomology (if the dimension is odd), resp. the transcendental lattice (if the dimension is

Received by the editors September 19, 2005; revised January 30, 2006.
We are grateful to the DFG for support under grant Hu 337/5-2 in the frame of the Schwerpunktprogramm SPP 1084 "Globale Methoden in der komplexen Geometrie".
AMS subject classification: Primary: 14G10; secondary: 14J32, 11G40.

even), is a tensor product of the middle cohomologies of modular Calabi–Yau manifolds of lower dimension. In some cases this tensor product (or, more precisely, its semi-simplification) splits into 2-dimensional modular pieces.

In order to obtain higher dimensional Calabi–Yau manifolds with small, *e.g.*, 2-dimensional, middle cohomology, one has to refine the Kummer construction by taking quotients with respect to bigger groups. We consider suitable actions of the groups $G = \mathbb{Z}_3^n$ or $\mathbb{Z}_4^n$ and discuss this in particular in the case of quotients of the form $(E \times \cdots \times E)/G$, where E is an elliptic curve with extra automorphisms (see §3,4). We show that these quotients have a smooth Calabi–Yau model, whose middle cohomology (if the dimension is odd), resp. the transcendental lattice (if the dimension is even), is 2-dimensional. Moreover, we show modularity and determine the corresponding cusp forms.

The final example which we discuss goes back to Ahlgren [1]. He considers a 5-dimensional affine variety X which is a double cover of 5-space branched along 12 hyperplanes and relates the number of points of $X(\mathbb{F}_p)$ to the cusp form $g_6(q) = \eta(q^2)^{12}$ of weight 6 and level 4. We prove in Theorem 5.1 that X has a smooth projective model which is a 5-dimensional Calabi–Yau manifold with $b_1 = b_3 = 0$ and $b_5 = 2$, whose L-series of the middle cohomology is that of the weight form g_6.

2 The Kummer Construction

We start by generalizing the Kummer construction, which has been used to construct Calabi–Yau threefolds as quotients of the product of a $K3$ surface with an involution and an elliptic curve modulo the diagonal involution. To begin with, let Y be a projective manifold of dimension n with $H^q(\mathcal{O}_Y) = 0$ for $q > 0$ and let $D \in |-2K_Y|$ be a smooth divisor. The line bundle $-K_Y$ defines a double covering $\pi\colon X \to Y$ branched along the divisor D, and $K_X = \pi^*(K_Y + (-K_Y)) = 0$. Moreover, since $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus K_Y$ it follows that for $0 < q < n$,

$$H^q(\mathcal{O}_X) \cong H^q(\mathcal{O}_Y) \oplus H^q(K_Y) \cong H^q(\mathcal{O}_Y) \oplus H^{n-q}(\mathcal{O}_Y) = 0,$$

and therefore the variety X is a Calabi–Yau manifold.

Now assume that we have a pair $Y_i, i = 1, 2$ of algebraic manifolds of dimension n_i, together with smooth divisors $D_i \in |-2K_{Y_i}|$. Moreover, assume that $H^q(\mathcal{O}_{Y_i}) = 0$ for $i = 1, 2$ and $q > 0$ and let X_i be the double covers described above. By construction, the product $X_1 \times X_2$ admits an action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proposition 2.1 *Under the above assumptions the quotient of the product $X_1 \times X_2$ by the diagonal involution admits a crepant resolution X, which is a (smooth) Calabi–Yau manifold. Moreover, there is a double cover $X \to Y$, branched along a smooth divisor D with $H^q(\mathcal{O}_Y) = 0$ for $q > 0$.*

Proof The resolution may be described as follows: we start with the blow-up

$$\sigma\colon Y \to Y_1 \times Y_2$$

https://doi.org/10.4153/CMB-2007-049-9 Published online by Cambridge University Press

of $Y_1 \times Y_2$ along $D_1 \times D_2$. Denote the exceptional divisor by E and let

$$D = \sigma^*(D_1 \times Y_2 + Y_1 \times D_2) - 2E$$

be the strict transform of $D_1 \times Y_2 \cup Y_1 \times D_2$. Since $D_1 \times Y_2$ and $Y_1 \times D_2$ intersect transversally along $D_1 \times D_2$, the divisor D is smooth and isomorphic to the disjoint union of $D_1 \times Y_2$ and $Y_1 \times D_2$.

Moreover

$$\begin{aligned} D &= \sigma^*(D_1 \times Y_2 + Y_1 \times D_2) - 2E \sim \sigma^*(\pi_1^*(-2K_{Y_1}) + \pi_2^*(-2K_{Y_2})) - 2E \\ &= \sigma^*(-2K_{Y_1 \times Y_2}) - 2E \sim -2K_Y. \end{aligned}$$

Since Y and $Y_1 \times Y_2$ are smooth birational projective manifolds, $H^q(\mathcal{O}_Y) \cong H^q(\mathcal{O}_{Y_1 \times Y_2}) = 0$ for $q > 0$ (by the Künneth decomposition), and hence the double cover X of Y branched along D is a Calabi–Yau manifold.

Clearly, X is birational to the quotient of $X_1 \times X_2$ by the action of the diagonal in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. More precisely, the fixed point set of the diagonal involution is the inverse image B of $D_1 \times D_2$. The quotient by the diagonal involution has transversal A_1-singularities along the image of B. Let Z be the blow-up of $X_1 \times X_2$ along B. The involution lifts to Z with fixed point set equal to the exceptional divisor $\tilde{B}$, which is ruled over B. The quotient of Z by this involution is isomorphic to X, *i.e.,* we have a commutative diagram of the form

$$\begin{array}{ccc} X_1 \times X_2/\mathbb{Z}_2 & \longleftarrow & Z/\mathbb{Z}_2 = X \\ \downarrow & & \downarrow \pi \\ Y_1 \times Y_2 & \longleftarrow & Y \end{array}$$

where the horizontal lines are inverse maps to blow-ups and the vertical lines are branched double covers. ■

The above proposition allows us to use the covering $X \to Y$ inductively, and thus to construct higher-dimensional Calabi–Yau manifolds.

The Euler characteristic of X depends not only on the Euler characteristics of X_1 and X_2, but also on the involution. By standard topological arguments we obtain

$$\begin{aligned} e(X) &= \frac{1}{2}e(X_1)e(X_2) + \frac{3}{2}e(D_1)e(D_2), \\ e(D) &= \frac{1}{2}e(X_1)e(D_2) + \frac{1}{2}e(D_1)e(X_2) + e(D_1)e(D_2). \end{aligned}$$

In the special case when Y_2 is an elliptic curve branched over 4 points in $\mathbb{P}^1$, we have $e(X) = 6e(D_1)$ and $e(D) = 2e(X_1) + 4e(D_1)$.

Example 2.1 The case where X_1 is a $K3$-surface with an involution whose quotient is rational and X_2 is an elliptic curve was studied independently by Borcea [4] and Voisin [19] in the context of mirror symmetry. Already in this case we have several possibilities leading to different Euler numbers. Namely, $e(D_1)$ is an even integer ranging from -18 (for a smooth plane sextic) to 20 (10 lines coming from the resolution of six lines in $\mathbb{P}^2$ with four triple points). If X_2 is an elliptic curve, we get Calabi–Yau threefolds with Euler numbers equal to -108, -96, -84, -72, -60, -48, -36, -24, -12, 0, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120.

The Hodge numbers of X cannot be computed in a similarly straightforward way. If we know the Hodge numbers of X_1 and X_2, we can compute the Hodge numbers of $X_1 \times X_2$. The involution will kill the skew-symmetric part of the Hodge groups and preserve the symmetric part. But we also have to take into account the contribution to the cohomology coming from the blow-up of B and describe the action of the involution on it.

Proposition 2.2 *Let $X_1, \dots, X_n$ be Calabi–Yau manifolds with involutions as above. The quotient of the product $X_1 \times \cdots \times X_n$ by the action of*

$$\{(m_1, \dots, m_n) \in \mathbb{Z}_2^n \mid m_1 + \cdots + m_n = 0\} \cong \mathbb{Z}_2^{n-1}$$

has a crepant resolution of singularities which is a Calabi–Yau manifold.

Proof We shall proceed by induction on n. The case $n = 2$ follows from Proposition 2.1. Since the resulting Calabi–Yau manifold has again an involution, we can iterate the procedure. For a sequence of Calabi–Yau manifolds X_i with involution we have the following factorization

$$(X_1 \times \cdots \times X_n)/\mathbb{Z}_2^{n-1} \cong \left((X_1 \times \cdots \times X_n)/\mathbb{Z}_2^{n-2}\right)/(\mathbb{Z}_2^{n-1}/\mathbb{Z}_2^{n-2}),$$

where $\mathbb{Z}_2^{n-2}$ denotes the group $\{(m_1, \dots, m_n) \in \mathbb{Z}_2^n | m_1 + \cdots + m_{n-1} = m_n = 0\}$. Consequently,

$$(X_1 \times \cdots \times X_n)/\mathbb{Z}_2^{n-1} \cong \left(\left((X_1 \times \cdots \times X_{n-1})/\mathbb{Z}_2^{n-2}\right) \times X_n\right)/\mathbb{Z}_2,$$

which proves the proposition. ∎

Corollary 2.3 *Let $E_i, i = 1, \dots, n$ be elliptic curves. The quotient $E_1 \times \cdots \times E_n$ by the action of $\mathbb{Z}_2^{n-1}$ has a smooth model X^n which is a Calabi–Yau manifold with Euler characteristic $e(X^n) = \frac{1}{2}(6^n + 3(-2)^n)$.*

We would like to remark that quotients of the form $(E_1 \times E_2 \times E_3)/\mathbb{Z}_2^2$ were first considered by Borcea [3], who also proved that the resulting Calabi–Yau threefolds have CM if and only if the factors E_i have CM.

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Lemma 2.4 *If n is odd, then*

$$H^n(X^n) \cong H^n(E_1 \times \cdots \times E_n)^{\mathbb{Z}_2^{n-1}} \cong H^1(E_1) \otimes \cdots \otimes H^1(E_n).$$

For n even the (invariant) submotive $H^n(E_1 \times \cdots \times E_n)^{\mathbb{Z}_2^{n-1}}$ of $H^n(X^n)$ is isomorphic to the direct sum of a submotive generated by cycles of products of $n/2$ fibres and a submotive $I(X^n) \cong H^1(E_1) \otimes \cdots \otimes H^1(E_n)$. The motive $I(X^n)$ contains the transcendental submotive, i.e., *the orthogonal complement to the algebraic cycles of X^n.*

Proof We first consider the invariant part of the middle cohomology of $E_1 \times \cdots \times E_n$. Any tensor product $\bigotimes_j H^{i_j}(E_j)$ which contributes to this must have $\sum_j i_j = n$. Now assume that at least one $i_j = 1$. Then we must have that all $i_j = 1$, since otherwise one can find some $\varepsilon \in \mathbb{Z}_2^{n-1}$ which acts by -1 on $\bigotimes_j H^{i_j}(E_j)$. If n is odd, then $\sum_j i_j = n$ can only occur if at least one, and hence, by the above argument, all $i_j = 1$. We finally remark that X^n is of the form $Z_n/\mathbb{Z}_2^{n-1}$ where Z_n arises from the product $E_1 \times \cdots \times E_n$ by blowing up rational submanifolds. This only contributes to the even cohomology, and this contribution is spanned by algebraic cycles. ■

This discussion easily implies the following.

Proposition 2.5 *Assume that the E_i are defined over $\mathbb{Q}$ with the involution given as $x \mapsto -x$, and let $L(X^n, s)$, resp. $L(I(X^n), s)$, be the L-series associated with the Galois action on $H^n(X^n)$ for n odd and the submotive $I(X^n)$ for n even. Then $L(X^n, s) = L(g_{E_1} \otimes \cdots \otimes g_{E_n}, s)$, resp. $L(I(X^n), s) = L(g_{E_1} \otimes \cdots \otimes g_{E_n}, s)$, where the g_{E_i} are the cusp forms associated with E_i.*

Proof The only statement which requires a proof is that X_n is defined over $\mathbb{Q}$. But this is clear, since the factors E_i, the involutions, and the locus which is blown up are all defined over $\mathbb{Q}$. ■

Here we consider $g_{E_1} \otimes \cdots \otimes g_{E_n}$ as the tensor product of Galois-modules. For the analytic properties of (some) tensor products see [10].

Remark 2.6 For a generic choice of elliptic curves, $I(X^n)$ equals the transcendental submotive of X^n, whereas in special cases it may be strictly bigger. For instance, if the factors E_{2i-1} and E_{2i} $(i = 1, \dots, n/2)$ are isogeneous, then $I(X^n)$ contains the product of the graphs of isogenies. If, moreover, the E_i's have complex multiplication, then $I(X^n)$ also contains the product of graphs of complex multiplications. Note that this is in agreement with the appearance of the factors $L(s - \frac{n}{2})$ and $L(\chi_{-d}, s - \frac{n}{2})$ in the L-series given below.

We now specialize the situation even further and assume that all E_i are isomorphic to an elliptic curve E with complex multiplication in $\mathbb{Q}(\sqrt{-d})$. If n is odd, then

$$L(X, s) = L(g_{n+1}, s)^{\binom{n}{0}} L(g_{n-1}, s-1)^{\binom{n}{1}} \cdots L\big(g_2, s - \tfrac{n-1}{2}\big)^{\binom{n}{(n-1)/2}},$$

https://doi.org/10.4153/CMB-2007-049-9 Published online by Cambridge University Press

and if n is even, then

$$L(I(X),s) = L(g_{n+1},s)^{\binom{n}{0}} L(g_{n-1},s-1)^{\binom{n}{1}} \cdots L\big(g_3, s-\tfrac{n-2}{2}\big)^{\binom{n}{(n-2)/2}} \times L(\chi_{-d}, s-\tfrac{n}{2})^{\frac{1}{2}\binom{n}{n/2}} L(s-\tfrac{n}{2})^{\frac{1}{2}\binom{n}{n/2}}.$$

Here $\zeta(\chi_{-d},s)$ is the Dirichlet L-function defined by the character associated with the number field $K = \mathbb{Q}(\sqrt{-d})$, *i.e.*, $\chi_{-d}(p) = (\frac{-d}{p})$, and g_k is the cusp form corresponding to the $(k-1)$-st power of the Grössencharakter ψ of the elliptic curve E [15]. The cusp form g_k has weight k and complex multiplication in the same field as E. The Fourier coefficient $a_n(g_k)$ is given by the sum of the values of the Grössencharakter ψ^{k-1} at the ideals in the ring $\mathcal{O}_K$ of integers in K of norm n, relatively prime to the conductor of E. For a prime p which is inert in $\mathcal{O}_K$, we get $a_p = 0$, because there is no ideal in $\mathcal{O}_K$ with norm p. For a split prime p we have $p = \alpha_p\bar{\alpha}_p$ for some $\alpha_p \in \mathcal{O}_K$, which is determined by E. Then $a_p(g_k) = \alpha_p^{k-1} + \bar{\alpha}_p^{k-1}$, more explicitly, we have $a_p(g_3) = a_p^2 - 2p$, $a_p(g_4) = a_p^3 - 3pa_p$, $a_p(g_5) = a_p^4 - 4pa_p^2 + 2p^2$, and so on.

In terms of the associated Galois representations, the connection between the forms g_k and g_2 can be described as follows. Consider the representation associated with g_2 and let $(\alpha_p, \bar{\alpha}_p)$ be the eigenvalues of Frob_p for primes p with $\chi_{-d}(p) = 1$. If $\chi_{-d}(p) = -1$, then the corresponding eigenvalues are $(ip^{1/2}, -ip^{1/2})$. The eigenvalues of g_k are then $({\alpha_p}^{k-1}, \bar{\alpha}_p^{k-1})$ for $\chi_{-d}(p) = 1$ and $(p^{(k-1)/2}, -p^{(k-1)/2})$ for k odd and $\chi_{-d}(p) = -1$, resp. $(ip^{(k-1)/2}, -ip^{(k-1)/2})$ for k even and $\chi_{-d}(p) = -1$.

We want to conclude this section by discussing one further example of our Kummer construction. As the first factor we choose the rigid Calabi–Yau threefold X_3, constructed as a resolution of singularities of the double covering of $\mathbb{P}^3$ branched along the following arrangement of eight planes

$$xt(x-z-t)(x-z+t)y(y+z-t)(y+z+t)(y+2z) = 0.$$

For a discussion of the properties of this (and other) double octics see [5] and [13, Octic Arr. No. 19]. As the second factor we take the $K3$ surface S which is obtained as a desingularization of the double sextic branched along the following arrangement of six lines $xy(x+y+z)(x+y-z)(x-y+z)(x-y-z) = 0$. Both X_3 and S come with natural involutions which allow us to apply Proposition 2.1. In this way we obtain a smooth Calabi–Yau fivefold X_5, which is the quotient of a blow-up $\widetilde{X_3 \times S}$ of $X_3 \times S$ by an involution. So the Hodge groups of X_5 are the invariant part of the Hodge groups of $\widetilde{X_3 \times S}$. Since we blow up products of lines and blown up planes, the odd-dimensional cohomology groups of $\widetilde{X_3 \times S}$ and $X_3 \times S$ are the same.

Now, the odd-dimensional cohomology groups of S vanish, whereas the only odd-dimensional cohomology of X_3 is $H^3(X_3) = H^{3,0} \oplus H^{0,3}$, which is anti-invariant. The anti–invariant part of the cohomology of S is $H^{2,0} \oplus H^{0,2} \cong T(S) \otimes_{\mathbb{Z}} \mathbb{C}$, where $T(S)$ is the transcendental lattice. Consequently, $b_1(X_5) = b_3(X_5) = 0$ and $b_5(X_4) = 4$, and moreover $H^5(X_5) \cong H^3(X_3) \otimes T(S)$. Recall that (see [2] and [13, p. 57])

$$L(T(S),s) \triangleq L(g_3,s), \quad L(X_3,s) \triangleq L(g_4,s)$$

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where g_3 and g_4 are the unique weight 3, resp. weight 4, Hecke eigenforms of level 16 and 32 with complex multiplication by i. As usual, $\stackrel{\circ}{=}$ denotes equality up to a finite number of Euler factors. In concrete terms

$$g_3(q) = \eta(q^4)^6 = q - 6q^5 + 9q^9 + 10q^{13} - 30q^{17} + \cdots,$$
$$g_4(q) = q + 22q^5 - 27q^9 - 18q^{13} - 94q^{17} + 359q^{25} + \cdots,$$

where $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind η-function. Both of these forms can be derived from the unique weight 2 level 32 newform

$$g_2(q) = \eta(q^8)^2\eta(q^4)^2 = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} + \cdots$$

by taking the second, resp. third, power of the Grössencharakter of $\mathbb{Q}[i]$ given as $\psi((\alpha)) = \alpha$ for $\alpha \in \mathbb{Z}[i]$, $\alpha \equiv 1 \mod 2+2i$.

Hence we obtain that the L-series of X_5 is the product of the L-series associated with X_3 and S, and we also find that it factors as

$$L(X_5, s) \stackrel{\circ}{=} L(g_4 \otimes g_3, s) \stackrel{\circ}{=} L(g_6, s)L(g_2, s-2),$$

where g_2 is as above and g_6 is a level 32 cusp form of weight 6, namely

$$g_6(q) = q - 82q^5 - 243q^9 - 1194q^{13} + 2242q^{17} + 3599q^{25} + \cdots$$

which can be derived from g_2 by taking the fifth power of the Grössencharakter. Obviously, we can iterate this procedure to obtain modular Calabi–Yau manifolds of higher dimension (with increasingly complex middle cohomology).

3 Calabi–Yau Manifolds with an Endomorphism of Order 3

We shall construct for any positive integer n a Calabi–Yau n-fold X_n with an endomorphism of order 3 such that $\dim H^n(X_n) = 2$ for n odd and $\dim T(X_n) = 2$ for n even, where $T(X_n) \subset H^n(X_n)$ is the transcendental part. Moreover, we shall show that the (semi-simplifications of) the Galois representation on $H^n(X_n)$ (resp. $T(X_n)$) and the Galois representation associated with a suitable cusp form with CM by $\sqrt{-3}$ are isomorphic.

Fix the primitive third root of unity $\zeta = e^{2\pi i/3}$. Let X_1 and X_2 be two Calabi–Yau manifolds admitting $\mathbb{Z}_3$-actions which do not preserve the canonical form. Moreover, assume that the fixed point set of the action on X_1 is a smooth divisor, whereas on X_2 it is a disjoint union of a smooth divisor and a smooth codimension two submanifold. Fix an automorphism η_1 of X_1 such that $\eta_1^*\omega_{X_1} = \zeta\omega_{X_1}$ and an automorphism η_2 of X_2 such that $\eta_2^*\omega_{X_2} = \zeta^2\omega_{X_2}$ such that they act on X_1 and X_2 as described above. Then η_1 is given locally near the branch-divisor on X_1 as $(\zeta, 1, 1\dots)$, whereas η_2 is given locally either as $(\zeta^2, 1, 1\dots)$ near the branch divisor on X_2 or as $(\zeta, \zeta, 1\dots)$ near the codimension 2 fixed locus.

On $X_1 \times X_2$ we have an action of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, and we consider the action of $\mathbb{Z}_3$ on $X_1 \times X_2$, given by the automorphism $\eta = \eta_1 \times \eta_2$.

Proposition 3.1 *Under the above assumptions, the quotient variety $X_1 \times X_2/\mathbb{Z}_3$ has a resolution of singularities X which is a Calabi–Yau manifold. The manifold X admits a $\mathbb{Z}_3$-action which satisfies the same assumptions as for X_2.*

Proof The singularities of $X_1 \times X_2/\mathbb{Z}_3$ correspond to the fixed locus of η, which is the cartesian product of the fixed point sets of η_1 and η_2. Consequently, we get two kinds of singularities: a singular codimension two stratum W_1, which is a transversal A_2-singularity, and a codimension three stratum W_2, which is a transversal cone over a triple Veronese surface. Both types of singularities admit a crepant resolution (described explicitly below), and we denote the resulting manifold by X. Since the canonical form on $X_1 \times X_2$ is η-invariant, it descends to the quotient and thus to the crepant resolution. Consequently, we get $\omega_X \cong \mathcal{O}_X$.

Denote by W_1 (resp. W_2) the union of the codimension two (resp. three) strata of the fixed point set of η and consider the blow-up Z_1 of $X_1 \times X_2$ along $W_1 \cup W_2$. Then η lifts to Z_1, and the fixed point set is a codimension two subvariety lying over W_1 and a divisor over W_2. Let Z_2 be the blow-up of Z_1 along the codimension two fixed submanifold. Again, the action of $\mathbb{Z}_3$ lifts to Z_2, and the fixed point set is a divisor. So the quotient Z of Z_2 by the action of $\mathbb{Z}_3$ is a smooth manifold, and it is a blow-up of X. (In terms of the A_2-singularity, the difference between Z and X is that we blow up the point of intersection of the two (-2)-curves which come from the resolution of the A_2-singularity.) Now observe that $H^0(Z_2, \Omega^q_{Z_2}) = H^0(X_1 \times X_2, \Omega^q_{X_1\times X_2}) = 0$ for $q \neq 0, n_1, n_2, n_1 + n_2$, and hence, by taking the invariant part with respect to the action of η, we obtain that $H^0(Z, \Omega^q_Z) = H^0(X, \Omega^q_X) = 0$ for $q \neq 0, n_1 + n_2$. This proves that X is a (smooth) Calabi–Yau manifold.

The action of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ on $X_1 \times X_2$ induces an action of $\mathbb{Z}_3$ on X generated by the induced action of $\mathrm{id} \times \eta_2$ to X. We shall study this action in local coordinates. For the transversal A_2-singularity we can find local coordinates on $X_1 \times X_2$ in which the action is given as (ζ, ζ^2). Note that for simplicity we shall omit the coordinates on which η acts trivially. The quotient map is given by $(x_1, x_2) \mapsto (u_1, u_2, u_3) = (x_1^3, x_2^3, x_1x_2)$, and the image has equation $u_3^3 = u_1u_2$. The resolution of singularities is given by blowing up the submanifold $u_1 = u_2 = u_3 = 0$. In suitable charts on the blown up surface, the quotient map is then given by

$$(y_1, y_2) \mapsto (y_1, y_1^2y_2^3, y_1y_2),$$
$$(y_1, y_2) \mapsto (y_1^3y_2^2, y_2, y_1y_2),$$
$$(y_1, y_2) \mapsto (y_1^2y_2, y_1y_2^2, y_1y_2).$$

The map from $X_1 \times X_2$ to the resolution of the quotient in local analytic terms is given by

$$\left(x_1^3, \frac{x_2}{x_1^2}\right), \left(\frac{x_1}{x_2^2}, x_2^3\right), \left(\frac{x_1^2}{x_2}, \frac{x_2^2}{x_1}\right),$$

depending on the charts we work in. The action of $\mathrm{id} \times \eta_2$ is given on $X_1 \times X_2$ as $(1, \zeta^2)$, so it lifts to X as $(1, \zeta^2)$, $(\zeta^2, 1)$ or (ζ, ζ) respectively, depending on the affine chart we consider.

For the cone over the Veronese triple embedding, the resolution is given by the so-called canonical resolution [18, (16.10, p. 199)]. If the action on $X_1 \times X_2$ is given in local coordinates as (ζ, ζ, ζ), then the map inverse to the resolution of the quotient is given as

$$\left(x_1^3, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right), \quad \left(\frac{x_1}{x_2}, x_2^3, \frac{x_3}{x_2}\right), \quad \text{or} \quad \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, x_3^3\right).$$

The action of $\mathrm{id} \times \eta_2$ is given on $X_1 \times X_2$ as $(1, \zeta, \zeta)$, so it lifts to X as $(1, \zeta, \zeta)$, $(\zeta^2, 1, 1)$, and $(\zeta^2, 1, 1)$, respectively.

In all cases X satisfies the assumptions made for X_2. ∎

Remark 3.2 We can now use the Calabi–Yau manifold X with the $\mathbb{Z}_3$-action on it to repeat this construction inductively.

We consider an elliptic curve defined over $\mathbb{Q}$ with an automorphism of order 3, which we can, without loss of generality, assume to be in Weierstrass form $y^2 = x^3 - D$. The automorphism η is given by $x \mapsto \zeta x$.

Theorem 3.3 *Let E be the elliptic curve with an automorphism η of order* 3*, and let $\bar{X}_n$ be the quotient of E^n by the action of the group*

$$\{(\eta^{a_1} \times \cdots \times \eta^{a_n}) \in \mathrm{End}(E^n) : a_1 + \cdots + a_n \equiv 0 \mod 3\}.$$

Then $\bar{X}_n$ has a smooth model X_n, which is a Calabi–Yau manifold, and $\dim(H^n(X_n)) =$ 2 *if n is odd, resp.* $\dim(T(X_n)) = 2$ *if n is even, where $T(X_n)$ is the transcendental part of the cohomology.*

Moreover, X_n is defined over $\mathbb{Q}$ and $L(H^n(X_n), s) \mathrel{\hat{=}} L(g_{n+1}, s)$, resp. $L(T(X_n)) \mathrel{\hat{=}} L(g_{n+1}, s)$, where g_{n+1} is the weight $n+1$ cusp form with complex multiplication in $\mathbb{Q}(\sqrt{-3})$, associated with the n-th power of the Grössencharakter of E.

Proof The claim about X_n being a Calabi–Yau manifold follows by repeated application of Proposition 3.1. To compute the middle cohomology, resp. its transcendental part, we first notice that it is enough to compute the invariant part of the cohomology of E^n. This follows since the divisors which we introduce by blowing up are linear spaces blown up in some subspaces, so their cohomology is generated by algebraic cycles. The subspace $\bigotimes H^{10}(E) \oplus \bigotimes H^{01}(E)$ is always invariant. If n is odd, then, by an argument similar to the one we used in the proof of Proposition 2.4, this is the only contribution to the invariant part of $H^n(E^n)$. If n is even, we have, in addition, summands of the form $H^{i_1}(E) \otimes \cdots \otimes H^{i_n}(E)$, where $i_k = 0$ or 2 and $\sum i_k = n$, which are also generated by algebraic cycles.

Now we turn to the arithmetic statements. We first note that $\bar{X}_n$ is defined over $\mathbb{Q}$, since it is defined over $\mathbb{Q}(\sqrt{-3})$ and invariant under the Galois group. Since we blow up in submanifolds defined over $\mathbb{Q}$, the resolution X_n is also defined over $\mathbb{Q}$.

The endomorphism η induces endomorphisms $\eta_p \colon E(\bar{\mathbb{F}}_p) \to E(\bar{\mathbb{F}}_p)$ (for $p \neq 3$, $p \nmid D$), also of order 3. The induced endomorphisms η_p have three fixed points, and hence the Lefschetz fixed point formula implies $\operatorname{tr} \eta_p^* = -1$. Now, if $l \equiv 1 \mod 6$, then $\mathbb{Q}_l$ contains a primitive root of unity ρ_l, and the eigenvalues of η_p^* are

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powers of ρ_l which sum up to -1 and are therefore equal to ρ_l and ρ_l^2. Denote by $v_1, v_2 \in H^1_{\text{ét}}(E_p)$ the corresponding eigenvectors. It is easy to see that the subspace of $H^1_{\text{ét}}(E_p)^{\otimes n}$ invariant under the action of $\mathbb{Z}_3^n$ is generated by $v_1^{\otimes n} = v_1 \otimes \cdots \otimes v_1$ and $v_2^{\otimes n} = v_2 \otimes \cdots \otimes v_2$, so we need to compute the images of the tensor power of Frobenius on $v_1^{\otimes n}$ and $v_2^{\otimes n}$. To this end, we shall need to compute the action of Frobenius Frob_p^* in the base v_1, v_2.

We shall consider the cases $p \equiv 1, 5 \mod 6$ separately. For $p \equiv 1 \mod 6$ the Frobenius map Frob_p^* commutes with η_p^*, so it acts as $v_1 \mapsto \alpha_p v_1$ and $v_2 \mapsto \bar{\alpha}_p v_2$, where α_p and $\bar{\alpha}_p$ are the eigenvalues of Frob_p^*. Consequently, the eigenvalues of Frobenius on the invariant part of $H^1_{\text{ét}}(E_p))^{\otimes n}$ equal α_p^n and $\bar{\alpha}_p^n$.

For $p \equiv 5 \mod 6$ we have $\mathrm{Frob}_p^* \circ \eta_p^* = (\eta_p^*)^{-1} \circ \mathrm{Frob}_p^*$, which easily implies that in the base v_1, v_2 Frobenius is given by the matrix $\begin{pmatrix} 0 & \lambda \\ -\frac{p}{\lambda} & 0 \end{pmatrix}$. Consequently, the action of Frobenius on the invariant subspace of $H^1_{\text{ét}}(E_p)^{\otimes n}$ equals $\begin{pmatrix} 0 & \lambda^n \\ (-\frac{p}{\lambda})^n & 0 \end{pmatrix}$, with eigenvalues equal to $\pm p^{n/2}$ for n even, and $\pm i p^{n/2}$ for n odd.

Taking all the cases together, we see that the Galois representation on $H^n(X^n)$ for n odd, resp. $T(X^n)$ for n even, has the same eigenvalues as the representation associated with the cusp form g_{n+1} associated with the n-th power of the Grössencharakter of the elliptic curve E. ∎

Remark 3.4 In the case where E is given by the equation $y^2 = x^3 - 1/4$, the form g_2 is the unique weight 2 newform of level 27, namely

$$g_2(q) = \eta(q^9)^2\eta(q^3)^2 = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots .$$

In this case

$$g_3(q) = q + 4q^4 - 13q^7 - q^{13} + 16q^{16} + 11q^{19} + 25q^{25} + \cdots ,$$
$$g_4(q) = \eta(q^3)^8 = q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} + \cdots ,$$

which are the unique level 27 and 9 forms of weight 3 and 4. The cusp forms g_k correspond to powers of the Grössencharakter of the field $\mathbb{Q}(\sqrt{-3})$ given by $\psi((\alpha)) = \alpha$ for $\alpha \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, $\alpha \equiv 1 \mod 3$. For other models of E one obtains appropriate twists of these forms.

4 Calabi–Yau Manifolds with an Endomorphism of Order 4

In this section we shall construct a similar example as in the previous section, but with an endomorphism of order 4. Let X_1 and X_2 be two Calabi–Yau manifolds admitting $\mathbb{Z}_4$-actions η_1 and η_2. Assume that the fixed point set of η_1 is a divisor, near which the action has a linearization of the form $(1, i)$, whereas the fixed point set of η_2 is a disjoint union of submanifolds of codimension one, two, or three, near which the action has a linearization as $(-i, 1, \dots)$, $(-1, i, 1, \dots)$, and $(i, i, i, 1 \dots)$, respectively.

On $X_1 \times X_2$ we have an action of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, and we consider the action of $\mathbb{Z}_4$ on $X_1 \times X_2$ given by the automorphism $\eta = \eta_1 \times \eta_2$.

Proposition 4.1 *Under the above assumptions, the quotient $X_1 \times X_2/\mathbb{Z}_4$ has a resolution of singularities X, which is a Calabi–Yau manifold. The manifold X admits a $\mathbb{Z}_4$-action which satisfies the same assumptions as for X_2.*

Proof We shall show that the quotient admits a crepant resolution of singularities. We shall consider separately the three cases depending on the codimension of the component of the fix-point set of η_2.

Near the fixed divisor of η_2 the action of η on $X_1 \times X_2$ is locally given by $(i, -i)$. (As in previous similar proofs, we omit the variables on which η acts trivially). Consequently, the quotient is a transversal A_3 singularity along the singular subvariety, which can be resolved by blowing up twice. In local coordinates, the map from $X_1 \times X_2$ inverse to the resolution is given in affine charts as

$$\Big(x^4, \frac{y}{x^3}\Big), \quad \Big(y^4, \frac{x}{y^3}\Big), \quad \Big(\frac{x^3}{y}, \frac{y^2}{x^2}\Big), \quad \Big(\frac{y^3}{x}, \frac{x^2}{y^2}\Big), \quad \text{or} \quad \Big(\frac{x^2}{y^2}, xy\Big).$$

The action of $\mathrm{id} \times \eta_2$ on $X_1 \times X_2$ has a linearization $(1, -i)$, so it lifts to the resolution as $(1, -i)$, $(1, -i)$, $(i, -1)$, $(i, -1)$, or $(-1, -i)$. In all the cases except the last one, the action is exactly as we assume for X_2, but in the last case the fixed point of the action is $(0, 0)$, which does not belong to the domain of the map.

Now consider the singularity corresponding to a codimension two fixed stratum of η_2. Then the action on $X_1 \times X_2$ has a local linearization of the form $(i, i, -1)$. We first divide by the square of η, which is an involution with fixed point set of codimension two resulting in transversal A_1-singularities. These we resolve by blowing-up the singular locus. The action of $\mathbb{Z}_4$ lifts to this resolution again as an involution with a codimension two fixed point set, leading once more to transversal A_1-singularities, which we resolve with a single blow-up. Simple computations show that in terms of local coordinates, the map from $X_1 \times X_2$ inverse to the resolution looks in local coordinates like

$$\Big(x^4, \frac{z}{x^2}, \frac{y}{x}\Big), \quad \Big(z^2, \frac{x^2}{z}, \frac{y}{x}\Big), \quad \Big(\frac{x^2}{z}, x^2 z, \frac{y}{x}\Big), \quad \Big(\frac{x}{y}, x^2 y^2, \frac{z}{xy}\Big), \quad \Big(\frac{x}{y}, z^2, \frac{xy}{z}\Big),$$
$$\Big(\frac{x}{y}, \frac{xy}{z}, xyz\Big), \quad \Big(y^4, \frac{z}{y^2}, \frac{x}{y}\Big), \quad \Big(z^2, \frac{y^2}{z}, \frac{x}{y}\Big), \quad \text{or} \quad \Big(\frac{y^2}{z}, y^2 z, \frac{x}{y}\Big).$$

The action of $\mathrm{id} \times \eta_2$ on $X_1 \times X_2$ is linearized by $(1, i, -1)$, so it lifts to the resolution as $(1, -1, i)$, $(1, -1, i)$, $(-1, -1, i)$, $(-i, -1, i)$, $(-i, 1, -i)$, $(-i, -i, -i)$, $(1, 1, -i)$, $(1, 1, -i)$, or $(1, 1, -i)$. The lifting satisfies the assumption made for X_2 in all except cases 3, 4, 5, and 6, when the fixed points do not lie in the domain of the map.

The last case is the fixed point stratum of η_2 of codimension 3, so the action on $X_1 \times X_2$ has a local linearization of the form (i, i, i, i). Here again, it is easier to resolve in one step. On the quotient we get a transversal cone over the Veronese fourfold embeding of $\mathbb{P}^3$. The crepant resolution is given by the so-called canonical resolution [18, (16.16, p. 199)] for which the inverse map is given as

$$\Big(x^4, \frac{y}{x}, \frac{z}{x}, \frac{t}{x}\Big), \Big(\frac{x}{y}, y^4, \frac{z}{y}, \frac{t}{y}\Big), \Big(\frac{x}{z}, \frac{y}{z}, z^4, \frac{t}{z}\Big), \Big(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, t^4\Big),$$

and so the action of id $\times\eta_2$ lifts as $(1,i,i,i)$, $(-i,1,1,1)$, $(-i,1,1,1)$, and $(-i,1,1,1)$ respectively, which completes the proof. ■

To produce an explicit example, consider the elliptic curve E given by the Weierstrass equation $y^2 = x^3 - Dx$, where D is a square-free integer. This curve has complex multiplication in the field $\mathbb{Q}[i]$, and the map $\rho\colon (x,y) \mapsto (-x, iy)$ is an endomorphism of E of order 4.

Theorem 4.2 *Let $\bar{X}_n$ be the quotient of E^n by the action of the group*

$$\{(\eta^{a_1} \times \cdots \times \eta^{a_n}) \in \operatorname{End}(E^n) : a_1 + \cdots + a_n \equiv 0 \mod 4\}.$$

Then $\bar{X}_n$ has a smooth model X_n, which is a Calabi–Yau manifold, and $\dim(H^n(X_n)) = 2$ *if n is odd, resp.* $\dim(T(X_n)) = 2$ *if n is even, where $T(X_n)$ is the transcendental part of the cohomology.*

Moreover, X_n is defined over $\mathbb{Q}$, and $L(H^n(X_n), s) \mathrel{\mathring{=}} L(g_{n+1}, s)$, resp. $L(T(X_n)) \mathrel{\mathring{=}} L(g_{n+1}, s)$, where g_{n+1} is a weight $n+1$ cusp form with complex multiplication in $\mathbb{Q}(i)$.

Proof The existence of a crepant resolution follows from repeated application of Proposition 4.1, and the remaining statements can be proved exactly in the same way as in the proof of Theorem 3.3. ■

5 The Example of Ahlgren

Let $\bar{X}$ be the double cover of $\mathbb{P}^5$ branched along the union of the twelve hyperplanes

$$x(x-u)(x-v)y(y-u)(y-v)z(z-u)(z-v)t(t-u)(t-v) = 0.$$

This is a projective closure of the fivefold studied by Ahlgren [1]. He proved that the number of points defined over $\mathbb{F}_p$ on the affine part ($u = 1$) of this variety equals

$$N(p) = p^5 + 2p^3 - 4p^2 - 9p - 1 - a_p,$$

where a_p is the p-th Fourier coefficient of the unique normalized weight 6 and level 4 cusp form (which is equal to $\eta^{12}(q^2)$).

Our goal here is to prove the following.

Theorem 5.1 *The variety $\bar{X}$ has a smooth model X (defined over $\mathbb{Q}$), which is a Calabi–Yau fivefold with Betti numbers $b_1(X) = b_3(X) = 0$, $b_5(X) = 2$. More precisely, $h^{50} = h^{05} = 1$, $h^{14} = h^{23} = h^{32} = h^{41} = 0$. The (semi-simplifications of the) Galois representation of the action of Frobenius on $H^5(X)$ and the Galois representation corresponding to the unique normalized cusp form of level* 4 *and weight* 6 *(which is $\eta^{12}(q^2)$) are isomorphic.*

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Type	dim	mult	#	N_1	N_2	N_3	N_4	N_5	N_6
T_1	3	2	66	0	0	0	0	0	0
T_2	2	3	148	3	0	0	0	0	0
T_3	2	4	18	6	0	0	0	0	0
T_4	1	4	117	6	4	0	0	0	0
T_5	1	5	36	10	6	1	0	0	0
T_6	1	6	18	15	8	3	0	0	0
T_7	0	5	12	10	10	0	5	0	0
T_8	0	6	18	15	16	1	6	2	0
T_9	0	7	12	21	23	3	8	3	1
T_{10}	0	8	3	28	32	6	16	0	4
T_{11}	0	9	4	36	21	9	9	9	6

Table 1

Before we can give the proof we need some preparations.

The variety $\bar{X}$ is a double cover of a degree twelve arrangement, in the sense of Definition 5.4 (see §5.1 where we collect the necessary statements). In Proposition 5.6 we describe a procedure to resolve singularities of such a double cover, and our goal here is to check that the arrangement satisfies the assumptions of that proposition.

We shall distinguish the singularities by their multiplicity and dimension and denote the resulting classes by T_k. Let N_k be the number of singularities of type T_k that contain a given singularity. Then the situation can be summed up by Table 1. We see that $T_2, T_4, T_5, T_7, T_8, T_9$ are near pencil, whereas T_1, T_3, T_6, T_{10}, and T_{11} satisfy $\lfloor \frac{m(C)}{2} \rfloor = n - d(C) - 1$. Hence $\bar{X}$ has a crepant resolution of singularities X, which is a smooth Calabi–Yau variety.

Studying the singularities in the above table, we see that the only prime of bad reduction is 2 (due to taking the double cover). The exterior powers of the matrix of coefficients of the arrangement of hyperplanes have coefficients equal to $0, \pm 1$, so the reduction modulo an odd prime has the same number and type of singularities as in characteristic 0. Consequently, the same blow-ups as in characteristic 0 give a resolution of singularities.

To prove modularity of X, we study the number of points of X_p in $\mathbb{F}_p$. In principle, it should be possible to give an explicit formula, as was done in the analogous situation in dimension 3. However, in this case there are many more different types of singularities, and so the computations would be very long and tedious. For our purpose it is enough to have the following information on the "shape" of that number.

Proposition 5.2 *For any odd prime p we have*

$$\#X(\mathbb{F}_p) = 1 + \sum_{i=1}^{4} \sum_{j=1}^{b_{2i}} \left(\frac{a_{i,j}}{p} \right) p^i + p^5 - a_p,$$

where the $a_{i,j}$ are square-free non-zero integers.

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Sketch of the Proof of Proposition 5.2 Using Ahlgren's result we only have to take into account the effect of adding the hypersurface at infinity and of all blow-ups. All these varieties are resolutions of certain double covers branched along divisors of small degrees. Using the projection formula for finite maps, it is not difficult to show that the Hodge spaces contributing to the odd cohomology groups are all zero. The even cohomology groups are spanned by algebraic cycles, which project (under the double covering) onto cycles defined over $\mathbb{Q}$. Consequently, the even cohomology groups can be generated by cycles which are either defined over $\mathbb{Q}$ or over some quadratic extension. Fix a non-invariant irreducible algebraic subvariety Z, and denote by Z' its image under the involution defined by the double cover. Clearly $Z+Z'$ is defined over $\mathbb{Q}$. Assume that Z (and hence also Z') is defined over a quadratic extension $\mathbb{Q}(\sqrt{a})$. Recall that p is an odd prime, and hence a prime of good reduction. Over $\bar{\mathbb{F}}_p$ the sum $(Z+Z')_p$ splits into a sum of two cycles Z_p and Z'_p. Frobenius maps the class Z_p to the class of $p^i Z_p$ or $p^i Z'_p$ ($i = 5 - \dim Z$) depending on whether a is a square in $\mathbb{F}_p$ or not. Consequently the class of the cycle $Z_p - Z'_p$ is an eigenvector with eigenvalue $(\frac{a}{p})p^i$, where $(\frac{a}{p})$ is the Legendre symbol. Using the Lefschetz fixed-point formula we obtain the proposition. ∎

Proof of Theorem 5.1 For every $1 \leq i \leq 4$ and $1 \leq j \leq b_{2i}$ we consider the one-dimensional Galois representation $\rho_{i,j}$ with eigenvalues $(\frac{a_{i,j}}{p})p^i$ and define $\tilde{\rho}$ to be the direct sum of all $\rho_{i,j}$. So $\tilde{\rho}$ is the Galois representation associated with the algebraic cycles. Let $\bar{\rho}_i$ be the Galois action on the i-th cohomology, and denote by $\bar{\rho}$ the direct sum of $\bar{\rho}_{2i}$, $i = 1, \dots, 4$. Finally, denote by ρ the Galois representation associated with the unique cusp form of level 4 and weight 6. By Proposition 5.2, we can write the number of points of $X(\mathbb{F}_p)$ as $1 + p^5 + \operatorname{tr}(\tilde{\rho}_p) - \operatorname{tr}\rho_p$. By the Lefschetz fixed point formula this is equal to

$$1 + p^5 + \operatorname{tr}(\bar{\rho}_p) - \operatorname{tr}(\rho_{1,p}) - \operatorname{tr}(\rho_{3,p}) - \operatorname{tr}(\rho_{5,p}) - \operatorname{tr}(\rho_{7,p}) - \operatorname{tr}(\rho_{9,p}).$$

Comparing the above two formulas and clearing the signs, we get

$$\operatorname{tr}(\bar{\rho}_p) + \operatorname{tr}(\rho_p) = \operatorname{tr}(\tilde{\rho}_p) + \operatorname{tr}(\rho_{1,p}) + \operatorname{tr}(\rho_{3,p}) + \operatorname{tr}(\rho_{5,p}) + \operatorname{tr}(\rho_{7,p}) + \operatorname{tr}(\rho_{9,p}).$$

So the representations $\bar{\rho} \oplus \rho$ and $\tilde{\rho} \oplus \rho_1 \oplus \rho_3 \oplus \rho_5 \oplus \rho_7 \oplus \rho_9$ have equal traces for any odd prime, and consequently they have isomorphic semi-simplifications (see [16, Lemma, p. I-11]). Semi-simplification preserves the eigenvalues. By construction and the Weil conjectures, the representation $\bar{\rho} \oplus \rho$ has no eigenvalue with absolute value equal to $p^{1/2}$ or $p^{3/2}$, and only two eigenvalues with absolute value equal equal to $p^{5/2}$. So $H^1(X) = H^3(X) = H^7(X) = H^9(X) = 0$, and the Galois representations ρ and ρ_5 have equal eigenvalues and hence isomorphic semi-simplifications. ∎

Remark 5.3 The Ahlgren variety is birational to the quotient of the fourfold fiber product of the Legendre family, resp. the extremal rational elliptic surface with three singular fibers of Kodaira types I_2, I_2, I_2^* (which in [14] is denoted by X_{222}) by the group $\mathbb{Z}_2^3$. In each fiber this is the construction described in Section 2 so it is fibered by Calabi–Yau fourfolds.

5.1 Resolution of Singularities of Double Arrangements

In this subsection we shall describe in detail the procedure which we use to resolve the singularities of Ahlgren's fivefold. Let Y be an n-dimensional smooth projective manifold.

Definition 5.4 A sum $D = \bigcup_{i=1}^{N} D_i$ of smooth hypersurfaces D_i in Y is called an *arrangement* if for each subset $\{i_i, \dots, i_r\} \subset \{1, \dots, N\}$ the (ideal-theoretic) intersection $C_{i_1,\dots,i_r} = D_{i_1} \cap \dots \cap D_{i_r}$ is smooth.

The following lemma is obvious from the definitions.

Lemma 5.5 *Let $D = D_1 \cup \dots \cup D_N \subset Y$ be an arrangement. Then*

(i) *If $\dim(D_{i_1} \cap \dots \cap D_{i_r}) = n - r$ for some $\{i_i, \dots, i_r\} \subset \{1, \dots, N\}$, then $D_{i_1} \dots D_{i_r}$ intersect transversally.*

(ii) *For any $\{i_i, \dots, i_r\} \subset \{1, \dots, N\}$ the tangent space to the intersection $D_{i_1} \cap \dots \cap D_{i_r}$ (at any point) equals the intersection of the tangent spaces to the divisors D_i.*

We now consider the decomposition of the singular locus of D by multiplicities. For this we take the set $\mathcal{S}$ of all components C of intersections $D_{i_1} \cap \dots \cap D_{i_r}$ where $r \geq 2$ and $\{i_i, \dots, i_r\} \subset \{1, \dots, N\}$. To each element $C \in \mathcal{S}$ we assign its multiplicity $m(C) = \operatorname{mult}_C D = \#\{i : C \subset D_i\}$ and dimension $d(C) = \dim C$. An element $C \in \mathcal{S}$ will be called *near-pencil* if it is contained in an element $C' \in \mathcal{S}$ with $d(C) = d(C') - 1$ and $m(C) = m(C') + 1$ (*i.e.*, C is cut out from C' by a single hypersurface).

If the arrangement $D \subset Y$ is even (as an element of the Picard group $\operatorname{Pic}(Y)$), then there exists a double cover $\pi \colon X \to Y$ of Y branched along D. Such a double cover is uniquely determined by fixing a line bundle $\mathcal{L}$ on Y with $\mathcal{O}(D) \cong \mathcal{L}^{\otimes 2}$.

Proposition 5.6 *Assume that for every variety $C \in \mathcal{S}$ either C is near-pencil or $\lfloor \frac{m(C)}{2} \rfloor = n - d(C) - 1$. Then X admits a projective crepant resolution of singularities.*

Proof Let $C \in \mathcal{S}$ be of dimension $d(C) = d$ and multiplicity $m(C) = m$. By the definition of an arrangement, this is a smooth subvariety of Y, and we consider the blow-up $\sigma \colon \tilde{Y} \to Y$ of Y along C with exceptional divisor E. Recall that C, and hence E, are irreducible, by the definition of $\mathcal{S}$. The pullback $\sigma^* D$ of D to $\tilde{Y}$ is even in the Picard group of $\tilde{Y}$, but it is in general not reduced. We define D^* as the unique reduced and even divisor satisfying $\tilde{D} \leq D^* \leq \sigma^* D$, where $\tilde{D}$ is the strict transform of D. In fact D^* is equal to $\tilde{D}$ or $\tilde{D} + \varepsilon E$ where $\varepsilon = 0$ if m is even and $\varepsilon = 1$ if m is odd. This means that when the multiplicity is even, we take the strict transform of the branch locus as the new branch locus, whereas when the multiplicity is odd we add the exceptional divisor. Equivalently $D^* = \sigma^* D - 2\lfloor \frac{m}{2} \rfloor E$. We have

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$K_{\tilde{Y}} + \frac{1}{2}D^* = \sigma^*(K_Y + \frac{1}{2}D) + (n - d - 1 - \lfloor \frac{m}{2} \rfloor)E$, and so $K_{\tilde{Y}} + \frac{1}{2}D^* = \sigma^*(K_Y + \frac{1}{2}D)$ exactly when $\lfloor \frac{m}{2} \rfloor = n - d - 1$. We shall call a blow-up for which this equality holds *admissible*.

Assume now that $C \in \mathcal{S}$ is a minimal element (with respect to inclusion) among those components which are not near pencil. Then, by assumption, the blow-up σ along C is admissible. We want to show that D^* is again an arrangement satisfying the assumptions of the theorem. Let $D_1, \dots, D_k$ be the components of D that contain C. Let us pick some other components $D_{k+1}, \dots, D_{k+p}$ and denote by C_1 the intersection $C_1 = C \cap D_{k+1} \cap \cdots \cap D_{k+p}$. As the problem is local, we can assume that C_1 is irreducible. Our aim is to show that the intersection $\tilde{D}_1 \cap \cdots \cap \tilde{D}_l \cap \tilde{D}_{k+1} \cap \cdots \cap \tilde{D}_{k+p}$ is smooth.

The intersection consists of two parts, namely the strict transform of the intersection and the intersection of the exceptional loci. The dimension of the former is less than or equal to $\dim C_1 + \operatorname{codim} C - 1 - l$, and so its codimension is greater than or equal to $\operatorname{codim} C_1 - \operatorname{codim} C + 1 + l$. Since all the intersections of C with D_{k+j} are near pencil, we obtain that $\operatorname{codim} C_1 - \operatorname{codim} C = p$ and that the codimension of the intersection of the exceptional loci is greater than $p + l$, and hence this is not a component of the intersection $\tilde{D}_1 \cap \cdots \cap \tilde{D}_l \cap \tilde{D}_{k+1} \cap \cdots \cap \tilde{D}_{k+p}$. Consequently, the intersection $\tilde{D}_1 \cap \cdots \cap \tilde{D}_l \cap \tilde{D}_{k+1} \cap \cdots \cap \tilde{D}_{k+p}$ equals the strict transform of the intersection $D_1 \cap \cdots \cap D_l \cap D_{k+1} \cap \cdots \cap D_{k+p}$, and hence is smooth. To conclude that D^* is an arrangement in the case of m odd, we also have to take the exceptional divisor of the blow-up into account. But this is transversal to any strict transform.

To show that the arrangement D^* satisfies the assumption of the proposition, we observe that in the case of m even the exceptional varieties for D^* are blow-ups of the exceptional varieties for D, with the same multiplicities and dimensions. In the case of m odd, we have to add the intersections with the exceptional divisors, but these are near-pencil singularities.

A resolution of singularities of X can now be obtained by blowing-up all the components $C \in \mathcal{S}$ which are not near-pencil, starting from the smallest dimension. Since every blow-up decreases the number of not near-pencil elements, the process will terminate. As the intersection of two hyperplanes cannot be near-pencil, the components of the final branch locus must be disjoint, and hence we get a resolution of singularities. Finally, since all blow-ups are admissible, the resulting resolution is crepant.

Denote by $\sigma\colon \tilde{Y} \to Y$ the composition of all inverse maps to the blow-ups, and by $\pi\colon X \to Y$ (resp. $\tilde{\pi}\colon \tilde{X} \to \tilde{Y}$) the double cover of Y (resp. of $\tilde{Y}$) branched along the divisor D (resp. along $\tilde{D}$). Then there exists a unique map $\tilde{\sigma}\colon \tilde{X} \to X$ making the following diagram commutative.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \tilde{\sigma}\ } & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{Y} & \xrightarrow{\ \sigma\ } & Y \end{array}$$

https://doi.org/10.4153/CMB-2007-049-9 Published online by Cambridge University Press

So the constructed crepant resolution of X is given by a proper birational morphism. ■

Clearly, the resolution is in general not unique, but depends on the order of the blow-ups.

Acknowledgements The first author would like to thank the Leibniz Universität Hannover for kind hospitality and excellent working conditions while staying there. We would also like to thank M. Schütt for numerous discussions.

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