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DECOMPOSITIONS OF GRAPHS OF MODULES OVER SEMISIMPLE RINGS

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In this paper we show that an R-module graph over a semisimple ring R can be written as a direct sum of graphic submodules that are uniquely determined up to isomorphism type. Moreover, this decomposition enables us to describe the R-module graph in graphic terms as a disjoint union of connected components, each of which consists of a complete directed graph on its vertices together with a set of loops at each vertex, determined by the loops at 0. We also give a graphic version of Maschke's Theorem.

0. Introduction

In [4] and [5] Ribenboim described a way of endowing an algebraic object with a compatible directed graph structure. For example, an R-module graph M_Γ is a quadruple $M_\Gamma = (M, V(M), o, t)$ where M is an ordinary R-module, $V(M)$ is a submodule of M, and $o, t: M \to V(M)$ are R-homomorphisms that restrict to the identity on $V(M)$. Thus, $M = \ker(o) \oplus V(M) = \ker(t) \oplus V(M)$. These decompositions are natural in an algebraic sense but unsatisfying categorically because $\ker(o)$ and $\ker(t)$ are submodules which are not R-module graphs in their own right.

When R is a field we are dealing with vector space graphs and in

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[3] we have studied decompositions of M_Γ into indecomposable R-subspaces. An analogous decomposition is possible when R is a semisimple ring in the usual (nongraphic) sense.

Before launching into the study of R-module graphs over a semisimple ring, R, we recall some observations made in [2] concerning arbitrary R-module graphs.

First, an R-module graph M_Γ is a directed graph whose vertices are the elements of $V(M)$ and whose edges are the elements of $E(M) = M \setminus V(M)$. An edge e is directed from $o(e)$ to $t(e)$.

A submodule M' of M is called a graphic submodule if $o(M') \cup t(M') \subseteq M'$.

Let $t(\ker(o)) = V_0$ be the set of vertices that are graph theoretically adjacent to 0. $C_0 = \ker(o) \oplus V_0$ is a graphic submodule of M and it is the graph theoretic connected component of 0.

$L_0 = \ker(o) \cap \ker(t)$ is a graphic submodule and it consists of the vertex 0 together with all loops at 0.

When R is semisimple, ordinary R-modules are completely reducible in the sense described in [1, Chapter II]. This is not the case for R-module graphs, as we shall see.

1. A graphic decomposition of an R-module graph

The decomposition $M = \ker(o) \oplus V(M)$ is module theoretic but not graph theoretic since $\ker(o)$ is not a subgraph of M_Γ unless $\ker(o) = \ker(t) = L_0$. Similarly the decomposition $C_0 = \ker(o) \oplus V_0$ has components which are not graphic.

When R is semisimple we can express C_0 and M as direct sums of graphic submodules. To do this, first write $\ker(o) = L_0 \oplus C$. This can be done because $L_0 \subseteq \ker(o)$ and $\ker(o)$ is completely reducible. The complement C of L_0 in $\ker(o)$ is not unique, but the next proposition shows that for any choice of C, $V_0 \approx C$.

Proposition 1.1 *If* $C \subseteq \ker(o)$ *and* C *satisfies* $\ker(o) = L_0 \oplus C$, *then* $C \approx V_0$.

Proof. Since $C \subseteq \ker(o)$, $t(c)$ is in V_0 for each c in C. Thus we can define $\varphi_C : C \to V_0$ by

$$\varphi_C(c) = t(c) \ . \tag{1.2}$$

If $\varphi_C(c) = 0$, then c is in $L_0 \cap C = \{0\}$. Thus φ_C is one-one. Let v be any element in V_0. We can write $v = t(x)$ for some x in $\ker(o)$ since v is adjacent to 0. By hypothesis, $x = \ell + c$ for unique ℓ in L_0 and c in C. Thus,

$$\varphi_C(c) = \varphi_C(x - \ell) = t(x - \ell) = t(x) - t(\ell) = v \ .$$

So, φ_C is onto.

Given C as in Proposition 1.1, let $K_{0,C} = C \oplus V_0$. The properties of $K_{0,C}$ are summarized in the next proposition.

Proposition 1.3 $K_{0,C}$ *is a graphic submodule of* M. *Moreover,* $K_{0,C}$ *is a complete directed graph in the sense that for any ordered pair,* (v,w) *of vertices,* $v \neq w$, *from* $K_{0,C}$, *there is a unique edge* e *in* $K_{0,C}$ *satisfying* $o(e) = v$ *and* $t(e) = w$.

Proof. Given any k in $K_{0,C}$, write $k = c + v$ where c is in C and v is in V_0. Since $t(c)$ is also in V_0, $t(k) = t(c) + v$ is in V_0. Also, $o(k) = o(c) + o(v) = v$ is in V_0. Thus, $t(K_{0,C}) \cup o(K_{0,C}) \subseteq V_0 \subseteq K_{0,C}$, which proves that $K_{0,C}$ is graphic.

Next consider $v \neq w$ in $V_0 = V(K_{0,C})$. From the proof of Proposition 1.1, we know that there exist c_v and c_w in C such that $v = t(c_v)$ and $w = t(c_w)$. The edge $e = v - c_v + c_w$ is in $K_{0,C}$ and satisfies

$$o(e) = o(v) - o(c_v) + o(c_w) = v \ ,$$

$$t(e) = t(v) - t(c_v) + t(c_w) = v - v + w = w \ .$$

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If e' in C also satisfies $o(e') = v$ and $t(e') = w$, then $e - e'$ is in $L_0 \cap K_{0,C} = \{0\}$. This proves the uniqueness of the edge from v to w.

Now let C be such that $\ker(o) = L_0 \oplus C$. Then

$$C_0 = \ker(o) \oplus V_0 = (L_0 \oplus C) \oplus V_0 = L_0 \oplus (C \oplus V_0) = L_0 \oplus K_{0,C} .$$

Thus C_0, the connected component of 0, is the direct sum of two graphic submodules, one characterized graphically as loops at 0 and having only one vertex, the other characterized as a complete directed graph on the vertices in the connected component of 0. While L_0 is unique, the component $K_{0,C}$ is determined only up to isomorphism.

Since R is semisimple and $V_0 \subseteq V(M)$, we can write $V(M) = V_0 \oplus W$. W is graphic because every submodule of $V(M)$ is graphic. Thus,

$$\begin{aligned} M &= \ker(o) \oplus V(M) \\ &= \ker(o) \oplus (V_0 \oplus W) \\ &= (\ker(o) \oplus V_0) \oplus W \\ &= C_0 \oplus W \\ &= L_0 \oplus K_{0,C} \oplus W \end{aligned}$$

is a way of expressing M as a direct sum of graphic submodules.

It is easy to see that the submodule $W \approx V(M)/V_0$ is determined up to isomorphism. It plays a role in helping to describe M as a graph.

PROPOSITION 1.4. *As a graph, M is the disjoint union of subgraphs, C_w, where C_w is the connected component of w, and w varies over W. Moreover, each C_w is graph theoretically isomorphic to C_0.*

Proof. Let w_1 and w_2 be in W and suppose $C_{w_1} = C_{w_2}$. Then $w_1 - w_2$ is in $C_0 \cap V(M)$. Thus, $w_1 - w_2$ is in V_0. But then $0 + w_1 = (w_1 - w_2) + w_2$ and since $V(M) = V_0 \oplus W$, we have $0 = w_1 - w_2$ and $w_1 = w_2$.

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This shows that

$$M = \bigcup_{\substack{\text{(disjoint)} \\ w \text{ in } W}} C_w .$$

To see that C_0 and C_w are isomorphic as graphs, define $F: C_w \to M$ by $F(x) = x - w$ and $G: C_0 \to M$ by $G(x) = x + w$. First we check that $F(C_w) \subseteq C_0$. Let v be a vertex in C_w and e an edge in C_w. There is a sequence of edges, $e_1, \ldots, e_k$ in C_w with $o(e_1) = w$, $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, k-1$, and $t(e_k) = v$, and there is another sequence, $f_1, \ldots, f_s$, with $o(f_1) = w$, $t(f_i) = o(f_{i+1})$ for $i = 1, \ldots, s-1$, and $t(f_s) = o(e)$. But then, $o(e_1 - w) = 0$, $t(e_i - w) = o(e_{i+1} - w)$ for $i = 1, \ldots, k-1$, and $t(e_k - w) = v-w = F(v)$. So $F(v)$ is in C_0. Also, $o(f_1 - w) = 0$, $t(f_i - w) = o(f_{i+1} - w)$ for $i = 1, \ldots, s-1$, and $t(f_s - w) = t(f_s) - w = o(e) - w = o(e - w) = o(F(e))$. Thus, $F(e)$ is in C_0.

A similar argument can be used to show that $G(C_0) \subseteq C_w$. Since

$$GF(x) = G(x - w) = (x - w) + w = x \quad \text{and}$$

$$FG(x) = F(x + w) = (x + w) - w = x,$$

F and G are one-one and onto.

To see that F and G preserve graphic structure, note that

$$oF(x) = o(x - w) = o(x) - w = Fo(x) \quad \text{and}$$

$$tF(x) = t(x - w) = t(x) - w = Ft(x)$$

and similarly, $oG(x) = Go(x)$ and $tG(x) = Gt(x)$.

In accordance with our observations we name the components L_0, $K_{0,C}$, and W of M as follows:

DEFINITION 1.5. L_0 is called the *loop component* of M, $K_{0,C}$ is called the *complete component*, and W is called the *partition component*.

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The loop and partition components of a graphic module M share the property that any ordinary submodule of either one is graphic. $K_{0,C}$ is different. C is a submodule of $K_{0,C}$ but it is not graphic because $t(C) = V_0$ but $V_0 \not\subseteq C$ since $C \cap V_0 \subseteq \ker(o) \cap V_0 = \{0\}$.

2. Decomposition of R-module graphs into indecomposable graphic R-submodules.

If an R-module graph M_Γ is indecomposable, then it must be comprised entirely of one of its graphic components.

Definition 2.1. An R-module graph M_Γ is called *loop type* if $M = L_0$, *complete type* if $M = K_{0,C}$, and *vertex type* if $M = W$.

If M_Γ is a loop type or vertex type indecomposable R-module graph then M must be irreducible as an R-module since any nontrivial submodule would be a direct summand and the summands would automatically be graphic. On the other hand, if M_Γ is a complete type indecomposable, then M is not indecomposable as an R-module since $M = C \oplus V_0$ with $C \approx V_0 \neq \{0\}$.

Proposition 2.2. *A complete type R-module graph* $K_{0,C} = C \oplus V_0$ *is indecomposable if and only if* V_0 *is an irreducible R-module.*

In order to prove this it will be useful to know the next fact:

Lemma 2.3. *If* $C \oplus V_0$ *is a complete type R-module graph and* V *is any submodule of* V_0, *then* $K = \varphi_C^{-1}(V) \oplus V$ *is a graphic submodule (where* φ_C *was defined in equation (1.2)).*

Proof. Each k in K can be written uniquely as $k = x + v$ where x is in $\varphi_C^{-1}(V)$ and v is in V. Thus,

$$o(k) = o(x) + o(v) = v \quad \text{and}$$

$$t(k) = t(x) + t(v) = \varphi_C(x) + v.$$

This shows that $o(K) \cup t(K) \subseteq V = V(K)$.

Proof (of Proposition 2.2.). First suppose V_0 is not irreducible. Since R is semisimple, $V_0 = V_1 \oplus \dots \oplus V_n$, where each V_i is an irreducible R-module. Let $K_i = \varphi_C^{-1}(V_i) \oplus V_i$. It is graphic by Lemma 2.3 and given the nature of φ_C, it is clear that $K_{0,C} = K_1 \oplus \dots \oplus K_n$. That is, $K_{0,C}$ is not indecomposable.

On the other hand, suppose V_0 is irreducible but $K_{0,C} = G \oplus H$ for some graphic submodules G and H. Since $V(K_{0,C}) = V_0 = V(G) \oplus V(H)$, we may assume that $V(G) = V_0$ and $V(H) = \{0\}$. Thus, $H \subseteq L_0$ which is $\{0\}$ for a complete type R-module. This shows that $K_{0,C}$ is indecomposable.

The next fact is an easy consequence of our understanding of the nature of indecomposable R-module graphs and the usual decomposition theorem for modules over semisimple rings (e.g. see [1]).

THEOREM 2.4. *If M_Γ is any R-module graph then*

$$M = L_{01} \oplus \dots \oplus L_{0\lambda} \oplus K_1 \oplus \dots \oplus K_\tau \oplus W_1 \oplus \dots \oplus W_\mu$$

where each L_{0i} is a loop type indecomposable, each K_j is a complete type indecomposable, and each W_k is a vertex type indecomposable. The numbers λ, τ, and μ are unique and if

$$M = L'_{01} \oplus \dots \oplus L'_{0\lambda} \oplus K'_1 \oplus \dots \oplus K'_\tau \oplus \dots \oplus W'_1 \oplus \dots \oplus W'_\mu$$

is another decomposition of M into indecomposable graphic submodules, the indices may be chosen so that $L_{0i} \approx L'_{0i}$, $K_j \approx K'_j$, and $W_k \approx W'_k$ for $i = 1, \dots, \lambda$, $j = 1, \dots, \tau$, and $k = 1, \dots, \mu$.

3. Modules over semisimple group rings.

In this section we suppose $R = k[G]$ where k is a field of characteristic not dividing the order of G. By Maschke's Theorem, R is known to be a semisimple ring. Each R-module graph M_Γ is also a

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k-vector space graph. Observe also that each g in G acts as a graphic k-linear operator, $T(g)$, on M_Γ .

The main theorem of this section shows how the graphic vector space structure of M_Γ influences its graphic module structure. It is a relative of Maschke's Theorem.

THEOREM 3.1. *Let* k *be a field and* G *a finite group satisfying* $char(k) \nmid [G:1]$. *Let* M_Γ *be a* $k[G]$*-module graph and* M'_Γ *a graphic submodule of* M_Γ . *If there is a graphic k-subspace* N *of* M *such that* $M = M' \oplus N$, *then there is a graphic* $k[G]$*-submodule* N' *of* M *such that* $M = M' \oplus N'$.

Proof. Let $E: M_\Gamma \to M'_\Gamma$ be the projection of M onto M' arising from the decomposition $M = M' \oplus N$. E is a graphic linear transformation and so is

$$F = \frac{1}{[G:1]} \sum_{g \text{ in } G} T(g)ET(g)^{-1} .$$

It is easy to check that for each g in G , $FT(g) = T(g)F$, which means F is actually a graphic $K[G]$-module homomorphism that maps M_Γ to M'_Γ . Also easy to verify is the fact that $F \mid M' = id_{M'}$.

Now let $N' = (id_M - F)(M)$. N' is a $k[G]$-submodule of M since id_M and F are $k[G]$-homomorphisms.

Let s be in $\{o,t\}$ and m in M. Then

$$s(m - F(m)) = s(m) - sF(m) = s(m) - F(s(m)) = (id_M - F)(s(m)) .$$

This shows that $o(N') \cup t(N') \subseteq N'$. Thus, N' is a graphic $k[G]$-submodule of M and $F: M_\Gamma \to M'_\Gamma$ may be interpreted as a projection homomorphism corresponding to the desired decomposition: $M_\Gamma = M'_\Gamma \oplus N'_\Gamma$.

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