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ON THE NONEXISTENCE OF ORTHOGONAL STEINER SYSTEMS OF ORDER 9

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ABSTRACT. It is shown that no pair of orthogonal Steiner triple systems of order 9 exists.

1. **Introduction.** Orthogonal Steiner triple systems (OSS) were first introduced by O'Shaughnessy [2] as a means of producing Room designs. He pointed out that it was relatively easy to find OSS's of orders congruent to 1 mod 6 (that an infinitude of such exist was shown in [1]) but he conjectured that they did not exist for orders congruent to 3 mod 6. It is shown here that, in fact, a pair of orthogonal Steiner systems of order 9 does not exist.

2. **The Main Result.** Two Steiner triple systems, SS1 and SS2, on the same set of symbols are said to be orthogonal if they satisfy the following two conditions:

(1) they have no blocks in common;

(2) if two pairs of elements appear with the same third element in one system, then they must appear with different elements in the other system, that is, if xyz and $xuv \in$ SS1 then syz and tuv belong to SS2 for some s and t, $s \neq t$.

Since all Steiner systems of order 9 are isomorphic and have the Kirkman property (they may be partitioned into resolution classes which contain each element of the design exactly once), we assume that one Steiner system, SS1, contains the resolution class 123, 456, and 789 and for the moment is undetermined beyond this.

The method used here is to complete a second Steiner system, SS2, subject to the condition that it is to be orthogonal to the 3 known blocks of SS1, and then attempt to complete SS1 orthogonal to SS2.

Since block $123 \in$ SS1, we see that

$$\left.\begin{matrix} 12x \\ 23y \\ 13z \end{matrix}\right\} \in \text{SS2 with } x \neq y \neq z \text{ and } x, y, z \neq 1, 2, \text{ or } 3.$$

There are two major cases to consider:

Case I. x, y, z, all belong to the same block of SS1;

Case II. x and y belong to one block of SS1 and z belongs to the other block of SS1.

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Since at this point 4, 5, and 6 are equivalent to each other and to 7, 8, 9, we choose

Case I. $x=4$, $y=5$, $z=6$,

Case II. $x=4$, $y=5$, $z=7$.

In Case I, SS2 contains 124, 235, 136, and $45x'$, $56y'$, $46z'$ with $x' \neq y' \neq z'$ and $x', y', z' \neq 1, 2, 3, 4, 5, 6$. Without loss of generality we choose $x'=7$, $y'=8$, and $z'=9$. Then,

$$\text{SS2 contains} \begin{matrix} 124 \\ 235, \\ 136 \end{matrix} \quad \begin{matrix} 457 \\ 568, \\ 469 \end{matrix} \quad \text{and} \quad \begin{matrix} 78x'' \\ 89y''. \\ 79z'' \end{matrix}$$

The element x'' may be determined by noting that 235 and 469 belong to SS2, and the Kirkman property then implies that 178 must belong to SS2. Likewise $y''=2$ and $z''=3$. We further use the pair replications of individual elements; for example, 1 appears with 2, 3, 4, 6, 7, 8, thus 5 and 9 do not yet appear paired with 1 and the triple 159 must belong to the system SS2. The entire system is now determined to be:

$$\text{SS2} = \begin{matrix} 124 & 136 & 159 & 178 \\ 379 & 289 & 267 & 235. \\ 568 & 457 & 348 & 469 \end{matrix}$$

A similar procedure applied to Case II yields two distinct possible Steiner systems orthogonal to the one Kirkman resolution class of SS1 yet determined. These are

$$\text{SS2*} = \begin{matrix} 124 & 137 & 156 & 189 \\ 368 & 269 & 278 & 235 \\ 579 & 458 & 349 & 469 \end{matrix}$$

and

$$\text{SS2**} = \begin{matrix} 124 & 137 & 159 & 168 \\ 389 & 269 & 278 & 235. \\ 567 & 458 & 346 & 479 \end{matrix}$$

In attempting to complete SS1 orthogonal to SS2, SS2*, or SS2**, we start with the triple 124 which belongs to each SS2 candidate. It must split in SS1 as follows:

$12\underline{\ 3\ }$ already determined by the fact that $123 \in \text{SS1}$,

24 [7, 8, or 9],

14 [7, 8, or 9].

Note that 24 or 14 appearing with a 1, 2, 3 or 4, 5, 6 would duplicate the pair replication of the Steiner system since both 123 and 456 already belong to SS1. This leads to the consideration of 6 possible cases:

A. 247 and $148 \in \text{SS1}$,
B. 247 and $149 \in \text{SS1}$,
C. 248 and $147 \in \text{SS1}$,
D. 248 and $149 \in \text{SS1}$,
E. 249 and $147 \in \text{SS1}$,
F. 249 and $148 \in \text{SS1}$.

Further, the addition of these two triples to the system does not uniquely determine it. Since 235 belongs to each of the systems SS2, SS2*, and SS2** and must split in SS1 we have, as above,

23 $\underline{1}$ already determined,
35 [7, 8, or 9]
25 [7, 8, or 9].

One additional triple will completely determine the system and thus we must consider 3 subcases within each of the 6 cases listed above: each of (*i*) 357, (*ii*) 358, or (*iii*) 359 determines a possible SS1 (all distinct).

Case A will be done in detail; the others are very similar.

THEOREM. *There does not exist a Steiner triple system,* SS1 *containing triples* 123, 456, 789, 247, *and* 148 *which is orthogonal to any of the systems* SS2, SS2*, *or* SS2**.

Proof. The proof involves the three subcases mentioned above.

(i) If SS1 contains in addition to the initial blocks 123, 456, 789, and 247, 148, the block 357, then 357 and 148 determine a Kirkman resolution class the third member of which is 269. Thus the assumption that 357 belong to SS1 leads to the fact that 269 must also belong to SS1. However, 269 already belongs to both SS2* and SS2** contradicting property (1) of the definition of orthogonal Steiner systems. In order to see that the SS1 determined by 357 cannot be orthogonal to SS2 we note that a 2 appears with 1, 3, 4, 6, 7, and 9 in triples already present in SS1 and thus 258 must belong to SS1. Also 5 appears with 2, 3, 4, 6, 7, and 8 implying that 159 must be a block of SS1. However, the block 159 belongs to SS2 contradicting orthogonality again. Thus the triple system containing 123, 456, 789, 247, 148, and 357 is not orthogonal to any of SS2, SS2*, or SS2**.

(ii) If SS1 contains 358 in addition to its initial blocks 123, 456, 789, and 247, 148, then 358 and 247 belonging to SS1 imply by the Kirkman condition that $169 \in$ SS1. The element 1 appears in three triples, 123, 148, and 169 and thus to complete its replication it must appear with 5 and 7, that is, $157 \in$ SS1. Further replication of the element 5 implies that 259 also belongs to the system. 259 and 148 determine a Kirkman resolution class also occupied by 367. The element 3 may now be used to determine that 349 must belong to SS1. This implies that SS1 is not orthogonal to SS2* since 349 also belongs to SS2*. SS1 is now completely determined and listed below.

$$\mathrm{SS1} = \begin{matrix} 123 & 148 & 157 & 169 \\ 456 & 259 & 268 & 247 \\ 789 & 367 & 349 & 358 \end{matrix}$$

This system is not orthogonal to either SS2 or SS2** since the second defining property of orthogonality is contradicted by the following pairs of triples. 18 and

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39 appear with the same element 4 in SS1 and also with the same element 7 in SS2; that is

418 and 439 belong to SS1 while 718 and 739 belong to SS2.

Further, 148 and 123 belong to SS1 while 548 and 523 belong to SS2**. Thus no Steiner system including triples 123, 456, 789, 247, 148, and 358 is orthogonal to SS2, SS2*, or SS2**.

(iii) Allowing 359 to belong to SS1 contradicts the Kirkman property since 359 would belong to two distinct classes, that is, 247 and 359 imply 168 belongs to SS1; and 148 and 359 imply 267 belongs to SS1. This further implies that pairs 27 and 18 are repeated twice, a contradiction to the fact that SS1 is to be a Steiner triple system. This completes the proof.

The following tabulation shows how the remaining cases are disposed of. O_1, O_2, and K mean contradiction of orthogonality condition 1, orthogonality condition 2, and the Kirkman condition, respectively.

	(i)	(ii)	(iii)	(i)	(ii)	(iii)	(i)	(ii)	(iii)
A.	O_1	O_2	K	O_1	O_1	K	O_1	O_2	K
B.	O_1	K	O_1	O_2	K	O_1	O_2	K	O_1
C.	O_1	O_1	K	O_1	O_1	K	O_2	O_1	K
D.	K	O_1	O_2	K	O_2	O_1	K	O_1	O_1
E.	O_1	K	O_1	O_2	K	O_2	O_1	K	O_2
F.	K	O_1	O_1	K	O_2	O_1	K	O_1	O_2

Orthogonal Steiner systems correspond to Room squares of a particular type. One of the unsolved problems in the theory of Room squares is the existence of squares of side n where n is a multiple of 3 (and not of 9). A multiplication theorem of Stanton and Horton [3] shows that if there exist squares of side v_1 and v_2, then there exists a square of side v_1v_2. However, the nonexistence of a square of side 3 leaves the existence question open for multiples of 3. The existence of orthogonal Steiner systems of orders congruent to 3 mod 6 would then imply the existence of Room squares of side n where n is a multiple of 3. We have shown that no such pair of OSS's exists of order 9; however a Room square of side 9 has been shown to exist [4].

References

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