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EXAMPLE OF AN INJECTIVE MODULE WHICH IS NOT NICE

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In [1] Lambek calls the injective R-module I *nice* if every torsionfree factor module of the ring of quotients Q of R with respect to I is divisible. If I is nice then Q is a dense subring of the bicommutator $\mathrm{Bic}_R I$ of I with respect to the finite topology (see [1, Proposition 2]). We now give an example of an injective R-module over an Artinian ring R which is *not* nice. Since R is Artinian, $Q=\mathrm{Bic}_R I$, by Proposition B of [1].

Before we give the example, we state the following, which depends on [2] for terminology.

LEMMA. *Let P be a maximal ideal of the right Artinian ring R. Suppose R is P-torsionfree and I_P is the unique indecomposable P-torsionfree injective right R-module with associated prime ideal P. If I_P is nice then every minimal P-closed right ideal $E\neq 0$ of R is a minimal right ideal of R_P.*

Proof. Let $E\neq 0$ be a minimal P-closed right ideal of R. Then $eR_P\neq 0$ for every $e\neq 0$ of E. Furthermore eR_P is a torsionfree epimorphic image of R_P. Since I_P is nice, eR_P is P-divisible. Thus $E=eR_P$ for every $0\neq e \in E$, and E is a minimal right ideal of R_P.

Example. Let D be a field and

$$R=\left\{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & a \end{pmatrix} \middle| \, a, b, c \in D\right\}.$$

Then R is right and left Artinian. Clearly

$$e=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a primitive idempotent of R. Let N be the Jacobson radical of R. Then

$$0 < eN = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \middle| \, d \in D\right\} < eR = \left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & a \end{pmatrix} \middle| \, a, d \in D\right\}$$

is the unique composition series of eR.

Clearly

$$P=\left\{\begin{pmatrix}0&0&0\\b&c&0\\d&0&0\end{pmatrix}\middle|\; b,c,d\in D\right\}$$

is a maximal two-sided ideal of R, and

$$ReR=\left\{\begin{pmatrix}a&0&0\\b&0&0\\c&0&a\end{pmatrix}\middle|\; a,b,c\in D\right\}$$

is the minimal ideal in the filter of all dense right ideals of R in the P-torsion theory. It is easy to see that the left annihilator of ReR is zero. Hence R is P-torsionfree. Thus

$$R_P=End_R(ReR)=\left\{\begin{pmatrix}a&0&0\\b&c&0\\d&e&a\end{pmatrix}\middle|\; a,b,c,d,e\in D\right\}.$$

Let $V=eR/eN$. Then $I_P=I_R(V)$, where $I_R(V)$ denotes the injective hull of V.

Suppose I_P is nice. Since eN is a minimal right ideal of R, the P-closure $cl_P(eN)$ of eN is a minimal P-closed right ideal of R. By the lemma it follows that $cl_P(eN)$ is a minimal right ideal of R_P. But

$$cl_P(eN)=\left\{\begin{pmatrix}0&0&0\\0&0&0\\d&e&0\end{pmatrix}\middle|\; d,e\in D\right\}>eNR_P\neq 0,$$

because

$$eNR_P=\left\{\begin{pmatrix}0&0&0\\0&0&0\\d&0&0\end{pmatrix}\middle|\; d\in D\right\}.$$

This contradiction proves that I_P is not nice.

References

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2. J. Lambek and G. Michler, *The torsion theory at a prime ideal of a right Noetherian ring*, (submitted for publication).
3. J. Lambek, *Torsion theories, additive semantics and rings of quotients*, Springer Lecture Note in Math. **177**, 1971.

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