Can. J. Math., Vol. XXVII, No. 3, 1975, pp. 725–735

ON DECOMPOSABILITY OF COMPACT PERTURBATIONS OF NORMAL OPERATORS

M. RADJABALIPOUR AND H. RADJAVI

The main purpose of this paper is to show that a bounded Hilbert-space operator whose imaginary part is in the Schatten class $C_p(1 \leqq p < \infty)$ is strongly decomposable. This answers affirmatively a question raised by Colojoara and Foias [**6**, Section 5(e), p. 218].

In case $0 \leqq T^* - T \in C_1$, it was shown by B. Sz.-Nagy and C. Foias [**2**, p. 442; **25**, p. 337] that T has many properties analogous to those of a decomposable operator and by A. Jafarian [**11**] that T is strongly decomposable. The authors of [**11**] and [**24**] employ the properties of the characteristic function of the contraction operator obtained from the Cayley transform of T; their method is not applicable to the general case where $T^* - T$ is merely an operator of class C_p $(1 \leqq p < \infty)$.

The techniques of the present paper are mainly inspired from the results of [**12**; **13**; **17**; **20**]. We state our results in a rather more general context. All we need is that the operators under consideration satisfy the following conditions.

Condition (I). Let J be a C^2 Jordan curve. A Hilbert-space operator T is said to satisfy Condition (I) if

(a) it is the sum of a normal operator with spectrum on J and an operator of the Schatten class C_p $(1 \leqq p < \infty)$, and

(b) $\sigma(T)$ does not fill the interior of J.

(The class C_p is the ideal of compact operators T such that $\sum(\mu_n)^p < \infty$ where $\mu_1, \mu_2, \ldots$ are the eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity for $1 \leqq p < \infty$; C_∞ is the ideal of all compact operators.)

Condition (II). We say T satisfies Condition (II) if $T|M$ and $(T^*|M^\perp)^*$ satisfy Condition (I) for all (trivial or non-trivial) hyperinvariant subspaces M of T.

We conjecture that Conditions (I) and (II) are equivalent; Lemma 3 below proves this equivalence in some special cases.

1. Main results.

We begin with some lemmas.

LEMMA 1. *If T satisfies Condition* (I) *then $\sigma(T)\backslash J$ consists of isolated points of $\sigma(T)$. Moreover if $\lambda \in \sigma(T)\backslash J$ and C is a sufficiently small circle around λ,*

Received February 12, 1974 and in revised form, November 4, 1974.

then the projection

$$E(\lambda) = (2\pi i)^{-1} \int_C (z - T)^{-1} dz$$

has a finite dimensional range.

Proof. Let $T = A + B$ where A is a normal operator, $\sigma(A) \subseteq J$, and $B \in C_p$. Since $(T - z)^{-1} = [I + (A - z)^{-1}B]^{-1}(A - z)^{-1}$, the first assertion follows from [**7**, Lemma VII. 6.13, p. 592]. Since $[I + (A - z)^{-1}B]^{-1} = I - (A - z)^{-1}B[I + (A - z)^{-1}B]^{-1}$, we have

$$E(\lambda) = (2\pi i)^{-1} \int_C (A - z)^{-1}B[I + (A - z)^{-1}B]^{-1}(A - z)^{-1} dz,$$

where C is a circle excluding λ from $\sigma(T)\backslash\{\lambda\}$. Thus $E(\lambda)$ is compact and hence has a finite dimensional range.

Remark 1. If T satisfies Condition (I) then $\sigma(T)$ is nowhere dense and thus T has the single valued extension property [**9**, Lemma XVI. 5.1, p. 2149]. (An operator T on a Banach space X is said to have the single valued extension property if there exists no non-zero X-valued analytic function f such that $(z - T)f(z) \equiv 0$.)

Remark 2. Lemma 1 remains true if p is replaced by ∞.

LEMMA 2. *Let A be a normal operator whose spectrum is a proper subset of a C^2 Jordan curve J and let B be an operator of the Schatten class $C_p(1 \leqq p < \infty)$. Then $T = A + B$ satisfies Condition* (I).

Proof. Since $\sigma(T)$ is bounded, there exists a Jordan curve J_1 such that $\sigma(A) \subseteq J_1$ and $\sigma(T)$ does not fill the interior of J_1. Thus $\sigma(T)\backslash J_1$ is countable and hence $\sigma(T)$ does not fill the interior of J. (See the proof of Lemma 1.)

LEMMA 3. (a) *If $T^* - T \in C_p$ $(1 \leqq p < \infty)$ then T satisfies Condition* (II).

(b) *If $T^*T - I \in C_p$ $(1 \leqq p < \infty)$ and $\sigma(T)$ does not fill the unit disc then T satisfies Condition* (II).

Proof. The proof of (a) follows from Lemma 2 and the fact that the property $S^* - S \in C_p$ is inherited by the restrictions of S to arbitrary invariant subspaces. For part (b) assume $T^*T - I \in C_p$ $(1 \leqq p < \infty)$ and $\sigma(T)$ does not fill the unit disc. Since the image of T in the Calkin algebra is unitary, it follows that $\lambda - T$ is a Fredholm operator for $|\lambda| \neq 1$. Let $g(\lambda) = \text{index}$ $(\lambda - T) = \dim\ N(\lambda - T) - \dim\ R(\lambda - T)^{\perp}$ for $|\lambda| \neq 1$. Since g is an integer-valued continuous function of λ and $g(\lambda) = 0$ for $\lambda \in \rho(T)$, $g(\lambda) = 0$ for $|\lambda| \neq 1$. In particular $\dim N(T) = \dim R(T)^{\perp} < \infty$. Let $T = U(T^*T)^{1/2}$ where U can be chosen to be a unitary operator, because $\dim N((T^*T)^{1/2}) = \dim\ N(T) = \dim\ R(T)^{\perp}$. Now the relation $(T^*T)^{1/2} - I = (T^*T - I)$-$[(T^*T)^{1/2} + I]^{-1}$ implies that $(T^*T)^{1/2} - I \in C_p$. Thus T (and consequently

https://doi.org/10.4153/CJM-1975-080-8 Published online by Cambridge University Press

T^*) satisfies Condition (I) with J = unit circle. It follows that $TT^* - I \in C_p$; and since the condition $S^*S - I \in C_p$ is inherited by the restrictions of S to arbitrary invariant subspaces, we conclude that T satisfies Condition (II). (For the material related to index theory we refer the reader to [**4**, p. 70-71] and the references cited there.)

Notations. Let F be a closed subset of the plane and let T be a (bounded linear) operator defined on a Hilbert space H. We will fix the following notations throughout the paper.

(1) $N_T(F) = \text{Span}\,\{x \in H : (T - \lambda)^n x = 0$ for some $\lambda \in F$ and some positive integer $n\}$.

(2) $\sigma_T(x) = \mathbf{C}\backslash\rho_T(x) = \mathbf{C}\backslash \bigcup\{G \subseteq \mathbf{C} : G$ is open and there exists an analytic function $f : G \to H$ such that $(z - T)f(z) \equiv x\}$ where $x \in H$ and T has the single valued extension property.

(3) $X_T(F) = \{x \in H : \sigma_T(x) \subseteq F\}$.

(4) $\bar{G} = G^- =$ the closure of a set $G \subseteq \mathbf{C}$.

(5) For two subspaces M and N of H we write $H = M \oplus N$ if for each $x \in H$ there exists a unique pair $(x_1, x_2) \in M \times N$ such that $x = x_1 + x_2$.

Definition. A subspace M is called a spectral maximal subspace of an operator T if

(a) M is an invariant subspace of T, and

(b) $N \subseteq M$ for all invariant subspaces N of T such that $\sigma(T|N) \subseteq \sigma(T|M)$.

It is shown in [**6**, Theorem 3.8, p. 23] that if T has the single valued extension property and $X_T(F)$ is closed, then $X_T(F)$ is a spectral maximal subspace of T and $\sigma(T|X_T(F)) \subseteq F \cap \sigma(T)$. Moreover every spectral maximal subspace of T is also a hyperinvariant subspace of T [**6**, Theorem 3.2, p. 18].

LEMMA 4. *If T satisfies Condition* (I), *then $\sigma(V) \cap F \subseteq J$ where V is the operator induced on $H/N_T(F)(= N_T(F)^\perp)$ by T.*

Proof. Since $N_T(F)$ is a hyperinvariant subspace of T, it follows from [**1**, Lemma I.3.1] that $\sigma(V) \subseteq \sigma(T)$. Let $\lambda \in (F \cap \sigma(T))\backslash J$. Since $N_T(F)$ includes the range of the projection $E(\lambda)$ of Lemma 1, we have

$$\int_C (z - V)^{-1} dz = 0.$$

Thus $\lambda \notin \sigma(V)$ and hence $\sigma(V) \cap F \subseteq J$.

LEMMA 5. *Let T satisfy Condition* (I). *Let J_1 be a* (*non-trivial*) *closed subarc of J such that $J_1 \cap (\sigma(T)\backslash J)^- = \emptyset$. Then*

$$M_1 = X_T(J_1) \text{ and } M_2 = X_T([\sigma(T)\backslash J_1]^-)$$

are closed and

https://doi.org/10.4153/CJM-1975-080-8 Published online by Cambridge University Press

(a) $\sigma(T|M_1) \cup E = (\sigma(T) \cap J_1) \cup E$,
(b) $\sigma(T|M_2) \cup E = (\sigma(T)\backslash J_1) \cup E$,
(c) $\sigma(V) \subseteq [\sigma(T)\backslash J_1]^-$,

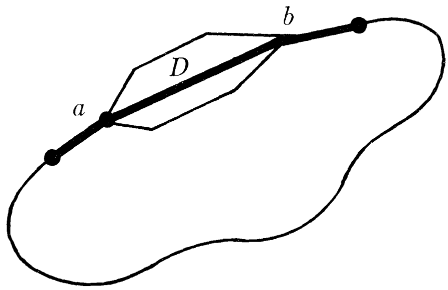
where E is the set of endpoints of J_1 and V is the operator induced on H/M_1 by T.

Proof. Let J_2 be an open subarc of J containing J_1 such that $J_2 \cap (\sigma(T)\backslash J)^- = \emptyset$. It follows from [**19**, Lemma 6.11, p. 104] that for each point $a \in J_2$ and each closed bounded line segment L with a as endpoint which is not tangent to J and satisfies $L \cap J = \{a\}$, there is a constant M such that

$$(*) \quad ||(z - T)^{-1}|| \leqq \exp \{M|z - a|^{-q}\} \quad \text{for } z \in L\backslash\{a\},$$

where q is a positive constant independent of a.

We shall show that M_2 is closed; the proof for M_1 is similar. Let x_n be an arbitrary Cauchy sequence in M_2 converging to x. Let f_n be the analytic function such that $(z - T)f_n(z) = x_n$ for $z \notin (\sigma(T)\backslash J_1)^-$. Let a and b be two points on J_1 both distinct from its endpoints, and let J_{ab} denote the open subarc of J_1 with endpoints a and b. Let D be a Jordan domain with the following properties: (i) D contains J_{ab} and $\bar{D} \cap \sigma(T) \subseteq \bar{J}_{ab}$, and (ii) in a neighbourhood of a (respectively b) the boundary of D consists of two line segments starting from a (respectively b) and making positive angles less than $\pi/2q$ with the tangent to J at a (respectively b) which points toward b (respectively a).



By [**23**, Example 2] or [**19**, proof of Theorem 6.3, p. 97] there exists a function g analytic in D and continuous on $\bar{D}$ such that

$$\sup \{||g(z)(z - T)^{-1}|| : z \in \partial D\backslash\{a, b\}\} < \infty$$

and $g(z) \neq 0$ for all $z \in D$. (See also the proofs of [**17**, Lemma 3 and Corollary 3] in this direction.) Let $h_n(z) = (z - a)(z - b)g(z)f_n(z)$, $z \in D$. By a proof similar to the proof of [**8**, Lemma XVI. 5.4, p. 2151] we can show that $h(z) = \lim_n (z-a)(z-b)g(z)f_n(z)$ is analytic in D and $(z-T)\{(z-a)(z-b)g(z)\}^{-1} h(z) = x$, $z \in D$. This shows that $\sigma_T(x) \subseteq \mathbf{C}\backslash J_{ab}$. Letting J_{ab} converge to J_1 we deduce that $\sigma_T(x) \subseteq \mathbf{C}\backslash J_1$. Thus M_2 is closed and hence $\sigma(T|M_2) \subseteq [\sigma(T)\backslash J_1]^-$. Now let a and b be the endpoints of J_1 and let g, D be as described

https://doi.org/10.4153/CJM-1975-080-8 Published online by Cambridge University Press

above. Since $g(z) \neq 0$ for all $z \in D$, it follows from [**18**, Remark on Theorem 1] that $(\sigma(T) \cap J_1) \cup E = (\sigma(T) \cap D) \cup E \subseteq \sigma(T|M_1) \cup E$. Thus $\sigma(T|M_1) \cup E = (\sigma(T) \cap J_1) \cup E$ which completes the proof of (a). A similar argument finishes the proof of (b).

For (c) let $S = T|M_1$ and let

$$T = \begin{bmatrix} S & R \\ 0 & V \end{bmatrix} \begin{matrix} M_1 \\ M_1^{\perp}. \end{matrix}$$

Since M_1 is a hyperinvariant subspace of T, $\sigma(V) \subseteq \sigma(T)$. Thus $||(z - V)^{-1}|| \leq ||(z - T)^{-1}||$ for $z \in \rho(T)$ and hence $||(z - V)^{-1}||$ also satisfies the above growth condition (*) at all points $a \in J_2$.

Therefore $X_V(J_1)$ is closed and $\sigma(V|X_V(J_1)) \cup E = (\sigma(V) \cap J_1) \cup E$. Let $W = T|M_1 \oplus X_V(J_1)$. Since $\sigma(W) \subseteq J_1$ and M_1 is a spectral maximal subspace of T, we have $X_V(J_1) = \{0\}$ and thus $\sigma(V) \cap J_1 \subseteq E$. This proves (c) and with it the lemma.

COROLLARY 1. *Lemma 5 remains true if J_1 is the disjoint union of a finite number of (non-trivial) closed subarcs of J.*

Proof. The proof of (a) and (b) follows from the fact that J_1 and $(J \backslash J_1)^-$ are the intersection of a finite number of closed subarcs of J together with the Riesz decomposition theorem; the proof of (c) is exactly the same as in Lemma 5.

LEMMA 6. *If T satisfies Condition* (II) *then $\sigma(T|N_T(F)) = L$ where $L = \{\lambda \in F : (T - \lambda)^n x = 0$ for some $x \neq 0$ and some positive integer $n\}^-$.*

Proof. The inclusion $L \subseteq \sigma(T|N_T(F))$ is obvious. Also if $\lambda \notin L \cup J$ it follows from the Riesz decomposition theorem and Lemma 1 that $H = E(\lambda)H \oplus [I - E(\lambda)]H$, $\lambda \notin \sigma(T|[I - E(\lambda)]H)$, and $N_T(F) \subseteq [I - E(\lambda)]H$. Since $\sigma(T|N_T(F)) \subseteq \sigma(T)$ is nowhere dense, we have $\lambda \notin \sigma(T|N_T(F))$ and thus $\sigma(T|N_T(F)) = L \cup \Delta$, where Δ is a subset of J. Let $S = T|N_T(F)$. Since $N_T(F)$ is a hyperinvariant subspace of T, S satisfies Condition (I). Let J_1 be an arbitrary closed subarc of J in the complement of L. In view of Lemma 5, $X_S(J_1)$ and $X_S([\sigma(S)\backslash J_1]^-)$ are closed. Since $N_T(F) = N_S(F) \subseteq X_S([\sigma(S)\backslash J_1]^-)$, it follows again from Lemma 5 that $\sigma(S) \subseteq [\sigma(S)\backslash J_1]^-$. Thus $\sigma(S) \cap J_1$ is a subset of the endpoints of J_1. This shows that $\Delta \subseteq L$ which completes the proof of the lemma.

LEMMA 7. *Let T satisfy Condition* (II). *Let G be an open subset of the plane such that $J \cap \partial G$ is a finite set. Then $X_T(\bar{G})$ is closed and $\sigma(V) \subseteq (\sigma(T)\backslash \bar{G})^-$ where V is the operator induced on $H/X_T(\bar{G})$ by T.*

Proof. The case $\bar{G} \cap J = \emptyset$ follows from the Riesz decomposition theorem and Lemma 4. Assume $\bar{G} \cap J \neq \emptyset$. Let G_n be a decreasing sequence of open sets converging to $\bar{G}$ such that $\bar{G}_n \cap J$ is the disjoint union of a finite number

https://doi.org/10.4153/CJM-1975-080-8 Published online by Cambridge University Press

of closed arcs, and $G_n \supseteq \bar{G}_{n+1}$. Let $S_n = T|N_T(\bar{G}_n)$ and let

$$T = \begin{bmatrix} S_n & R_n \\ 0 & V_n \end{bmatrix} \begin{matrix} N_T(\bar{G}_n) \\ N_T(\bar{G}_n)^\perp. \end{matrix}$$

The operator V_n satisfies Condition (I) and, by Lemma 4, $\sigma(V_n) \cap \bar{G}_n \subseteq J$. Thus, by Corollary 1, $X_{V_n}(\bar{G}_{n+1})$ is closed and $\sigma(V_n|X_{V_n}(\bar{G}_{n+1})) \subseteq \bar{G}_{n+1}$. We claim that $X_T(\bar{G}) = \cap H_n$, where $H_n = N_T(\bar{G}_n) \oplus X_{V_n}(\bar{G}_{n+1})$ $(n = 1, 2, \ldots)$. Let $x \in X_T(\bar{G})$. For each n let

$$T = \begin{bmatrix} S_n & R_{1n} & R_{2n} \\ 0 & V_{1n} & V_{2n} \\ 0 & 0 & V_{3n} \end{bmatrix} \begin{matrix} N_T(\bar{G}_n) \\ X_{V_n}(\bar{G}_{n+1}). \\ H_n^\perp \end{matrix}$$

Let y_n be the orthogonal projection of x on $H_n^\perp$. Obviously $(\lambda - V_{3n})^{-1}y_n$ has an analytic extension to $\mathbf{C}\backslash\bar{G}$. But Corollary 1 implies that $\sigma(V_{3n}) \subseteq \mathbf{C}\backslash G_{n+1}$. Therefore $(\lambda - V_{3n})^{-1}y_n$ has an analytic extension everywhere. Since V_{3n} has the single valued extension property, it follows that $y_n = 0$ and thus $x \in H_n$ for all n. Hence $X_T(\bar{G}) \subseteq \cap H_n$. Conversely if $x \in \cap H_n$ and $W_n = T|H_n$, then $(\lambda - T)^{-1}x$ has an analytic extension $(\lambda - W_n)^{-1}x$ to $\mathbf{C}\backslash\bar{G}_n$ and thus $x \in X_T(\bar{G}_n)$ for all n.

(By Lemma 5, $\sigma(W_n) = \sigma(T|N_T(\bar{G}_n)) \cup \sigma(V_n|X_{V_n}(\bar{G}_{n+1})) \subseteq \bar{G}_n$.)

Hence $x \in \cap X_T(\bar{G}_n) = X_T(\bar{G})$ which proves the equality of $X_T(\bar{G})$ and $\cap H_n$. This shows that $X_T(\bar{G})$ is closed.

Now let g_n be an increasing sequence of open sets converging to G such that $\bar{g}_n \cap J$ is the disjoint union of a finite number of closed arcs, and $\bar{g}_n \subseteq g_{n+1}$. Let $s_n = T|N_T(\bar{g}_{n+1})$ and let

$$T = \begin{bmatrix} s_n & r_n \\ 0 & v_n \end{bmatrix} \begin{matrix} N_T(\bar{g}_{n+1}) \\ N_T(\bar{g}_{n+1})^\perp. \end{matrix}$$

Here again $\sigma(v_n) \cap \bar{g}_{n+1} \subseteq J$ and v_n satisfies Condition (I). Thus, by Corollary 1, $X_{v_n}(\bar{g}_n)$ is closed and $\sigma(v_n|X_{v_n}(\bar{g}_n)) \subseteq \bar{g}_n$. Hence, by Lemma 6, $X_T(\bar{G}) \supseteq N_T(\bar{g}_{n+1}) \oplus X_{v_n}(\bar{g}_n) = K_n$, say. Let L_n be the orthogonal complement of K_n in $X_T(\bar{G})$ and let

$$v_n = \begin{bmatrix} v_{1n} & v_{2n} & v_{3n} \\ 0 & v_{4n} & v_{5n} \\ 0 & 0 & V \end{bmatrix} \begin{matrix} X_{v_n}(\bar{g}_n) \\ L_n \\ X_T(\bar{G})^\perp. \end{matrix}$$

By Corollary 1, the spectrum of the operator

$$\begin{bmatrix} v_{4n} & v_{5n} \\ 0 & V \end{bmatrix} \begin{matrix} L_n \\ X_T(\bar{G})^\perp \end{matrix}$$

is a subset of $(\sigma(v_n)\backslash\bar{g}_n)^-$. Since $(\sigma(v_n)\backslash\bar{g}_n)^-$ encloses no holes, it follows that $\sigma(V) \subseteq (\sigma(v_n)\backslash\bar{g}_n)^-$ for all n and thus $\sigma(V) \subseteq \sigma(T)\backslash G$. Finally, if possible, let $\lambda \in \sigma(V)$ and $\lambda \notin (\sigma(T)\backslash\bar{G})^-$. Since $J \cap \partial G$ is a finite set, it follows that

https://doi.org/10.4153/CJM-1975-080-8 Published online by Cambridge University Press

$\sigma(V)$ has an isolated point on ∂G which is impossible (because by applying the Riesz decomposition theorem to V we can find an invariant subspace M of T such that $M \supseteq X_T(\bar{G})$, $M \neq X_T(\bar{G})$, and $\sigma(T|M) \subseteq \bar{G}$).

LEMMA 8. *Let T satisfy Condition* (II). *Let $D_1, D_2, \ldots, D_n$ be n open discs such that ∂D_i is not tangent to ∂D_j, $(\partial D_i) \cap (\partial D_j) \cap J = \emptyset$ for all $i \neq j$, and $(\partial D_i) \cap J$ is a finite set for all i. Then*

$$X_T(\bar{D}_1 \cup \bar{D}_2 \cup \ldots \cup \bar{D}_n) = X_T(\bar{D}_1) + X_T(\bar{D}_2) + \ldots + X_T(\bar{D}_n).$$

Proof. We proceed by induction on n. The proof for $n = 1$ is trivial. Assume the lemma is true for $n = k$, we show that it is also true for $n = k + 1$. Since $D_1 \cup D_2 \cup \ldots \cup D_k$ and $D_1 \cup D_2 \cup \ldots \cup D_{k+1}$ satisfy the conditions of Lemma 7, the manifold $H_i = X_T(\bar{D}_1 \cup \ldots \cup \bar{D}_i)$ is closed and $\sigma(T|H_i) \subseteq \bar{D}_1 \cup \ldots \cup \bar{D}_i (i = 1, 2, \ldots, k + 1)$. Also since $T|H_{k+1}$ satisfies Condition (I) and $D_{k+1} \cap (D_1 \cup \ldots \cup D_k)$ satisfies the conditions of Lemma 7, it follows that the manifold $K = X_T(\bar{D}_{k+1} \cap [\bar{D}_1 \cup \ldots \cup \bar{D}_k])$ is closed, $\sigma(T|K) \subseteq \bar{D}_{k+1} \cap (\bar{D}_1 \cup \ldots \cup \bar{D}_k)$, and

$$\sigma(V) \subseteq \{\mathbf{C} \backslash [\bar{D}_{k+1} \cap (\bar{D}_1 \cup \ldots \cup \bar{D}_k)]\}^-$$

where V is the operator induced on H_{k+1}/K (the orthogonal complement of K in H_{k+1}) by $T|H_{k+1}$. Thus $\sigma(V)$ is the disjoint union of two closed sets E_1 and E_2 such that $E_1 \subseteq \bar{D}_{k+1}$ and $E_2 \subseteq \bar{D}_1 \cup \bar{D}_2 \cup \ldots \cup \bar{D}_k$ (see also Lemma 4 for points off J). Hence by the Riesz decomposition theorem $H_{k+1}/K = X_V(E_1) \oplus X_V(E_2)$. This shows that every $x \in H_{k+1}$ can be written in a (not necessarily unique) form $x = x_1 + x_2$ with $x_j \in K \oplus X_V(E_j)$, $j = 1, 2$. Since $K \oplus X_v(E_1) \subseteq X_T(\bar{D}_{k+1})$ and $K \oplus X_V(E_2) \subseteq H_k$, it follows that $H_{k+1} = H_k + X_T(\bar{D}_{k+1})$ and thus by the induction hypotheses $H_{k+1} = X_T(\bar{D}_1) + X_T(\bar{D}_2) + \ldots + X_T(\bar{D}_{k+1})$. The proof of the lemma is complete.

For convenience we accept the following definition of a decomposable operator [**9**].

Definition. An operator T defined on a Banach space X is called decomposable if for every finite open covering $G_i (i = 1, 2, \ldots, n)$ of $\sigma(T)$ there exists a set of spectral maximal subspaces $Y_i (i = 1, 2, \ldots, n)$ of T such that

(a) $\sigma(T|Y_i) \subseteq \bar{G}_i$, $(i = 1, 2, \ldots, n)$,

(b) $X = Y_1 + Y_2 + \ldots + Y_n$.

Moreover, T is called strongly decomposable if its restriction to an arbitrary spectral maximal subspace is again decomposable.

THEOREM 1. *If T satisfies Condition* (II), *then T is strongly decomposable.*

Proof. Let $G_1, G_2, \ldots, G_n$ be an arbitrary finite open covering of $\sigma(T)$. For each point $\lambda \in \sigma(T)$ there exists an open disc D_λ with center λ such that $\bar{D}_\lambda \subseteq G_i$ for some i. Moreover, since $\sigma(T)\backslash J$ consists of isolated points of $\sigma(T)$ (Lemma 1), we can assume $\bar{D}_\lambda \cap \sigma(T) = \{\lambda\}$ if $\lambda \in \sigma(T)\backslash J$ and $J \cap \partial D_\lambda$

has two points if $\lambda \in J$. Now since $\sigma(T)$ is compact and $\sigma(T) \subseteq \bigcup D_\lambda$, there exists a finite collection $\{D_{ij} : j = 1, 2, \ldots, n_i, i = 1, 2, \ldots, n\}$ of the discs D_λ such that $\sigma(T) \subseteq \bigcup_{i,j} D_{ij}$ and $G_i \supseteq \bigcup_j D_{ij}$, $i = 1, 2, \ldots, n$. Moreover, if necessary, by a slight expansion of the discs we can assume the discs D_{ij}, $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, n$ satisfy the conditions of Lemma 8. Thus, by Lemma 8, $H = \sum_{i,j} X_T(\bar{D}_{ij}) = \sum_i Y_i$ where $Y_i = \sum_j X_T(\bar{D}_{ij}) = X_T(\bigcup_j \bar{D}_{ij})$ and $\sigma(T|Y_i) \subseteq \bar{G}_i$. This shows that T is decomposable. Since $\sigma(T)$ is nowhere dense, it follows from [**3**] that T is strongly decomposable. The theorem is proved.

In view of Lemma 3 we have the following corollary.

COROLLARY 2. (a) *If* $T^* - T \in C_p (1 \leqq p < \infty)$ *then* T *is strongly decomposable.*

(b) *If* $T^*T - I \in C_p (1 \leqq p < \infty)$ *and* $\sigma(T)$ *does not fill the unit disc then* T *is strongly decomposable.*

2. Examples and open problems. The following example shows that if $T^*T - I \in C_p$ and $\sigma(T)$ fills the unit disc then T may not be decomposable.

Example 1. Let $\{e_n : n = 0, \pm 1, \pm 2, \ldots\}$ be an orthonormal basis for a Hilbert space H, A be the bilateral shift $Ae_n = e_{n+1}$, and let B be the rank one operator defined by $Bx = -(x|e_0)e_1$, $x \in H$. Let $T = A + B$. Obviously $T^*T - I \in C_p$ for all $p \geqq 1$. However T is not decomposable because the restriction of T to the invariant subspace: span $\{e_n : n = 0, -1, -2, \ldots\}$ does not have the single-valued extension property [**6**, p. 10, 31]. (Note that if an operator has the single valued extension property, then so does its restriction to any invariant subspace.)

The next example shows that Corollary 2(a) is not true if $p = \infty$.

A closed set Δ is called a spectral set for a Hilbert space operator T if $||u(T)|| \leqq \sup \{|u(z)| : z \in \Delta\}$ for all rational functions u with poles off Δ. If Δ is a convex spectral set for T, then $(Tx|x) \in \Delta$ for, $||x|| = 1$ [**21**, Lemma 4, p. 5].

Example 2. Let $V : L^2(0, 1) \to L^2(0, 1)$ be the Volterra operator

$$Vf(x) = \int_0^x f(t).$$

Let $W = (I + V)^{-1}$, ϕ be the conformal mapping from the unit disc onto the set $\Delta_1 = \{re^{i\theta} : 0 \leqq r \leqq 1, 0 \leqq \theta \leqq \pi/4\}$, $\phi(1) = 0$, and let $A = \phi(W)$ [**22**, proof of Theorem 8, p. 143]. (Note that W is a non-unitary contraction with $\sigma(W) = \{1\}$ [**10**, Problem 150] and thus A is a quasinilpotent operator.) Let $T = A_1 \oplus A_2 \oplus \ldots$ on $H = L^2(0, 1) \oplus L^2(0, 1) \oplus \ldots$ where $A_n = g_n(A)$ and $g_n(re^{i\theta}) = r^{1/n}e^{i\theta/n}$ for $re^{i\theta} \in \Delta_1$, $n = 1, 2, \ldots$. It follows from [**18**, proof of Proposition 1] that the set $\Delta_n = \{re^{i\theta} : 0 \leqq r \leqq 1, 0 \leqq \theta \leqq \pi/4n\}$ is a spectral set for A_n, $\sigma(T) = [0, 1]$, and $X_T(\{0\})$ is not closed. (Actually we showed in [**18**] that $0 \in \sigma(T) \subseteq [0, 1]$; but since $X_T(\{0\})$ is dense in H and T has spectral radius 1, it follows from the Riesz decomposition theorem that $\sigma(T)$ cannot be disconnected.) In particular T cannot be a decomposable operator. We show that Im (T) is compact. Let π be the canonical mapping from the algebra $B(H)$ of bounded operators on H onto the Calkin algebra $B(H)/C_\infty$. Let p_n be a sequence of polynomials converging to ϕ uniformly on the (closed) unit disc. We have $\pi(W) = (\pi(I) + \pi(V))^{-1} = \pi(I)$ and $\pi(A) = \pi(\phi(W)) = \pi(\lim p_n(W)) = \lim \pi(p_n(W)) = \lim p_n(\pi(W)) = \lim p_n(\pi(I)) = \lim p_n(1)\,\pi(I) = 0$ because the unit disc is a spectral set for the contraction W. Therefore A, and by a similar argument $A_n (n = 2, 3, \ldots)$, are compact. Since Im $(T) =$ Im $(A_1) \oplus$ Im $(A_2) \oplus \ldots$ and $\|\text{Im } (A_n)\| \leqq \tan (\pi/4n)$, it follows that Im (T) is a compact operator. Thus T is a non-decomposable operator with $\sigma(T) = [0, 1]$ and $T^* - T \in C_\infty$.

It is stated (without proof) in a paper of Macaev [**14**, p. 975] that there are operators T with compact imaginary parts such that $\sigma(T|M) = \Delta$ for all invariant subspaces $M \neq \{0\}$ of T where Δ is a closed set having more than one point. This is another way to show that Theorem 1 is not true if $p = \infty$. (Example 2 is completely different and shows that $X_T(\{0\})$ may not be closed.)

Next we discuss some open problems.

Problem 1. If T satisfies Condition (I), then must T be decomposable?

In the following we suggest two methods to attack this problem.

(a) To show that Conditions (I) and (II) are equivalent.

(b) To show that $\|(z - T|M)^{-1}\|$ satisfies the growth condition (*) of the proof of Lemma 5 along J_1 whenever T satisfies Condition (I), M is a hyperinvariant subspace of T, and J_1 is a subarc of J such that $J_1 \cap (\sigma(T)\backslash J)^- = \emptyset$.

As for (a) the following theorem may prove useful.

THEOREM 2. *Conditions* (I) *and* (II) *are equivalent if p is replaced by* ∞.

Proof. Let T satisfy Condition (I) for $p = \infty$ and let M be an arbitrary hyperinvariant subspace of T. Let K be the space of all sequences (x_n) in H such that $x_n \to 0$ weakly, where H is the underlying Hilbert space. Let glim be a Banach generalized limit function defined on the space of all bounded sequences of complex numbers. Let $N = \{(x_n) \in K : \text{glim } \|x_n\| = 0\}$ and let $H^\wedge$ be the complement of the pre-Hilbert space [**5**] K/N. Every operator S on H has a unique well-defined representation $S^\wedge$ on $H^\wedge$ determined by $S^\wedge(x_n) = (Sx_n)$ for $(x_n) \in K/N$. The collection of all operators $S^\wedge$ is a C^*-algebra isomorphic to the Calkin algebra $B(H)/C_\infty$ and $A^\wedge = B^\wedge$ if and only if $A - B$ is compact [**5**]. It is easy to see that $M^\wedge$ is an invariant subspace of $T^\wedge$ and $T^\wedge|M^\wedge = (T|M)^\wedge$. Since $\sigma(T|M)^\wedge) \subseteq \sigma(T|N)$ and $T^\wedge$ is normal, it follows from the Putnam's inequality for hyponormal operators [**16**] that $\|V^*V - VV^*\| \leqq (1/\pi)$ area $(\sigma(V)) = 0$ where $V = T^\wedge|M^\vee$. Thus $(T|M)^\wedge =$

$T^\wedge|M^\wedge$ is normal and $\sigma((T|M)^\wedge) \subseteq \sigma(T^\wedge) \subseteq J$. Since $\sigma(T|M)$ is nowhere dense, index $(\lambda - T|M) = 0$ for all $\lambda \notin J$. Thus by [**4**, Theorem 11.1, p. 118] $T|M$ is the sum of a normal operator with spectrum on J and a compact operator. A similar verification for $T^*|M^\perp$ completes the proof of the theorem.

Theorem 2 remains true if J is replaced by an arbitrary closed set of zero area. Lemma 3 above gives some special cases where Conditions (I) and (II) are equivalent for all $1 \leqq p \leqq \infty$.

To see that (b) works, note that the conclusion of (b) is all we need in proving Lemmas 6–8 and Theorem 1. Also it can be seen that if J is a C^2-Jordan curve, A is an operator satisfying

$$\sup \{[\text{dist}(z, J)]^n \|(z - A)^{-1}\| : z \notin J\} < \infty$$

for some positive integer n, and if B is an operator in $C_p(1 \leqq p < \infty)$ such that $(\sigma(A + B)\backslash J)^- \cap J$ is nowhere dense in J, then $T = A + B$ is decomposable. The proof follows from the fact that (i) $\|(z - T)^{-1}\|$ satisfies the growth condition (*) of the proof of Lemma 5 at each point a of J which is not an accumulation point of $\sigma(T)\backslash J$ [**2**, proof of Theorem 3.5; **17**, proof of Corollary 3], and (ii) the discs D_λ in the proof of Theorem 1 can be chosen such that $(\sigma(T)\backslash J)^- \cap J \cap \partial D_\lambda = \emptyset$. The second assertion allows us to assume in Lemmas 6, 7, 8 and their proofs that $(\sigma(T)\backslash J)^- \cap J \cap \partial\Gamma = \emptyset$ where Γ stands for J_1 (Lemma 6), $G, G_1, \ldots, g_1, g_2, \ldots$ (Lemma 7) and $D_1, D_2, \ldots, D_n$ (Lemma 8). For the proof in case $\sigma(T) = \sigma(A + B) \subseteq J$ see [**17**, Corollary 3].

Corollary 2 gives a new class of concrete examples of decomposable operators which are like other known ones strongly decomposable [**6**, p. 217]. It seems that this new class of decomposable operators is the only one in which the question of $\mathfrak{A}$-spectrality is not answered [**6**, p. 78, 217]. We mention that if T satisfies the conditions of part (a) (respectively (b)) of Corollary 2 and $\sigma(T)$ is on the real line (respectively unit circle) then T is an $\mathfrak{A}$-selfadjoint (respectively $\mathfrak{A}$-unitary) operator [**6**, Theorem 5.2, p. 166]. The following problem is a special case of [**6**, Problem 5 (c), p. 217].

Problem 2. If T satisfies Condition (II), then must T be an $\mathfrak{A}$-spectral operator?

Let us mention that if T satisfies the part (a) of Condition (I) and T is reductive, then T is a normal operator [**15**, Theorem 3.2].

References

1. C. Apostol, *Spectral decomposition and functional calculus*, Rev. Roumaine Math. Pures Appl. *13* (1968), 1481–1528.

2. ——— *On the growth of resolvent, perturbation and invariant subspaces*, Rev. Roumaine Math. Pures Appl. *16* (1971), 161–172.

3. I. Bacalu, *On restrictions and quotients of decomposable operators*, Rev. Roumaine Math. Pures Appl. *18* (1973), 809–813.

4. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, Lecture notes in mathematics #345, (Springer-Verlag, 1973), 58–128.

5. J. W. Calkin, *Two sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. *42* (1941), 839–873.

6. I. Colojoara and C. Foias, *The theory of generalized spectral operators* (Gordon Breach, Science Publ., New York, 1968).

7. N. Dunford and J. Schwartz, *Linear operators, I* (Interscience, New York, 1958).

8. ——— *Linear operators, III* (Interscience, New York, 1971).

9. C. Foias, *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math. (Basel) *14* (1963), 341–349.

10. P. R. Halmos, *A Hilbert space problem book* (D. Van Nostrand Co., Princeton, 1967).

11. A. A. Jafarian, *Weak contractions of Sz.-Nagy and Foias are decomposable*, Rev. Roumaine Math. Pures Appl. (to appear).

12. A. A. Jafarian and F. H. Vasilescu, *A characterization of 2-decomposable operators*, Rev. Roumaine Math. Pures Appl. *19* (1974), 769–771.

13. K. Kitano, *Invariant subspaces of some non-selfadjoint operators*, Tôhoku Math. J. *20* (1968), 313–322.

14. V. I. Macaev, *A class of completely continuous operators*, Soviet Math. Dokl. *2* (1961), 972–975.

15. E. Nordgren, H. Radjavi, and P. Rosenthal, *On operators with reducing invariant subspaces*, Amer. J. Math. (to appear).

16. C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. *116* (1970), 323–330.

17. M. Radjabalipour, *Growth conditions and decomposable operators*, Can. J. Math. *26* (1974), 1372–1379.

18. ——— *On decomposition of operators*, Michigan Math. J. *21* (1974), 265–275.

19. H. Radjavi and P. Rosenthal, *Invariant subspaces* (Springer Verlag, Berlin, 1973).

20. J. Schwartz, *Subdiagonalization of operators in Hilbert space with compact imaginary part*, Comm. Pure Appl. Math. *15* (1962), 159–172.

21. J. G. Stampfli, *A local spectral theory for operators*, J. Functional Analysis *4* (1969), 1–10.

22. ——— *A local spectral theory for operators, III; Resolvents, spectral sets and similarity*, Trans. Amer. Math. Soc. *168* (1972), 133–151.

23. ——— *A local spectral theory for operators, IV; Invariant subspaces*, Indiana Univ. Math. J. *22* (1972), 159–167.

24. B. Sz.-Nagy and C. Foias, *Décomposition spectrale des contractions presque unitaires*, C. R. Acad. Sci. Paris Sér. A-B *262* (1966), 440–442.

25. ——— *Harmonic analysis of operators on Hilbert space* (North Holland, Amsterdam, 1970).

Dalhousie University,
Halifax, Nova Scotia