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AN ELEMENTARY PROOF OF A THEOREM ON UNILATERAL DERIVATIVES

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ABSTRACT. We give an elementary proof of the theorem of Saks that states that most functions in $C[0, 1]$ have infinite unilateral derivatives at a continuum many points.

Let $C[0, 1]$ be the set of all continuous real valued functions on the unit interval $[0, 1]$ and endow $C[0, 1]$ with the sup metric. Then $C[0, 1]$ is a complete metric space. S. Saks [2] proved by sophisticated means that most functions in $C[0, 1]$ have infinite unilateral derivatives at a continuum many points in $[0, 1]$. (By "most functions have a property" we mean that the set of all functions that do not have the property is a first category subset of $C[0, 1]$.) A simpler proof of Saks' Theorem [1] is attributed to D. Preiss, but his argument employed approximate derivatives and functions approximately derivable almost everywhere. In this note we give an easy elementary proof that does not require approximate differentiation, and indeed requires only a casual acquaintance with ordinary derivatives and Dini derivates. Unlike [1], our arguments do not require the fact that most functions in $C[0, 1]$ are nowhere approximately derivable. Our arguments will also give an elementary proof of the well known fact that most functions in $C[0, 1]$ have knot points almost everywhere. (By a "knot" point of a function f we mean a point x where $D^+f(x) = D^-f(x) = \infty$ and $D_+f(x) = D_-f(x) = -\infty$.)

In the 1930s it was shown that most functions in $C[0, 1]$ had no finite or infinite two-sided derivative at any point, and had no finite unilateral derivative at any point. Analysts attempted to prove that most functions had no infinite unilateral derivative at any point until Saks [2] proved that this was not true. His argument was complicated and his methods were deep. The recent proof cited in [1] is simpler, but it also requires deep methods. Our argument is simple, and students familiar with Baire category should be able to follow it.

Let F denote the family of all functions $f \in C[0, 1]$ for which $m(A_f) > 0$ where m denotes Lebesgue measure and $A_f = \{x \in (0, 1): D^+f(x) < 0\}$. Fix $f \in F$. Then there is a rational number $d > 0$ such that the closed set $A = \{x \in [0, 1): f(x) =$

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$\sup f[x, x+d]\}$ satisfies $m(A) > 0$. Let $U \supset A$ be an open set with $m(A) > 3m(u)/4$. Then there is a component interval I of U such that $m(A \cap I) > 3m(I)/4$. By altering the endpoints of I if necessary, we can produce rational numbers a and b such that $m(A \cap (a, b)) \geq \frac{1}{2}(b-a)$.

For rational numbers a, b and d ($a < b$, $d > 0$), we let $F(a, b, d)$ denote the set of all functions $g \in C[0, 1]$ such that

$$m\{x \in (a, b): g(x) = \sup g[x, x+d]\} \geq \tfrac{1}{2}(b-a).$$

LEMMA 1. *F is a first category subset of $C[0, 1]$.*

PROOF. In view of the preceding remarks, it suffices to prove that $F(a, b, d)$ is a closed nowhere dense subset of $C[0, 1]$. To show that it is closed, let $g_n \in F(a, b, d)$ such that $\lim g_n = g \in C[0, 1]$ uniformly, and put

$$E_n = \{x \in (a, b): g_n(x) = \sup g_n[x, x+d]\}.$$

Then $m(E_n) \geq \frac{1}{2}(b-a)$ for all n. Put $E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$. Then $m(E) \geq \frac{1}{2}(b-a)$. If $x \in E$, then $x \in E_j$ and $g_j(x) = \sup g_j[x, x+d]$ for infinitely many j and consequently $g(x) = \sup g[x, x+d]$. Thus

$$E \subset \{x \in (a, b): g(x) = \sup g[x, x+d]\}$$

and $g \in F(a, b, d)$.

To prove that $F(a, b, d)$ is nowhere dense, fix $f \in C[0, 1]$ and $\epsilon > 0$. Let $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ be a partition of the interval $[a, b]$ such that $t_j - t_{j-1} < d$ and such that $\sup f[t_{j-1}, t_j] - \inf f[t_{j-1}, t_j] < \frac{1}{4}\epsilon$ for all $j = 1, \ldots, n$. We construct a nonnegative function $h \in C[0, 1]$ such that $\sup h = \frac{1}{2}\epsilon$, $h(t_j) = \frac{1}{2}\epsilon$ for all j, and $m(X) > \frac{1}{2}(b-a)$ where $X = h^{-1}(0) \cap (a, b)$.

Now for any $x \in X$ satisfying $t_{j-1} < x < t_j$ we have $t_j \in [x, x+d]$ and $(f+h)(x) = f(x) + h(x) = f(x) < f(t_j) + \frac{1}{4}\epsilon < f(t_j) + h(t_j) = (f+h)(t_j)$. So

$$X \subset \{x \in (a, b): (f+h)(x) < \sup (f+h)[x, x+d]\}.$$

Since $m(X) > \frac{1}{2}(b-a)$, it follows that $f + h \notin F(a, b, d)$. But $|(f+h) - f| = h < \epsilon$. This completes the proof.

Thus most $f \in C[0, 1]$ satisfying $m(A_f) = 0$. The set of functions in $C[0, 1]$, monotone on an interval (a, b) (a, b rational), is obviously closed and nowhere dense in $C[0, 1]$, so most $f \in C[0, 1]$ are nowhere monotone and satisfy $m(A_f) = 0$. At this juncture we are essentially through, because a standard elementary theorem [1] states that such a function f satisfies $f'_+(x) = -\infty$ at a continuum many x in any subinterval J of $[0, 1]$. For the sake of completeness, and to show that all our arguments are elementary, we sketch a proof here.

Let $U_n \supset A_f$ be an open set with $m(U_n) < 2^{-n}$. Let F be the increasing function

$$F(x) = x + \sum_{n=1}^{\infty} \int_0^x \chi_{U_n}(t)\, dt$$

where χ denotes characteristic function. By direct computation we see that $D_+F(x) \geq 1$ for all x, and $F'(x) = \infty$ for all $x \in A_f$. Now choose $r, s \in J$ such that $r < s$ and $f(r) > f(s)$. Choose $k > 0$ so small that $(f + kF)(r) > (f + kF)(s)$, and let $(f + kF)(r) > y > (f + kF)(s)$. Let x_0 be the largest $x \in (r, s)$ for which $(f + kF)(x) = y$. Clearly $D^+(f + kF)(x_0) \leq 0$. Now $D_+F(x_0) \geq 1$, so $D^+f(x_0) \leq -k < 0$ and $x_0 \in A_f$. Hence $F'(x_0) = \infty$ and it follows from this and $D^+(f + kF)(x_0) \leq 0$ that $D^+f(x_0) = -\infty$. Thus $f'_+(x_0) = -\infty$ and $(f + kF)(x_0) = y$. It follows that the set

$$\{x \in J : f'_+(x) = -\infty\}$$

has as many points as there are real numbers between $(f + kF)(r)$ and $(f + kF)(s)$.

Finally, it is clear that most functions $f \in C[0, 1]$ satisfy $m\{x : D^+f(x) < n\} = 0$ for any integer n (just use the homeomorphism $f(x) \rightarrow f(x) - nx$ of $C[0, 1]$ onto $C[0, 1]$). Thus most $f \in C[0, 1]$ satisfy $D^+f = \infty$ almost everywhere. Clearly most $f \in C[0, 1]$ have knot points almost everywhere.

References

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