

## UNIONS OF WELL-ORDERED SETS

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### Abstract

In Zermelo-Fraenkel set theory weakened to permit the existence of atoms and without the axiom of choice we investigate the deductive strength of five statements which make assertions about the cardinality of the union of a well-ordered collection of sets. All five of the statements considered are consequences of the axiom of choice.

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### 1. Introduction

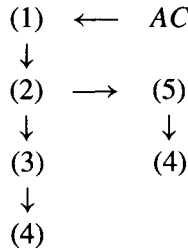
We will work in Zermelo-Fraenkel set theory without the axiom of choice ( $ZF$ ) or in  $ZF$  weakened to permit the existence of atoms ( $ZFU$ ). Our concern will be with the deductive strength of several theorems of the theory  $ZF + AC$  (that is, provable under the assumption of the axiom of choice) which make assertions about the cardinality of the union of a family of well-ordered sets. In general  $UT(\textit{condition 1}, \textit{condition 2}, \textit{condition 3})$  will denote the sentence which asserts that whenever  $X$  satisfies condition 1 and each element of  $X$  satisfies condition 2, then  $\bigcup X$  satisfies condition 3. If  $d$  is a cardinal number then the condition *has cardinal number*  $d$  will be denoted  $d$ . In addition  $WO$  will stand for the condition *is well-orderable*. This is the notation of [3].

The theorems of  $ZF + AC$  whose deductive strength we shall study are

- (1)  $(\forall$  infinite cardinals  $d$ )  $UT(d, d, d)$

- (2)  $(\forall \text{ ordinals } \alpha) UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$
- (3)  $(\forall \text{ ordinals } \alpha) UT(\aleph_0, \aleph_\alpha, \aleph_\alpha)$
- (4)  $UT(\aleph_0, \aleph_0, \aleph_0)$
- (5) If  $\{A_n : n \in \omega\}$  and  $\{B_n : n \in \omega\}$  are disjoint families of sets with  $|A_n| = |B_n|$  for all  $n \in \omega$  then  $|\bigcup_{n \in \omega} A_n| = |\bigcup_{n \in \omega} B_n|$

Theorem (1) was considered by Sierpinski in [7] while Theorem (3), Theorem (4) and Theorem (5) were first employed by Cantor. (See [5, p. 9, 36] and [2, p. 117].) All five of these statements appear in [5, p. 324] as part of an implication diagram in the following relative positions

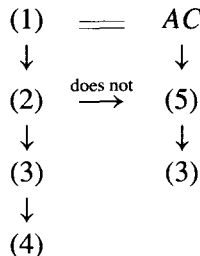


We first want to point out that (5) implies AC is not provable in ZF since the axiom of choice for countable collections of sets implies (5) but does not imply AC, [5, p. 324, 325].

Secondly, the fact that the implication (4) implies (3) is not provable in ZF follows from the results of [1] where it is shown that for any ordinal  $\beta$ ,  $UT(\aleph_\beta, \aleph_\beta, \aleph_\beta)$  does not imply the axiom of choice for countable collections of sets each of cardinality  $\aleph_{\beta+1}$ . Since  $UT(\aleph_0, \aleph_{\beta+1}, \aleph_{\beta+1})$  clearly implies the choice principle mentioned above, we obtain the fact that  $UT(\aleph_\beta, \aleph_\beta, \aleph_\beta)$  implies  $UT(\aleph_0, \aleph_{\beta+1}, \aleph_{\beta+1})$  is not provable in ZF.

Thirdly, (1) implies that for every infinite set A,  $|A \times A| = |A|$  which is known to be equivalent to AC (see [6, p. 140, CN 6]).

In the following section of the paper we will prove that (5) implies (3) and that the implications (2) implies (5) and (3) implies (2) cannot be proved in the theory ZFU. This will give the following diagram, where none of the downward arrows can be reversed.



The problems remaining unsolved are

PROBLEM 1. Does  $ZF \vdash (5) \rightarrow (2)$ ?

PROBLEM 2. Can the results  $ZFU \not\vdash (2) \rightarrow (5)$  and  $ZFU \not\vdash (3) \rightarrow (2)$  be strengthened to  $ZF \not\vdash (2) \rightarrow (5)$  and  $ZF \not\vdash (3) \rightarrow (2)$ ?

(We suspect that the implication  $(2) \rightarrow (5)$  from the first diagram is a misprint in [5] since no references to the result are given. We have therefore included the question of whether or not  $ZF \vdash (2) \rightarrow (5)$  as an open problem.)

## 2. The Proofs

THEOREM 2.1.  $(5)$  implies  $(3)$ .

PROOF. Assume that  $\{A_n : n \in \omega\}$  is a countable family of sets such that  $(\forall n \in \omega)(|A_n| = \aleph_\alpha)$ . By  $(5)$  with  $B_n = \{(n, \beta) : \beta < \aleph_\alpha\}$  we obtain  $|\bigcup_{n \in \omega} A_n| = |\aleph_0 \times \aleph_\alpha| = \aleph_\alpha$  by [4, p. 29, Theorem 10.12].

THEOREM 2.2.  $ZFU \not\vdash (3) \rightarrow (2)$ .

PROOF. Let  $M'$  be a model of  $ZFU + AC$  with a set  $A$  of atoms of cardinality  $\aleph_1$ . Let  $a$  be a fixed bijection from  $\aleph_1 \times \aleph_1$  onto  $A$  ( $a \in M'$ ). We use  $a$  to partition  $A$  into  $\aleph_1$  blocks each size  $\aleph_1$  as follows: For each  $\alpha < \aleph_1$  let  $A_\alpha = \{a(\alpha, \beta) : \beta < \aleph_1\}$ ;  $A_\alpha$  is called the  $\alpha^{\text{th}}$  block. Let  $G$  be the group of all permutations of  $A$  which fix the set of blocks, that is

$$G = \{ \phi : A \xrightarrow[\text{onto}]{} A : (\forall \alpha < \aleph_1)(\phi(A_\alpha) = A_\alpha) \}.$$

For each countable subset  $E$  of  $\aleph_1$ , let

$$G(E) = \{ \phi \in G : (\forall \alpha \in E)(\phi \text{ fixes } A_\alpha \text{ pointwise}) \}$$

and let  $\Gamma$  be the filter of subgroups generated by  $\{G(E) : E \subseteq \aleph_1 \wedge |E| = \aleph_0\}$ . Finally, let  $M$  be the permutation model determined by  $M'$  and  $\Gamma$ .

Theorem  $(2)$  fails in  $M$  since  $\{A_\alpha : \alpha < \aleph_1\}$  is a family of  $\aleph_1$  sets each of cardinality  $\aleph_1$  whose union is not well-ordered in  $M$ .  $(3)$  holds in  $M$ , for suppose that  $\{B_n : n \in \omega\}$  is a countable collection of sets each of cardinality  $\aleph_\alpha$ . For each  $n \in \omega$ , assume  $E_n \subseteq \aleph_1$ ,  $E_n$  is countable and  $G(E_n)$  fixes a

well-ordering of  $B_n$ . Then  $G(E_n)$  fixes  $B_n$  pointwise so if we let  $E = \bigcup_{n \in \omega} E_n$  then  $E \subseteq \aleph_1$ ,  $E$  is countable and  $G(E)$  fixes  $\bigcup_{n \in \omega} B_n$  pointwise and therefore, since  $|\bigcup_{n \in \omega} B_n| = \aleph_\alpha$  in  $M'$ ,  $|\bigcup_{n \in \omega} B_n| = \aleph_\alpha$  in  $M$ .

**THEOREM 2.3.**  $ZFU \not\vdash (2) \rightarrow (5)$ .

**PROOF.** We construct a permutation model with the idea in mind that we want (5) to fail.

Assume  $M'$  is a model of  $ZFU + AC$  with a countable set of atoms  $A$ . Write  $A$  as a disjoint union  $A = (\bigcup_{i \in \omega} A_i) \cup (\bigcup_{i \in \omega} B_i)$  where for all  $i \in \omega$ ,  $|A_i| = |B_i| = \aleph_0$ . Let  $f_i$  be a fixed bijection between  $A_i$  and  $B_i$  for each  $i \in \omega$ . The idea is to construct the model  $M$  so that for each  $i \in \omega$ ,  $f_i$  is in  $M$  but  $\bigcup_{i \in \omega} f_i$  is not in  $M$ . Let

$$G = \left\{ \phi : \begin{array}{l} \phi \text{ is a permutation of } A \\ \wedge (\forall i \in \omega)(\phi(A_i) = A_i \wedge \phi(B_i) = B_i) \\ \wedge \{a \in A : \phi(a) \neq a\} \text{ is finite} \\ \wedge (\forall i \in \omega)(\phi \upharpoonright A_i \text{ and } \phi \upharpoonright B_i \text{ are even permutations}) \end{array} \right\}.$$

For each finite subset  $x \subseteq \omega$  and finite  $E \subseteq A$  let

$$G(x, E) = \{ \phi \in G : (\forall a \in E)(\phi(a) = a) \wedge (\forall i \in x)(\phi(f_i) = f_i) \}$$

(where  $\phi(f_i) = f_i$  means  $(\forall a \in A_i)(\phi(f_i(a)) = f_i(\phi(a)))$ ). Let  $\Gamma$  be the filter of subgroups of  $G$  generated by the groups  $G(x, E)$  where  $x \subseteq \omega$ ,  $E \subseteq A$  and  $x$  and  $E$  are finite and let  $M$  be the permutation model determined by  $M'$  and  $\Gamma$ .

Then  $\{A_i : i \in \omega\}$  and  $\{B_i : i \in \omega\}$  are in  $M$  and are denumerable in  $M$ . Further  $(\forall i \in \omega)(|A_i| = |B_i|)$  in  $M$  since  $G(\{i\}, \emptyset)$  fixes  $f_i$ . However it is clear that there is no one to one function from  $\bigcup_{i \in \omega} A_i$  onto  $\bigcup_{i \in \omega} B_i$  in  $M$ . (Assuming  $G(x, E)$  fixes such a function, consider an  $a \in A - [E \cup (\bigcup_{i \in x} (A_i \cup B_i))]$ .) So (5) is false in  $M$ .

To show that (2) is true in  $M$ , it is enough to show that the union of a well-ordered collection of well-orderable sets in  $M$  is well-orderable in  $M$  since this implies (2) in all permutation models.

Assume  $Z \in M$  is well ordered in  $M$  and that  $G(x, E)$  fixes  $Z$  pointwise. We assume without loss of generality that

$$(6) \quad (\forall i \in \omega)(\forall a \in A_i)(a \in E \iff f_i(a) \in E).$$

Also assume that each element of  $Z$  is well-orderable in  $M$ . We will show that  $G(x, E)$  fixes  $\bigcup Z$  pointwise. Therefore assume  $t \in y \in Z$ . We will show that  $G(x, E)$  fixes  $t$ . By our assumption,  $G(x, E)$  fixes  $y$  and for some  $x' \subseteq \omega$  and  $E' \subseteq A$ ,  $G(x', E')$  fixes  $y$  pointwise. We may assume without loss of generality that  $x \subseteq x'$  and  $E \subseteq E'$ .

LEMMA 2.4. *If  $i \in x' - x$  and there is a  $\phi \in G(x, E)$  such that*

- (i)  $(\forall a \in A)(a \notin A_i \rightarrow \phi(a) = a)$  and
- (ii)  $\phi(t) \neq t$

*then  $\exists \psi \in G(x', E)$  and  $\exists t' \in y$  such that*

- (iii)  $(\forall a \in A)(a \notin A_i \cup B_i \rightarrow \psi(a) = a)$  and
- (iv)  $\psi(t') \neq t'$ .

PROOF. Assume that  $\phi$  satisfies conditions (1) and (2) and that the conclusion is false. Then for every  $\psi \in G(\emptyset, E)$  if

- (a)  $(\forall a \in A)(a \notin A_i \cup B_i \rightarrow \psi(a) = a)$  and
- (b)  $(\forall a \in A_i)(\psi(f_i(a)) = f_i(\psi(a)))$

then  $(\forall t' \in y)(\psi(t') = t')$  and in particular,  $\psi(t) = t$ .

If we let  $K = \{ \psi \in G(\emptyset, E) : \text{(a) and (b) hold} \}$  then we could rephrase our assumption as  $(\forall \psi \in K)(\forall t' \in y)(\psi(t') = t')$

Let  $H = \{ \eta : \eta \in G(x, E) \wedge (\forall a \in A)(a \notin A_i \rightarrow \eta(a) = a) \}$ . We identify  $H$  with the group of (finite) even permutations of  $A_i - E$  and we let  $H_t = \{ \eta \in H : \eta(t) = t \}$ . By hypothesis  $H_t \neq H$ . Further  $H_t$  is a normal subgroup of  $H$  for suppose  $\sigma \in H$  and  $\eta \in H_t$ . We argue that  $\sigma \eta \sigma^{-1} \in H_t$ , that is that  $\sigma \eta \sigma^{-1}(t) = t$ , as follows: Let  $\psi$  be defined by

$$\psi(a) = \begin{cases} \sigma(a), & a \in A_i \\ f_i(\sigma(f_i^{-1}(a))), & a \in B_i \end{cases}$$

so that  $\forall a \in A_i, \psi(f_i(a)) = f_i(\sigma(f_i^{-1}(f_i(a)))) = f_i(\sigma(a)) = f_i(\psi(a))$  so  $\psi$  satisfies (b).  $\psi$  clearly satisfies (a) so  $\psi(t) = t$ . Also  $\sigma \eta \sigma^{-1}(a) = \psi(a)$  for all  $a \in A$  (by considering two cases  $a \in A_i$  and  $a \in B_i$ ) so  $\sigma \eta \sigma^{-1}(t) = \psi(t) = t$ .

Since the alternating group on an infinite set has no non-trivial, proper, normal subgroups and  $H_t \neq H, H_t = \{ 1 \}$  where 1 denotes the identity permutation.

Assume now that  $\psi \in G(\emptyset, E)$  and  $\psi$  satisfies  $(\forall a \in A)(a \notin A_i \cup B_i \rightarrow \psi(a) = a)$ . We first want to show

(\*)  $\psi(t) = t \iff \psi \in K.$

By assumption  $\psi \in K \rightarrow \psi(t) = t$ . Assume  $\psi(t) = t$  and  $\psi \notin K$ . Then for some  $a_0 \in A_i$ ,  $\psi(f_i(a_0)) \neq f_i(\psi(a_0))$ . Define  $\psi' \in G(\emptyset, E)$  by

$$\psi'(a) = \begin{cases} \psi^{-1}(a), & a \in B_i \\ f_i^{-1}\psi^{-1}f_i(a), & a \in A_i \\ a, & \text{otherwise.} \end{cases}$$

We first note that  $\psi'$  fixes  $E$  pointwise since  $\psi$  does. (This uses our assumption (6).)  $\psi'$  clearly satisfies (a). We argue that  $\psi'$  satisfies (b) as follows. Suppose  $a \in A_i$ . Then  $a = f_i^{-1}(a')$  for some  $a'$  in  $B_i$ . Therefore

$$\psi'(a) = f_i^{-1}\psi^{-1}f_i f_i^{-1}(a') = f_i^{-1}\psi^{-1}(a') = f_i^{-1}\psi^{-1}f_i(a).$$

Hence  $f_i(\psi'(a)) = \psi^{-1}(f_i(a)) = \psi^{-1}(f_i(a))$ . Since  $\psi'$  satisfies (a) and (b) we conclude that  $\psi' \in K$  and therefore that  $\psi'(t) = t$ . Furthermore  $\psi'\psi(a) = a$  for all  $a \in B_i$ . Since  $\psi$  and  $\psi'$  are both equal to the identity outside of  $A_i \cup B_i$ , it follows that  $\psi'\psi \in H$ . Now  $\psi'\psi$  is not the identity since

$$\psi'\psi(a_0) = f_i^{-1}\psi^{-1}f_i\psi(a_0) \neq f_i^{-1}\psi^{-1}\psi f_i(a_0) = a_0.$$

So  $\psi'\psi \notin H_i$ , that is,  $\psi'\psi(t) \neq t$ . But this contradicts  $(\psi(t) = t \wedge \psi'(t) = t)$ . This establishes (\*).

By our assumption,  $\phi \in H$  and  $\phi(t) \neq t$ . Therefore for some  $a_0 \in A_i$ ,  $\phi(a_0) \neq a_0$ . We claim there exists  $\psi \in K$  such that

$$(**) \quad \phi\psi\phi^{-1} \notin K.$$

Choose  $a_1 \in A_i - E$  such that  $\phi(a_1) = a_1$  and let  $\psi = (a_0, a_1)(f_i(a_0), f_i(a_1))$  (the product of two disjoint cycles). Clearly  $\psi \in K$ . We also note that  $\psi\phi^{-1}(a_0) \neq a_1$  since  $\psi(a_0) = a_1$  and  $\phi^{-1}(a_0) \neq a_0$ . Now we calculate

$$\phi\psi\phi^{-1}(f_i(a_0)) = \phi\psi(f_i(a_0)) = \phi(f_i(a_0)) = f_i(a_1)$$

and

$$f_i(\phi\psi\phi^{-1}(a_0)) \neq f_i(\phi(a_1)) = f_i(a_1)$$

showing  $\phi\psi\phi^{-1} \notin K$ .

Now the proof of Lemma 2.4 is completed by observing that  $t' = \phi^{-1}(t) \in y$  and that  $\psi(\phi^{-1}(t)) \neq \phi^{-1}(t)$  (since otherwise  $\phi\psi\phi^{-1}(t) = t$ , and by (\*),  $\phi\psi\phi^{-1} \in K$  contradicting (\*\*).)

Now continuing with the proof that  $G(x,E)$  fixes  $t$ , assume that there is some  $\phi \in G(x, E)$  such that  $\phi(t) \neq t$ .

LEMMA 2.5.  $(\exists \psi \in G(x', E))(\exists t' \in y)(\psi(t') \neq t')$ .

PROOF. To see this we first write  $\phi = \phi_1\phi_2$  where

$$\begin{aligned} \phi_1(a) &= \begin{cases} \phi(a), & \text{if } (\exists i \in x')(a \in A_i \cup B_i), \\ a, & \text{otherwise;} \end{cases} \\ \phi_2(a) &= \begin{cases} a, & \text{if } (\exists i \in x')(a \in A_i \cup B_i), \\ \phi(a), & \text{otherwise.} \end{cases} \end{aligned}$$

If  $\phi_2(t) \neq t$  then  $\psi = \phi_2$  will work. Therefore we assume that  $\phi_2(t) = t$ . Then  $\phi_1(t) \neq t$  since  $\phi(t) \neq t$ . For each  $i \in x'$ , let

$$\eta_i(a) = \begin{cases} \phi_1(a), & \text{if } a \in A_i \cup B_i, \\ a, & \text{if } a \notin A_i \cup B_i. \end{cases}$$

Then  $\phi_1 = \prod_{i \in x'} \eta_i$ . Therefore for at least one  $i_0 \in x'$ ,  $\eta_{i_0}(t) \neq t$ . We can write  $\eta_{i_0} = \sigma_1\sigma_2$  where

$$\begin{aligned} \sigma_1(a) &= \begin{cases} \eta_{i_0}(a), & \text{if } a \in A_{i_0}, \\ a, & \text{otherwise;} \end{cases} \\ \sigma_2(a) &= \begin{cases} \eta_{i_0}(a), & \text{if } a \in B_{i_0}, \\ a, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that  $\sigma_1(t) \neq t$  or  $\sigma_2(t) \neq t$ . Assume without loss of generality that  $\sigma_1(t) \neq t$ . (The case  $\sigma_2(t) \neq t$  would require a version of Lemma 2.4 obtained by interchanging  $A_i$  and  $B_i$ .) Applying Lemma 2.4 with  $\phi = \sigma_1$  gives us the desired conclusion. This completes the proof of Lemma 2.5.

Assume that  $\psi$  and  $t'$  satisfy the conditions of Lemma 2.5. Let  $E'' = \{a \in A : \psi(a) \neq a\}$  and choose  $\eta \in G(x', E)$  such that  $\eta(E'') \cap E' = \emptyset$ . Then  $\eta(t') \in y$  since  $\eta \in G(x', E) \subseteq G(x, E)$ . Let  $\sigma = \eta\psi\eta^{-1}$ . Then

$$(7) \quad \sigma \in G(x', E').$$

( $\sigma$  is clearly in  $G(x', E)$  since  $\eta$  and  $\psi$  are. We need only show  $\sigma(a) = a$  for all  $a \in E'$ . If  $a \in E'$  then  $\eta^{-1}(a) \notin E''$ , so  $\psi(\eta^{-1}(a)) = \eta^{-1}(a)$ , and so  $\eta\psi\eta^{-1}(a) = a$ .)

$$(8) \quad \sigma(\eta(t')) \neq \eta(t').$$

(For if not,  $\eta\psi\eta^{-1}(\eta(t')) = \eta(t')$ , that is,  $\eta\psi(t') = \eta(t')$ , giving  $\psi(t') = t'$ .)

(7) and (8) contradict our choice of  $x'$  and  $E'$ , completing the proof of Theorem 2.3.

## References

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