Proceedings of the Edinburgh Mathematical Society (1976-77), 20, 225-228 ©

ON A PROBLEM ABOUT CYCLIC SUBGROUPS OF FINITE GROUPS

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(Received 16th May 1975)

1. Introduction

Let G be a finite group and let S be a subgroup of G with core $(S) = \bigcap_{x \in G} x^{-1}Sx = 1$.

We say that (G, S) has property $(*)$ if there exists $x \in G$ such that $S \cap x^{-1}Sx = 1$.

Conditions on G, S which ensure that (G, S) has property $(*)$ have been found by N. Ito (**4**), J. S. Broadkey (**2**), M. Herzog (**3**) and T. J. Laffey (**5**). In particular, T. J. Laffey has considered the case where S is cyclic and has shown that (G, S) has property $(*)$ if G is simple and conjectured that (G, S) has property $(*)$ if $S \cap F(G) = 1$.

In this note we consider again the case where S is cyclic. We show that (G, S) has property $(*)$ if $F(G) = 1$ from which the aforementioned result of T. J. Laffey follows.

Our notation is as in (**5**). We recall that $F(G)$ denotes the maximal nilpotent normal subgroup of G; an E-group is the product of a certain set of its subgroups $E_1, \ldots, E_n$ satisfying $[E_i, E_i] = E_i$, $E_i/Z(E_i)$ is simple and $[E_i, E_j] = 1$ $(1 \leq i, j \leq n, i \neq j)$; and, finally, $F^*(G)$ denotes the generalised Fitting subgroup of G, namely $F^*(G) = F(G) \cdot E(G)$ where $E(G)$ is the (unique) maximal normal E-subgroup of G. $F^*(G)$ contains its centraliser in G (cf. (**1**)).

2. The proof of the theorem

An error in part III of my original proof was kindly pointed out to me by Thomas J. Laffey together with the idea of the alternative part III given here.

Theorem. *Let G be a finite group with $F(G) = 1$ and let S be a cyclic subgroup of G. Then there exists $x \in G$ such that $S \cap x^{-1}Sx = 1$.*

Proof. By induction on $|G| + |S|$.

(I) We may assume that $|S|$ is square free, i.e. $S = S_1 \times \cdots \times S_r$ with $|S_i| = p_i$ prime, $p_i \neq p_j$ for $i \neq j$ $(1 \leq i, j \leq r)$ and $r > 1$.

Because S is cyclic and so it contains a subgroup $S_0 = S_1 \times \cdots \times S_r$ as claimed and $S_0 \cap x^{-1}S_0x = 1$ if and only if $S \cap x^{-1}Sx = 1$. If $S_0 \neq S$ we have the result by induction. Also if $r=1$ the result follows from our assumption that $F(G)=1$.

(II) We may assume $G=(N_1\times\cdots\times N_t)S$ where $N_1,\ldots,N_t$ are the minimal normal subgroups of G.

Because, since $F(G)=1$, it follows that the generalised Fitting subgroup of G is $N_1\times\cdots\times N_t$ where $N_1,\ldots,N_t$ are the minimal normal subgroups of G and by a theorem of Bender (1), $N_1\times\cdots\times N_t$ contains its centralizer in G. We consider $G_0=(N_1\times\cdots\times N_t)S$. If S has a core C in G_0 then by (I) we may write $C=C_1\times C_2$ where $C_1\leqslant N_1\times\cdots\times N_t$ while $C_2\cap (N_1\times\cdots\times N_t)=1$. But then $C_2(N_1\times\cdots\times N_t)=C_2\times N_1\times\cdots\times N_t$ contradicting the fact that $N_1\times\cdots\times N_t$ contains its centralizer. So $C=C_1\lhd N_1\times\cdots\times N_t$ and if c is a generator of C, say $c=c_1\ldots c_t$ with $c_1\in N_1,\ldots,c_t\in N_t$, and if $x\in N_1$ then $c^x=c_1^x\ldots c_t^x=c_1^xc_2\ldots c_t=c_1^sc_2^s\ldots c_t^s$ for some $s\in N$. Thus $c_1^x=c_1^s$ and $\langle c_1\rangle\lhd N_1$ so $c_1=1$. Similarly $c_2=\cdots=c_t=1$. Hence $C=1$, S has no core in G_0 and $F(G_0)=1$. So if $|G|+|S|>|G_0|+|S|$ we have the result by induction.

(III) We may assume $t=1$, so that $G=MS$ where M is a minimal normal subgroup of G.

Because otherwise, in view of (I) and (II), all the following is well-defined:

(1) $S=C_1\times((N_1\times\cdots\times N_t)\cap S)$.

(2) C_2 is the product of all the subgroups of prime order of S which are not contained in C_1, but are contained in $C_G(N_2\times\cdots\times N_t)$.

(3) C_3 is the product of all the subgroups of prime order of S which are not contained in C_1, but are contained in $C_G(N_1)$.

(4) If Q is a subgroup of prime order of S, but Q is not a subgroup of $C_1\times C_2\times C_3$, then since Q must be contained in $N_1\times\cdots\times N_t$, we can fix a generator a_Qb_Q of Q with $a_Q\in N_1$ and $b_Q\in N_2\times\cdots\times N_t$.

(5) D is the product of all the subgroups $\langle a_Q\rangle$ of N_1 defined in (4).

(6) E is the product of all the subgroups $\langle b_Q\rangle$ of $N_2\times\cdots\times N_t$ defined in (4).

(7) $S^{(1)}=C_1\times C_2\times D$; $S^{(2)}=C_1\times C_3\times E$; $G_1=S^{(1)}N_1$;
$G_2=S^{(2)}\ (N_2\times\cdots\times N_t)$.

From these definitions we see that $S^{(1)}$ is cyclic of an order dividing $|S|$ and so of square-free order by (I).

If $F_2(G_1)$ denotes the inverse image of $F(G_1/F(G_1))$ in G_1, then $F_2(G_1)$ is a normal solvable subgroup of G_1 and as such it intersects N_1 trivially by (II). Thus it is cyclic and therefore equals $F(G_1)$. Hence $G_1/F(G_1)$ satisfies the assumption of the theorem and by the induction assumption $S^{(1)}F(G_1)/F(G_1)$ has disjoint conjugates. That is to say $S^{(1)}\cap S^{(1)x}\leqslant F(G_1)$ for some $x\in N_1$. But since $F(G_1)$ is cyclic of order dividing $|S|$, this means $S^{(1)}\cap S^{(1)x}$ is normal in G_1 and any subgroup of $S^{(1)}\cap S^{(1)x}$ is normal in G_1.

https://doi.org/10.1017/S0013091500026316 Published online by Cambridge University Press

The argument above applies also to G_2 and $S^{(2)}$ so that we have:

(8) There are $x \in N_1$ and $y \in N_2 \times \cdots \times N_t$ such that any subgroup of $S^{(1)} \cap S^{(1)x}$ is normal in G_1 and any subgroup of $S^{(2)} \cap S^{(2)y}$ is normal in G_2.

Let Q be a subgroup of prime order of $S \cap S^{xy^{-1}}$. Then by (I) we have

(9) $Q^x = Q^y$.

If Q is a subgroup of C_1 and s is a generator of Q, then by (9) $x^{-1}sx = y^{-1}s^iy$ for some integer $i \geq 1$; but then $[s, x] = s^{i-1}[s^i, y]$ so that $s^{i-1} = [s, x][s^i, y]^{-1} \in Q \cap G' = Q \cap (N_1 \times \cdots \times N_t) = 1$. Thus $i = 1$ and $[s, x] = [s, y] \in N_1 \cap N_2 \times \cdots \times N_t = 1$. Therefore $Q = Q^x = Q^y$ so that $Q \leq S^{(1)} \cap S^{(1)x} \cap S^{(2)} \cap S^{(2)y}$ and by (8) $Q \lhd G$, a contradiction. If Q is a subgroup of C_2 then by (2) and (8) $Q = Q^y$; by (9) $Q = Q^x$; by (7) $Q \leq S^{(1)} \cap S^{(1)}x$; by (8) $Q \lhd G_1$ which together with $Q \leq S$ and $Q \leq C_G(N_2 \times \cdots \times N_G)$ imply $Q \lhd G$ a contradiction.

A similar argument shows Q cannot be a subgroup of C_3 either, so we have that Q is not a subgroup of $C_1 \times C_2 \times C_3$ and by (4) and (9) we get $(a_Qb_Q)^x = [(a_Qb_Q)^i]^y$ for some integer $i \geq 1$; but then by (5), (6) and (8) we get $\langle a_Q \rangle^x = \langle a_Q \rangle \leq D \cap D^x \leq S^{(1)} \cap S^{(1)x}$ so $\langle a_Q \rangle \lhd G_1$ by (8) and N_1 contain a normal abelian subgroup, a contradiction unless $a_Q = 1$, in which case the same argument shows $\langle b_Q \rangle^y = \langle b_Q \rangle$ is a normal abelian subgroup of $N_2 \times \cdots \times N_t$ again a contradiction. Thus $S \cap S^{xy^{-1}} = 1$ and the assertion (III) is proved.

(IV) We now get the final contradiction as follows: Since S has no conjugate which intersects it trivially then if M is as in (III), we have $M = \bigcup_{i=1}^r N_M(S_i)$ where the S_i are as in (I). Let the notation be chosen so that $n_0 = |M : N_M(S_1)| \leq |M : N_M(S_i)|$ for $i = 2, \ldots, r$. Then

$$|M| \leq \sum_{i=1}^r |N_M(S_i)| = \sum_{i=1}^r \frac{|M|}{|M : N_M(S_i)|} \leq \frac{r|M|}{|M : N_M(S_1)|}$$

and it follows that $n_0 \leq r$.

Let p be the maximum prime divisor of $|S|$. Then by (I), $r < p$ and so $n_0 < p$.

Now if n_1 is the number of conjugates of S_1 in G then $n_1 = |G : N_G(S_1)|$ and by (III) $N_G(S_1) = N_M(S_1) \cdot S$ so that

$$n_1 = \frac{|G|}{|N_G(S_1)|} = \frac{|M||S||N_M(S_1) \cap S|}{|M \cap S||N_M(S_1)||S|} \leq \frac{|M|}{|N_M(S_1)|} = n_0.$$

It follows that $n_1 < p$. But G acts by conjugation on the n_1 conjugates of S_1 and since $p | G$ but $p \nmid n_1!$, this action has a non-trivial kernel N; $N \cap M$ is a normal subgroup of G; so by (III) $N \cap M = 1$ in which case N centralises M against the proof of (II), or $N \cap M = M$ and so M normalises S_1, $S_1 \lhd G$ against our assumption $F(G) = 1$.

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