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ON PRIME SEMILATTICES

BY
Y. S. PAWAR* AND N. K. THAKARE

ABSTRACT. Several characterizations for prime semilattices are obtained. Prime semilattices that are compactly packed by filters have been characterized. Solution to the problem, "Find a condition on a semilattice by which every filter can be expressed as the intersection of all prime filters containing it", is furnished.

1. Introduction. Throughout this paper a semilattice will mean a meet semilattice i.e. a partially ordered set in which any two elements a and b have a greatest lower bound denoted by $a \wedge b$. The least upper bound for any subset $\{x_1, x_2, \ldots, x_n\}$ of S, if it exists will be denoted by $x_1 \vee x_2 \vee \cdots \vee x_n$. The least and the greatest elements of S when they exist will be denoted by 0 and 1 respectively.

A filter of a semilattice S is a non-empty subset F of S such that $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$. A proper filter F of S is prime if, whenever $x_2 \vee \cdots \vee x_n$ exists and is an element of F then $x_i \in F$ for some $i \in \{1, 2 \cdots n\}$. For any non-empty subset A of S the filter generated by A is denoted by $[A)$ where

$$[A) = \{x \in S : x \geq a_1 \wedge a_2 \wedge \cdots \wedge a_n \text{ for some } a_1, a_2 \cdots a_n \text{ in } A\}.$$

An ideal I of a semilattice S is a non-empty subset of S satisfying

(i) $y \leq x$ and $x \in I$ imply $y \in I$,

(ii) if join of any finite number of elements of I exists in S then it must be in I.

Balbes [1] introduced the prime semilattice as the semilattice S satisfying any one of the following equivalent conditions:

(1) If $x_1 \vee \cdots \vee x_n$ exists in S then for each x in S $(x \wedge x_1) \vee (x \wedge x_2) \vee \cdots \vee (x \wedge x_n)$ exists and equals $x \wedge (x_1 \vee x_2 \vee \cdots \vee x_n)$.

(2) If F is a filter in S and J a non-empty subset of S disjoint with F and such that $x_1 \vee \cdots \vee x_n$ exists whenever $x_1, x_2 \cdots x_n \in J$, then there exists a prime filter F' such that $F \subseteq F'$ and $F' \cap J = \emptyset$.

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(3) If $x \not\leq y$ then there exists a prime filter F' such that $x \in F'$ and $y \notin F'$.

In the first part of this paper we show that prime semilattices are characterized by Stone's theorem (Theorem 1). In the sequel we also consider the relative annihilators in the context of prime semilattices. In the last section we study prime semilattices that are compactly packed by filters.

2. Characterizations. Along the lines of Stone's Theorem for distributive lattices we have

THEOREM 1. *A semilattice S is prime if and only if for any ideal I and a filter F disjoint with I in S there exists a prime filter Q in S containing F and disjoint with I.*

Proof. Let S be a prime semilattice. The existence of a filter Q containing F and disjoint with I follows by Zorn's Lemma. To prove Q is prime let $x_1 \vee x_2 \vee \cdots \vee x_n$ exist and is in Q with $x_i \notin Q$ for every $i \in \{1, 2 \cdots n\}$. But then

$$[Q \cup \{x_1\}), [Q \cup \{x_2\}), \ldots, [Q \cup \{x_n\})$$

will have a non-void intersection with I; which in turn will imply that there exist $i_1, i_2 \cdots i_n$ in I such that

$$i_1 \geq q_1 \wedge x_1, i_2 \geq q_2 \wedge x_2, \ldots, i_n \geq q_n \wedge x_n$$

for some $q_1, q_2, \ldots, q_n$ in Q. I being an ideal, $i_1 \wedge i_2 \wedge \cdots \wedge i_n$ is in I. Further

$$i_1 \wedge i_2 \wedge \cdots \wedge i_n \geq (q_1 \wedge q_2 \wedge \cdots \wedge q_n) \wedge (x_1 \vee x_2 \vee \cdots \vee x_n),$$

by primeness of S. As

$$(q_1 \wedge q_2 \wedge \cdots \wedge q_n) \wedge (x_1 \vee \cdots \vee x_n)$$

is an element of $Q, \emptyset \neq Q \cap I$; this contradicts the choice of Q. Hence Q is prime.

Conversely, let $x \not\leq y$. Let $I = (y]$ and $F = [x)$. As $I \cap F = \emptyset$, by assumption, there exists a prime filter Q containing F and disjoint with I, i.e. Q contains x but not y, proving that S is prime.

It follows immediately from the above theorem

THEOREM 2. *A semilattice S is prime if and only if any one of the following separation properties hold.*

(1) *A filter and an element not belonging to it are separated by a prime filter.*

(2) *An ideal and an element not belonging to it are separated by a prime filter.*

In a distributive lattice every filter is the intersection of all prime filters containing it. The following theorem gives a solution to the problem:

"Find the condition on a semilattice S such that every filter of S can be expressed as the intersection of all prime filters containing it".

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In any semilattice, for a filter Q, define the radical of Q ($= rad\ Q$) to be the intersection of all prime filters containing Q.

Theorem 1 allows us to have

THEOREM 3. *A semilattice S is prime if and only if $rad\ Q = Q$, for any filter Q in S.*

Balbes [1, Theorem 2.4] proves that in a prime semilattice S every maximal filter is prime. Hence any prime ideal in a prime semilattice will be a minimal prime ideal if and only if its set complement is a maximal filter.

Recall that a semilattice S with 0 is called weakly complemented if for $a \neq b$ in S there exists $c \neq 0$ in S such that exactly one of $a \wedge c$, $b \wedge c$ is zero (see [4]). In a weakly complemented semilattice converse of Theorem 2.4 in [1] holds; this we prove in the following

THEOREM 4. *Let S be a semilattice with 0. If S is a prime semilattice then every maximal filter is prime. The converse holds when S is weakly complemented.*

Proof. Though the proof of first part follows from (Balbes [1], Theorem 2.4), we give here a rather simple proof. Let $x_1 \vee \cdots \vee x_n$ exist and

$$x_1 \vee x_2 \vee \cdots \vee x_n \in M$$

with $x_i \notin M$ for each i in $\{1, 2 \cdots n\}$, where M is a maximal filter in a prime semilattice S. As M is maximal, $x_i \notin M$ imply that there exist y_i in M such that $x_i \wedge y_i = 0$ for every $i \in \{1, 2 \cdots n\}$; (see [4]). But then

$$(x_1 \vee x_2 \vee \cdots \vee x_n) \wedge (y_1 \wedge y_2 \wedge \cdots \wedge y_n) = 0$$

by primeness of S, implies that $0 \in M$, contradicting the maximality of M. Hence M is a prime filter.

Now for the second assertion, as S is weakly complemented, $x \nleq y$ in S implies that there exists a maximal filter F containing x and not containing y (see [4]). As F is prime, by assumption, we conclude that S is a prime semilattice. This completes the proof.

As the complement of a maximal filter in a prime semilattice is a minimal prime ideal we get

COROLLARY 5. *In a prime semilattice with 0 a prime ideal M is a minimal prime ideal if and only if for every x in M there exists y not in M such that $x \wedge y = 0$.*

Varlet [3] defined an annihilator $\langle a, b \rangle$ of a relative to b in a semilattice S to be the set of all elements x in S such that $a \wedge x \leq b$. Mandelkar [2] characterized distributive lattices in terms of such annihilators. The following theorem shows that this very characterization applies well to prime semilattices.

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THEOREM 6. *For any semilattice S the following are equivalent.*
(1) *S is prime*
(2) $\langle a, b\rangle$ *is an ideal for all* a, b *in* S
(3) $\langle a, b\rangle$ *is an ideal for all* $b \le a$.

Proof. $(1) \Rightarrow (2)$.
Let S be a prime semilattice and $x_1, x_2, \ldots, x_n$ are in $\langle a, b\rangle$ such that $x_1 \vee x_2 \vee \cdots \vee x_n$ exists. Then as S is prime

$$a \wedge (x_1 \vee x_2 \vee \cdots \vee x_n) = (a \wedge x_1) \vee (a \wedge x_2) \vee \cdots \vee (a \wedge x_n) \le b$$

as $x_i \in \langle a, b\rangle$ for every i in $\{1, 2 \cdots n\}$.

This proves that $x_1 \vee x_2 \vee \cdots \vee x_n \in \langle a, b\rangle$. If $x \le x_1$ and $x_1 \in \langle a, b\rangle$ then $a \wedge x \le a \wedge x_1 \le b$ implies that $x \in \langle a, b\rangle$. Hence $\langle a, b\rangle$ is an ideal.

$(2) \Rightarrow (3)$ Obvious.

$(3) \Rightarrow (1)$.

Let $x_1 \vee \cdots \vee x_n$ exist in S and a be any element in S. As

$$(a \wedge x_1) \vee (a \wedge x_2) \vee \cdots \vee (a \wedge x_n) \le a$$

we get

$$\langle a, (a \wedge x_1) \vee (a \wedge x_2) \vee \cdots \vee (a \wedge x_n)\rangle$$

is an ideal in S. As $x_1, x_2, \ldots, x_n$ are in

$$\langle a, (a \wedge x_1) \vee \cdots \vee (a \wedge x_n)\rangle$$

and $x_1 \vee x_2 \vee \cdots \vee x_n$ is an element of

$$\langle a, (a \wedge x_1) \vee \cdots \vee (a \wedge x_n)\rangle$$

hence $a \wedge (x_1 \vee \cdots \vee x_n) \le (a \wedge x_1) \vee \cdots \vee (a \wedge x_n)$. As the reverse inclusion always holds we get

$$a \wedge (x_1 \vee \cdots \vee x_n) = (a \wedge x_1) \vee \cdots \vee (a \wedge x_n)$$

proving that S is a prime semilattice.

Let us denote by $a * b$ the pseudocomplement of a relative to b in a semilattice S, i.e. $a * b$ is an element satisfying

$$a \wedge x \le b \quad \text{if and only if} \quad x \le a * b.$$

If $a * b$ exists for all a, b in S then S is called an implicative or relatively pseudocomplemented semilattice (see [1]).

It follows from the definition of relative annihilator and implicative semilattice that a semilattice S is implicative if and only if the relative annihilator

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$\langle a, b\rangle$ is a principal ideal $(a*b]$, for all a, b in S. Hence we have

Corollary 7. *Every implicative semilattice is prime.*

A sufficient condition for a prime semilattice to satisfy $\langle a, b\rangle \cup \langle b, a\rangle = S$ is given in the following theorem. It will be interesting to see whether the condition is necessary also.

Theorem 8. *In a prime semilattice if the filters containing the given filter F form a chain then F is prime and*

$$\langle a, b\rangle \cup \langle b, a\rangle = S.$$

Proof. Let $a_1, a_2, \ldots, a_n$ be elements in S which are not in F. Fix up any element a_j, $1 \le j \le n$. Then as $[F \cup \{a_i\})$ and $[F \cup \{a_j\})$ are comparable for every i in $\{1, 2 \cdots n\}$ we may assume without loss of generality that

$$[F \cup \{a_i\}) \subseteq [F \cup \{a_j\}), \qquad i \in \{1, 2 \cdots n\}.$$

Hence for some f in F,

$$f \wedge a_i \le a_j; \qquad i \in \{1, 2 \cdots n\}.$$

If $a_1 \vee a_2 \vee \cdots \vee a_n$ exists and is in F then

$$f \wedge (a_1 \vee a_2 \vee \cdots \vee a_n) = (f \wedge a_1) \vee (f \wedge a_2) \vee \cdots \vee (f \wedge a_n)$$

by primeness of S. But as $f \in F$ and $a_1 \vee a_2 \vee \cdots \vee a_n \in F$ we have

$$f \wedge (a_1 \vee a_2 \vee \cdots \vee a_n) \in F$$

i.e.

$$(f \wedge a_1) \vee (f \wedge a_2) \vee \cdots \vee (f \wedge a_n) \in F.$$

But then

$$(f \wedge a_1) \vee (f \wedge a_2) \vee \cdots \vee (f \wedge a_n) \le a_j$$

implies that $a_j \in F$, which is a contradiction. Hence $a_1 \vee a_2 \vee \cdots \vee a_n \notin F$, proving that F is prime. Let us assume that $J = \langle a, b\rangle \cup \langle b, a\rangle$ be proper. Hence by Theorem 2, there exists a prime filter Q disjoint with J. By hypothesis we may assume that

$$[Q \cup \{a\}) \subseteq [Q \cup \{b\}).$$

Hence for some q in Q we have $q \wedge a \le b$, i.e. $q \in \langle a, b\rangle$ and hence $q \in Q \cap J$, contradicting the choice of J and hence $\langle a, b\rangle \cup \langle b, a\rangle = S$.

Definition (see [4]). Let P be a bounded poset. An element b is called a complement of a if the only upper bound of a and b is 1 and their only lower

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bound is 0. This fact is expressed by $a \vee b = 1$ and $a \wedge b = 0$ even when the operations $\wedge$ and $\vee$ are undefined for some pairs of P. Interestingly we have,

THEOREM 9. *Let S be a prime semilattice with* 0 *and* 1 *in which complement of every maximal ideal is a maximal filter. Then S is complemented.*

Proof. As S is prime semilattice, by Theorem 8, $\langle a, 0\rangle$ is an ideal for any a in S. Assume that a has no complement. Then $a \vee x \neq 1$ for all x in $\langle a, 0\rangle$. Consider the set

$$A = \{\{a, x\}^u : x \in \langle a, 0\rangle\}$$

and let $J = A^l$, where for any non-empty subset Y of S, Y^u and Y^l denote the set of all upper bounds of Y and the set of all lower bounds of Y respectively. As J is a proper ideal in S and $1 \in S$, J must be contained in some maximal ideal say M in S. Hence by assumption $S - M$ is a maximal filter. As $a \in S - M$ there exists b in $S - M$ such that $a \wedge b = 0$. Hence $b \in \langle a, 0\rangle \subseteq M$ gives that b is in M. Thus $b \in M \cap (L - M)$, a contradiction. So our assumption that a has no complement is false. Hence the proof.

3. Prime semilattices that are compactly packed by filters.

In any semilattice S with 1, if a filter Q is contained in $\bigcup_{i=1}^{n} F_i$, where F_i's $(1 \leq i \leq n)$ are prime filters of S then it can be verified that $Q \subseteq F_i$ for some i. Define a semilattice S with 1 to be compactly packed by filters if S satisfies $(*)$ if a filter $Q \subseteq \bigcup_{\lambda \in \Delta} F_\lambda$, where F_λ's are prime filters of S, then $Q \subseteq F_\lambda$ for some λ.

For prime filters in a semilattice which is compactly packed by filters we have

THEOREM 9'. *Every prime filter of a semilattice which is compactly packed by filters, is a radical of a principal filter.*

Proof. Let there be a filter Q such that $Q \neq rad\,[x)$ for any x in S. Hence, by definition of $rad\,[x]$, there exists a prime filter F_x in S containing x but not Q. Clearly $Q \subseteq \bigcup_{x \in Q} F_x$. S being compactly packed by filters, $Q \subseteq F_x$ for some x in Q, which is a contradiction. Hence the result.

Equivalent formulation of the condition $(*)$ of compactly packedness by filters in a prime semilattice with 1, is given in the following

THEOREM 10. *A prime semilattice S with* 1 *is compactly packed by filters if and only if it satisfies* $(**)$. *If a prime filter* $F \subseteq \bigcup_{\lambda \in \Delta} F_\lambda$, *where* F_λ *are prime filters of S then* $F \subseteq F_\lambda$ *for some* $\lambda \in \Delta$.

Proof. As only if part follows directly from the definition, we will prove if part only. Let S satisfy $(**)$ and $Q \subseteq \bigcup_{\lambda \in \Delta} F_\lambda$. Clearly $I = S - \bigcup_{\lambda \in \Delta} F_\lambda$ is an ideal and $I \cap Q = \emptyset$. By Theorem 1 there exists a prime filter F' such that

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$F' \cap I = \emptyset$ and $Q \subseteq F'$, i.e. $Q \subseteq F' \subseteq \bigcup_{\lambda \in \Delta} F_\lambda$. Hence $Q \subseteq F' \subseteq F_\lambda$ for some $\lambda \in \Delta$, by $(**)$, proving that S is compactly packed by filters.

By using the property $(**)$ we now give the characterization of prime semilattices which are compactly packed by filters.

THEOREM 11. *A prime semilattice S with 1 is compactly packed by filters if and only if every prime filter of S is a principal filter.*

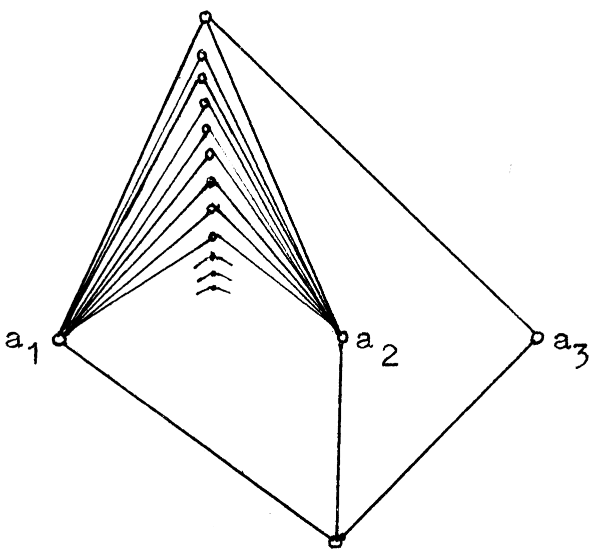
Proof. In view of Theorem 2 and Theorem 10 we need only prove the "if" part. Let every prime filter of S be a principal filter. If the prime filter $Q \subseteq \bigcup_{\lambda \in \Delta} F_\lambda$ we get $[x) \subseteq \bigcup_{\lambda \in \Delta} F_\lambda$, where $Q = [x)$, for some x in S. Hence $x \in F_\lambda$ for some $\lambda \in \Delta$, i.e. $Q \subseteq F_\lambda$ for some $\lambda \in \Delta$. Hence S is compactly packed by filters.

If $J(S)$ denotes the poset of all join-irreducible elements of a semilattice S then $J(S)$ is isomorphic with $\{[x) : x \in J(S)\}$ when S is a prime semilattice. Hence by Theorem 11, if $\mathcal{P}$ denotes the family of all prime filters of S, we have

COROLLARY 12. *If S is a prime semilattice with 1 that is compactly packed by filters then $\mathcal{P}$ is isomorphic with $J(S)$.*

The authors are thankful to the referee for making various suggestions which helped to improve the earlier version of the paper.

REMARK. The referee has very kindly supplied the following counterexample; it says in essence that the existence of $a_1 \vee \cdots \vee a_n$, $n \geq 3$, in a meet semilattice does not imply in general the existence of $a_1 \vee a_2$.



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This led the referee to suggest the following interesting question which we state here for the consideration of others also:

"Is the following condition sufficient for a meet semilattice S to be prime? If $x_1 \vee x_2$ exists then for each $x \in S$, $(x \wedge x_1) \vee (x \wedge x_2)$ exists and equals $x \wedge (x_1 \vee x_2)$."

The authors are also indebted to the referee for inviting their attention towards the above question.

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PAWAR, Y. S.
DEPARTMENT OF MATHEMATICS
SHIVAJI UNIVERSITY
KOLHAPUR-416004, INDIA

THAKARE, N. K.
DEPARTMENT OF MATHEMATICS AND STATISTICS
MARATHWADA UNIVERSITY
AURANGABAD-413004, INDIA

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