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On Domination of Zero-divisor Graphs of Matrix Rings

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Abstract. We study domination in zero-divisor graphs of matrix rings over a commutative ring with 1.

1 Introduction

By the zero-divisor graph $\Gamma(R)$ of a ring R we mean the graph with vertices $Z(R) \setminus \{0\}$ such that there is an (undirected) edge between vertices a and b if and only if $a \neq b$ and $ab = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. The concept of zero-divisor graphs has been studied extensively by many authors. For a list of references and the history of this topic the reader is referred to [1, 2].

A (simple) *directed graph* $D = (V, A)$ consists of a set V of vertices and a set A of directed edges, called *arcs*, where $A \subseteq V \times V$. The *outset* of a vertex u is the set $O(u) = \{v : (u, v) \in A\}$, and the *closed outset* of u is $O[u] = O(u) \cup \{u\}$. The *out-degree* $\deg^o(u)$ of a vertex u is the cardinality of $O(u)$. For a subset S of V, $O(S) = \bigcup_{u \in S} O(u)$ and $O[S] = \bigcup_{u \in S} O[u]$. A set $S \subseteq V$ is a *out-dominating set* of D if $O[S] = V$. The *out-domination number* $\gamma^o(D)$ of D is the minimum cardinality of an out-dominating set of D. The in-dominating sets and in-domination number γ^i are defined similarly, but considering the *insets*, where the inset, $I(v)$, of a vertex v is the set $\{w : (w, v) \in A\}$. The *in-degree* $\deg^i(v)$ of a vertex v is the cardinality of $I(v)$. For references on domination we refer [4].

Zero-divisor graphs of non-commutative rings are studied in [1]. The zero-divisor graph of a non-commutative ring R is the directed graph $\Gamma(R)$ whose vertices are all non-zero zero-divisors of R in which for any two distinct vertices x and y, $x \to y$ is an edge if and only if $xy = 0$.

In this note, we study domination in zero-divisor graphs of matrix rings over commutative rings with 1. We also let $Z(R)$ be the set of all zero-divisors of R.

We recall that if R is a commutative ring, then for a subset (or an element) X of R the *annihilator* of X is the ideal $\operatorname{ann}(X) = \{a \in R : aX = 0\}$. We note that by $G \leq H$ for two graphs we mean that G is a subgraph of H, while by $R \leq S$ for two rings we mean that R is a subring of S.

Let $M_n(R)$ be the ring of all $n \times n$ matrices over the ring R. Throughout the paper R is always a commutative ring with 1. In Section 2 we consider $\Gamma(M_n(R))$, where R

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is a field. In Section 3 we consider $\Gamma(M_n(R))$, where R is an Artinian commutative ring with 1.

2 Matrix Rings on Fields

All vector spaces in this section are finite-dimensional over a field F. We begin with the following trivial lemmas.

Lemma 2.1 *For any vector spaces V and W of finite dimension n over a field F, $L(V,V) \cong M_{n\times n}(F)$ as ring isomorphism.*

For two vector space V, W over field F of finite dimensions m, n, respectively, $L(V,W) \cong M_{m\times n}(F)$ as module isomorphism.

Lemma 2.2 *Let V be a vector space of dimension n and let $T, U \in L(V,V)$. Then $TU = 0$ if and only if $\operatorname{Im}(T) \subseteq \operatorname{Ker}(U)$.*

Corollary 2.3 (Akbari, et al. [1]) *Let V be a vector space of dimension n over a field F with $|F| = q$, and let $T \in L(V,V)$ with $rank(T) = k$. Then $\deg^o(T) = \deg^i(T) = 2q^{n(n-k)} - a$ and $\deg(T) = 2q^{n(n-k)} - q^{(n-k)^2} - a$, where $a = 1$, unless $T^2 = 0$, in which case $a = 2$.*

Proof By Lemma 2.2,

$$\deg^o(T) = |\{U : \operatorname{Im}(T) \subseteq \operatorname{Ker}(U)\}| - a = \left|L\Big(\frac{V}{\operatorname{Im}(T)}, V\Big)\right| - a,$$

$$\deg^i(T) = |\{U : \operatorname{Im}(U) \subseteq \operatorname{Ker}(T)\}| - a = \left|L\big(V, \operatorname{Ker}(T)\big)\right| - a.$$

On the other hand,

$$\left|\{U : \operatorname{Im}(T) \subseteq \operatorname{Ker}(U)\} \cap \{U : \operatorname{Im}(U) \subseteq \operatorname{Ker}(T)\}\right| = \left|L\Big(\frac{V}{\operatorname{Im}(T)}, \operatorname{Ker}(T)\Big)\right|.$$

The result follows. ∎

Recall that a directed graph is Eulerian if and only if for any vertex v, $\deg^i(v) = \deg^o(v)$.

Corollary 2.4 *For any integer n, $\Gamma(M_n(F))$ is Eulerian.*

We recall that a graph isomorphism from a graph G to a graph H is a bijection function $f\colon V(G) \to V(H)$ such that if $xy \in E(G)$, then $f(x)f(y) \in E(H)$.

Lemma 2.5 *For any commutative ring R, $\gamma^o(M_n(R)) = \gamma^i(M_n(R))$.*

Proof Notice that $\phi\colon \Gamma(M_n(R)) \to \Gamma(M_n(R))$ defined by $\phi(A) = A^t$ is a graph isomorphism. ∎

Lemma 2.6 *If A is an out-dominating set for $\Gamma(M_n(R))$, then there exists an out-dominating set B for $\Gamma(M_n(R))$ such that $|B| \le |A|$ and any element of B is of rank 1.*

Proof Let $T \in A$, and let v be a non-zero element of $\operatorname{Im}(T)$. There exists $T_1 \in L(V,V)$ such that $\operatorname{Im}(T_1) = \langle v \rangle$. Notice that any element that is dominated by T is also dominated by T_1. ■

Theorem 2.7 *If $|F| = q$, then $\gamma^o(L(V,V)) = \frac{q^n-1}{q-1}$.*

Proof Let S be a $\gamma^o(L(V,V))$-set. Let $v \in V \setminus \{0\}$. There exist $T \in L(V,V)$ such that $\operatorname{Ker}(T) = \langle v \rangle$. Since $T \in \Gamma(L(V,V))$, there is $T_1 \in S$ such that $T_1 T = 0$. Then $\operatorname{Im}(T_1) \subseteq \operatorname{Ker}(T)$. But $T_1 \neq 0$. Thus, $\operatorname{Im}(T_1) = \operatorname{Ker}(T)$. It follows that $|S| \geq |\{\langle v \rangle : v \neq 0\}|$. Now the result follows by Lemma 2.6. ■

Corollary 2.8 *Let $R = M_n(F)$, where F is a finite field. Then*

$$S = \{A = (a_{ij})_{n\times n} : a_{ij} = \delta_{1j}(\lambda_j), \text{ where } \lambda_1, \lambda_2, \dots, \lambda_n \in F \text{ and } \lambda_j = 1 \text{ for some } j\}$$

is a $\gamma^o(\Gamma(R))$-set.

Proof Notice that A is a zero-divisor if and only if the rows of A are linearly dependent. So there are $\lambda_1, \lambda_2, \dots, \lambda_n$ in F such that $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n = 0$, where $A_1, \dots, A_n$ are the rows of A. Without loss of generality assume that $\lambda_j = 1$ for some j. Then there is $B \in S$ such that $BA = 0$. This implies that S is a dominating set. On the other hand, $|S| = \frac{q^{n-1}}{q-1} = \gamma^o(\Gamma(R))$. Hence, S is a $\gamma^o(\Gamma(M_n(F)))$-set. ■

We refer to $M_n(F)$ with $|F| = q$ as $M_n(q)$.

Corollary 2.9 *For any n, $\gamma^o(M_n(q)) = \gamma^i(M_n(q)) = \frac{q^n-1}{q-1}$.*

3 Matrix Rings Over Artinian Commutative Rings

In this section we will study matrix rings over Artinian commutative rings with 1. From the structure theorem for Artinian rings we have the following lemmas.

Lemma 3.1 *Let R be a finite commutative ring with 1. Then $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a local ring for $i = 1, 2, \dots, t$.*

Lemma 3.2 *Let R_1, R_2 be commutative rings with 1. Then*

$$M_n(R_1 \times R_2) \cong M_n(R_1) \times M_n(R_2).$$

Lemma 3.3 *Let (R, M) be a local commutative ring with 1. If $M \neq 0$, then there is $x \in R$ such that $M = \operatorname{ann}(x)$.*

Lemma 3.4 *Let R be a commutative ring with 1. If $A \in M_n(R)$, then $A(\operatorname{adj}(A)) = (\operatorname{adj}(A))A = \det(A)I$.*

For Artinian rings that we handle in this section we have the following lemma.

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Lemma 3.5 *Let (R, M) be a local commutative ring with 1, and let $\phi: M_n(R) \to M_n(\frac{R}{M})$ be the natural epimorphism. Then $A \in Z(M_n(R))$ if and only if $\phi(A) \in Z(M_n(\frac{R}{M}))$.*

Proof Let $\phi(A) = \overline{A}$. By Lemma 3.3, $M = \mathrm{ann}(x)$.

($\Rightarrow$) Let $A \in Z(M_n(R))$. Then by Lemma 3.4, $\det(A) \in M$ and $\det(\overline{A}) = 0$, so $\overline{A} \in Z(M_n(\frac{R}{M}))$.

($\Leftarrow$) Let $\overline{A} \in Z(M_n(\frac{R}{M}))$. There is $\overline{B} \neq 0$ such that $\overline{AB} = 0$. So $AB = C$, where $C \in M_n(M)$. Now $B \notin M_n(M)$, and so $xB \neq 0$. On the other hand, $A(xB) = x(AB) = xC = 0$. Thus, $A \in Z(M_n(R))$. ∎

Theorem 3.6 *Let (R, M) be a local commutative ring with 1 and let $\frac{R}{M}$ be finite. Then $\gamma^o(M_n(R)) \le \gamma^o(M_n(\frac{R}{M}))$.*

Proof By Lemma 3.3, we have $M = \mathrm{ann}(x)$. Let

$$S = \{\overline{A_1}, \overline{A_2}, \dots, \overline{A_t}\}$$

be a $\gamma^o(\Gamma(M_n(\frac{R}{M})))$-set. By Lemma 3.5, for any $B \in Z(M_n(R))$, $\overline{B} \in Z(M_n(\frac{R}{M}))$. So there is $\overline{A} \in S$ such that $\overline{AB} = 0$. Then $AB = C$, where $C \in M_n(M)$. Thus, $(xA)B = xC = 0$. Therefore, $\{xA_1, xA_2, \dots, xA_t\}$ is a out-dominating set. ∎

Theorem 3.7 *Let (R, M) be a finite local commutative ring with 1, and let M be cyclic as an R-module. Then $\gamma^o(M_n(R)) = \gamma^o(M_n(\frac{R}{M}))$.*

Proof Let S be an out-dominating set for $\Gamma(M_n(R))$. Let K be a free R-module of rank n, and let $\{e_1, e_2, \dots, e_n\}$ be a basis of M. Let $\psi: M_n(R) \to \mathrm{Hom}(K, K)$ be the natural isomorphism. Let N be a maximal subspace of $\frac{K}{MK}$ as a vector space over $\frac{R}{M}$. There are $y_1, y_2, \dots, y_n$ in K such that $\{y_1 + MK, y_2 + MK, \dots, y_n + MK\}$ is a basis for $\frac{K}{MK}$, and $N = \langle y_1 + MK, y_2 + MK, \dots, y_{n-1} + MK\rangle$. Since M is a finite free R-module, $\{y_1, y_2, \dots, y_n\}$ is a basis for M. Let $N_1 = \langle y_1, y_2, \dots, y_{n-1}, \lambda y_n\rangle$, where $M = \langle\lambda\rangle$. Then N_1 is a maximal submodule of M. Further, there is $\phi \in \mathrm{Hom}(M, M)$ such that $\phi(e_i) = y_i$ for $i = 1, 2, \dots, n-1$ and $\phi(e_n) = \lambda y_n$. There is $B \in S$ such that $\phi\phi(B) = 0$.

If ϕ_1 is another homomorphism of M such that $\mathrm{Im}(\phi_1)$ is maximal and $\mathrm{Im}(\phi_1) \neq \mathrm{Im}(\phi)$, then $\phi_1\phi(B) \neq 0$. This implies that

$$|S| \le \Big|\big\{\phi \in \mathrm{Hom}(K, K) : \mathrm{Im}(\phi) \text{ is a maximal submodule}\big\}\Big|.$$

On the other hand

$$\Big|\big\{\phi \in \mathrm{Hom}(K, K) : \mathrm{Im}(\phi) \text{ is a maximal submodule}\big\}\Big| \ge \Big|\mathrm{Max}\Big(\frac{K}{MK}\Big)\Big|.$$

Thus

$$|S| \ge \Big|\mathrm{Max}\Big(\frac{K}{MK}\Big)\Big| = \gamma^o\bigg(\Gamma\Big(L\Big(\frac{K}{MK}, \frac{K}{MK}\Big)\Big)\bigg).$$ ∎

Lemma 3.8 *Let V be a vector space with $\dim(V) \ge 2$. Then $\Gamma(L(V, V))$ has a γ^o-set S such that for any $T \in S$, $T^2 = 0$.*

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Proof For any 1-dimensional subspace $W = \langle v \rangle$, there is a basis of V containing v. Since $\dim(V) \geq 2$, there is a linear transformation T_W such that $\mathrm{Im}(T_W) = \langle v \rangle$ and $v \in \mathrm{Ker}(T_W)$. So $T_W^2 = 0$, and by Theorem 2.7, $S = \{T_W : \dim(W) = 1\}$ is a γ^o-set. ∎

Lemma 3.9 *Let (R, M) be a commutative local ring with 1 such that M is cyclic as an R-module. Then $\Gamma(M_n(R))$ has a γ^o-set S such that for any $A \in S$, $A^2 = 0$.*

Proof By Lemmas 2.1 and 3.8, $\Gamma(M_n(\frac{R}{M}))$ has a γ^o-set S such that for any $\overline{A} \in S$, $\overline{A}^2 = 0$. By Theorem 3.7, $\{\lambda A : A \in S\}$ is a γ^o-set such that for any $A \in S$, $(\lambda A)^2 = 0$. ∎

Theorem 3.10 *Let $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a commutative ring with 1 such that the unique maximal ideal of R_i is principal. Then*

$$\gamma^o\big(\Gamma(M_n(R))\big) = \gamma^o\big(\Gamma(M_n(R_1))\big) + \gamma^o\big(\Gamma(M_n(R_2))\big) + \cdots + \gamma^o\big(\Gamma(M_n(R_t))\big).$$

Proof For $i = 1, 2, \ldots, t$, let $R_i = M_n(\mathbb{Z}_{p_i^{t_i}})$, let and S_i be a $\gamma^o(\Gamma(R_i))$-set such that for any $A \in S_i$, $A^2 = 0$. Let $S_i' = B_1 \times B_2 \times \cdots \times B_t$ such that $B_i = S_i$ and $B_j = 0$ for $j \neq i$. We show that $S_1' \cup S_2' \cup \cdots \cup S_t'$ is a γ^o-set. Notice that $(A_1, A_2, \ldots, A_n) \in Z(R)$ if and only if $A_i \in Z(R_i)$ for some i. Let $(A_1, A_2, \ldots, A_n) \in Z(R)$. Without loss of generality assume that $A_1 \in Z(R_1)$. There is $C_1 \in S_1$ such that $C_1 A_1 = 0$. Then $(C_1, 0, 0, \ldots, 0)(A_1, A_2, \ldots, A_n) = 0$. We deduce that $S_1' \cup S_2' \cup \cdots \cup S_t'$ is an out-dominating set. Let X be an out-dominating set. For any $A \in V(\Gamma(R_1))$, $(A, I, I, \ldots, I) \in V(\Gamma(R))$. So there is $Y \in X$ such that $Y(A, I, I, \ldots, I) = 0$. Then $Y = (Y_1, 0, 0, \ldots, 0)$. Thus $\{Y_1 : (Y_1, 0, 0, \ldots, 0) \in S\}$ is an out-dominating set for $\Gamma(R_1)$. Applying this on $j \geq 2$ we obtain

$$|X| \geq \gamma^o\big(\Gamma(R_1)\big) + \gamma^o\big(\Gamma(R_2)\big) + \cdots + \gamma^o\big(\Gamma(R_t)\big). \qquad ∎$$

Corollary 3.11 *Let $R = \mathbb{Z}_{p_1^{t_1}} \times \mathbb{Z}_{p_2^{t_2}} \times \cdots \times \mathbb{Z}_{p_k^{t_k}}$. Then*

$$\gamma^o\big(\Gamma(M_n(R))\big) = \gamma^i\big(\Gamma(M_n(R))\big) = \sum_{i=1}^{t} \frac{p_i^n - 1}{p_i - 1}.$$

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