SOLUTION OF IRVING'S RAMSEY PROBLEM

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(Received 6 June, 1979)

In [**1**] the following question was posed by R. W. Irving (see also Conjecture 4.10 in [**4**]): Is there an edge 2-colouring of the complete bipartite graph $K_{13,17}$ with no monochromatic $K_{3,3}$? We give a negative answer in this note (Theorem 2). Furthermore we prove Conjecture 4.11 (i) of [**4**] (Theorem 1), that is, any edge 2-coloured $K_{2n+1,4n-3}$ contains a monochromatic $K_{2,n}$ with the 2 and n vertices a subset of the $2n+1$ and $4n-3$ vertices, respectively. Theorem 1 is a consequence of Satz 4 in [**3**], however, we give a direct proof here.

Instead of edge coloured complete bipartite graphs $K_{x,y}$ we use 0–1-matrices $M=(m_{i,j})$, where $m_{i,j}=0$ or 1 $(1\leqq i\leqq x, 1\leqq j\leqq y)$, if the edge (i, j) of $K_{x,y}$ is of the first or second colour, respectively.

LEMMA 1. *If p_{ij}, p_{ik}, and p_{jk} denote the numbers of equal columns (both entries 0 or both entries 1) in the three pairs of rows of any triple of rows (i, j, k) in any 0–1-matrix with c columns, then*

$$p_{ij}+p_{ik}+p_{jk}\equiv c \pmod 2. \tag{1}$$

Proof. Each column contributes 1 or 3 to the sum on the left-hand side.

In the following we denote by $[x]$ and $\{x\}$ the greatest integer $\leqq x$ and the smallest integer $\geqq x$, respectively.

THEOREM 1. *Any $(2n+1, 4n-3)$-0–1-matrix contains a $(2, n)$-submatrix with entries 0 only, or 1 only.*

Proof. Any column of a $(2n+1, 4n-3)$-0–1-matrix M contains at least $\binom{n}{2}+\binom{n+1}{2}$ pairs of equal entries. Thus for the total number A of equal pairs in all columns of M we obtain

$$A\geqq(4n-3)\left(\binom{n}{2}+\binom{n+1}{2}\right)=4n^3-3n^2. \tag{2}$$

Using the pigeonhole principle there is at least one pair of rows in M with

$$p=\left\{\frac{4n^3-3n^2}{\binom{2n+1}{2}}\right\}=\left\{2n-2-\frac{n-2}{2n+1}\right\}=2n-2 \tag{3}$$

equal columns for $n\geqq 2$ (if $n=1$, then Theorem 1 is trivial).

We assume that no pair of rows in M has more than p equal columns. In any triple of rows at most two pairs have p equal columns (Lemma 1). Then the famous theorem of Turán ([**2**], p. 17) implies that at most $[(2n+1)^2/4] = n^2+n$ pairs of rows in M have p equal columns. It follows that

$$A \leqq (n^2+n)(2n-2) + \left(\binom{2n+1}{2} - (n^2+n)\right)(2n-3) = 4n^3 - 3n^2 - 2n, \tag{4}$$

which contradicts (2). Thus at least one pair of rows in M has $p+1 = 2n-1$ equal columns, that means, $\left\{\frac{2n-1}{2}\right\} = n$ columns have entries 0 only, or 1 only.

Theorem 2. *Any* (13, 17)-0–1-*matrix contains a* (3, 3)-*submatrix with entries* 0 *only, or* 1 *only.*

Proof. We denote by $B = (b_{i,j})$ a (13, 17)-0–1-matrix, by N the (3, 3)-matrix with entries 0 only, and by $\bar{M}$ the matrix M with 0 and 1 interchanged. If M contains a submatrix S, we will write $S \subset M$. The proof is divided into the following Lemmas. Those parts of their proofs which follow by changing 0 and 1 are omitted.

We consider the matrices S_i $(1 \leqq i \leqq 10)$ shown opposite as submatrices of B up to exchanges of rows or columns.

Lemma 2. *If* S_1 *or* $\bar{S}_1 \subset B$, *then* N *or* $\bar{N} \subset B$,

Proof. If $S_1 \subset B$, then either $N \subset B$, or every row of that (11, 6)-submatrix M of B determined by the columns of S_1 contains at least 4 entries 1, that is, $\binom{4}{3} = 4$ triples of entries 1. Then any distribution of these 44 triples among the 6 columns of M guarantees $\bar{N} \subset M$, since $2\binom{6}{3} < 44$.

Lemma 3. *If* S_2, S_2^T, $\bar{S}_2$ *or* $\bar{S}_2^T \subset B$, *then* N *or* $\bar{N} \subset B$.

Proof. Let $S_2 \subset B$, or $S_2 \subset B^T$, which corresponds to $S_2^T \subset B$. The first 5 columns and the last 7 rows of S_2 determine $M \subset S_2$. Either $N \subset S_2$, or every row of M contains at least 3 entries 1, that is, $\binom{3}{2} = 3$ pairs of entries 1. In any distribution of these 21 pairs among the 5 columns of M there are 2 columns with 3 pairs of entries 1 in a row, since $2\binom{5}{2} < 21$. Together with column 6 of S_2, it follows that $\bar{N} \subset S_2$.

Lemma 4. *If* S_3 *or* $\bar{S}_3 \subset B$, *then* N *or* $\bar{N} \subset B$.

Proof. If S_3 is in the first rows and columns of B, then rows 6 to 13, and columns 3 to 17 determine a (8, 15)-submatrix M of B. If there is one row of M with more than 6 entries 1, then Lemma 3 can be used. Otherwise M has at least $8 \times 9 = 72$ entries 0. Let s_i

S_1:

0	0	0	0	0	0
0	0	0	0	0	0

S_2:

0	0	0	0	0	·
0	0	0	0	0	·
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1

S_3:

0	0
0	0
0	0
0	0
0	0

S_4:

0	0	0	0	0	·	·
0	0	0	0	0	·	·
·	·	1	1	1	1	1
·	·	1	1	1	1	1

S_5:

0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	·	·	·	·	·	·	·	·	·
·	·	·	·	·	0	0	0	0	0	1	1	1	1

S_6:

·	·	0	0	0
1	1	0	0	·
1	1	0	0	·
1	1	·	·	1
1	1	·	·	1

S_7:

0	0	0	0	0	·
0	0	0	0	0	·
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1
·	·	·	·	·	1

S_8:

1	1	0	0	1
1	1	0	1	0
1	0	1	0	1
1	0	1	1	0
0	1	1	0	1
0	1	1	1	0

S_9:

0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	·	·	·	·	1	1	1	1	·
·	·	·	·	0	0	0	0	0	·	·	·	·	1

S_{10}:

0	0	0	0	0	0	0	0
0	0	0	0	0	·	·	·
·	·	·	0	0	0	0	0

https://doi.org/10.1017/S0017089500004171 Published online by Cambridge University Press

$(0 \leqq i \leqq 8)$ denote the number of columns of M with exactly i entries 0. Then

$$\sum_{i=0}^{8} s_i = 15, \quad \text{and} \quad \sum_{i=1}^{8} i s_i \geqq 72$$

yield

$$s_5 + 2s_6 + 3s_7 + 4s_8 \geqq 12 + 4s_0 + 3s_1 + 2s_2 + s_3 \geqq 12. \tag{5}$$

Together with (5) there are

$$\sum_{i=3}^{8} \binom{i}{3} s_i \geqq 10(s_5 + 2s_6 + 3s_7 + 4s_8) \geqq 120 > 2\binom{8}{3}$$

triples of entries 0 in the columns of M, so that $N \subset M$.

LEMMA 5. *If a column of B has 9 entries 0 or 9 entries 1, then N or $\bar{N} \subset B$.*

Proof. Let $b_{i,1} = 0$ for all i with $1 \leqq i \leqq 9$. Rows 1 to 9 and columns 2 to 17 determine $M \subset B$. If there is a column of M with at least 5 entries 0, we use Lemma 4. Otherwise at least $16\binom{5}{2} = 160$ pairs with both entries 1 in the columns of M distributed among all pairs of rows of M guarantee 2 rows of M having 5 columns with both entries 1, since $4\binom{9}{2} < 160$. Then $\bar{S}_2 \subset B$, and we use Lemma 3.

LEMMA 6. *If S_4 or $\bar{S}_4 \subset B$, then N or $\bar{N} \subset B$.*

Proof. Let S_4 be in the first rows and columns of B. Rows 5 to 13 and columns 1 to 2, 3 to 5, 6 to 7 determine M_1, M_2, M_3, respectively. Either N or $\bar{N} \subset B$, or every row of M_2 has at most 2 entries 0, and at most 2 entries 1. In at least 5 rows of M_2 there are 2 entries 0 (1). Then either $N \subset B$ ($\bar{N} \subset B$), or $\bar{S}_3 \subset M_1$ ($S_3 \subset M_3$), and Lemma 4 can be used.

LEMMA 7. *If S_5 or $\bar{S}_5 \subset B$, then N or $\bar{N} \subset B$.*

Proof. Let S_5 occur in the first rows and columns of B. Rows 4 to 13 and columns 1 to 5 of B determine M_1, and rows 4 to 13 together with columns 11 to 14 of B determine M_2. At first $b_{3,j} = 1$ for all j with $1 \leqq j \leqq 5$, or Lemma 2 can be used. Next either $N \subset B$, or every row in M_1 has at least 3 entries 1. If one row of M_2 exists with more than one entry 1, then together with row 3 of B we have $S_4 \subset B$, and use Lemma 6. Otherwise every row of M_2 has 3 entries 0, and any distribution of 10 triples 000 among the 4 columns of M_2 guarantees $N \subset M_2$, since $2\binom{4}{3} < 10$.

LEMMA 8. *N or $\bar{N}$ exist in any $(5, 5)$-matrix obtained by changing rows or columns of S_6, S_6^{T}, $\bar{S}_6$ or $\bar{S}_6^{\mathrm{T}}$.*

Proof. If the second or third element of column 5 of S_6 is 1, then $\bar{N} \subset S_6$, and otherwise $N \subset S_6$.

LEMMA 9. *If $S_7 \subset B$ (respectively $\bar{S}_7 \subset B$), then either N or $\bar{N} \subset B$, or S_8 (respectively $\bar{S}_8$) are in rows 3 to 8 and columns 1 to 5 of S_7 (respectively $\bar{S}_7$), up to exchanges of rows or columns.*

Proof. Let $S_7 \subset B$, and M denotes the elements of S_7 in rows 3 to 8 and columns 1 to 5. At least 3 entries 1 exist in every row of M, or $N \subset S_7$. If one row has more than 3 entries 1, then $\binom{4}{2}+5\binom{3}{2}>2\binom{5}{2}$ implies that 2 columns exist in M with 3 pairs 11, and together with column 6 of S_7 it follows $\bar{N} \subset S_7$. It remains that M has exactly 18 entries 1. No column of M has more than 4 entries 1, since otherwise $\bar{S}_3 \subset B$ (Lemma 4). No column of M has less than 3 entries 1, since otherwise 4 columns have 4 entries 1, and this forces 2 rows of these 4 columns to have 2 entries 1 and 4 rows to have 3 entries 1, and then $2\binom{2}{2}+4\binom{3}{2}>2\binom{4}{2}$ together with column 6 of S_7 guarantees $\bar{N} \subset S_7$. So M can have only 3 columns with 4, and 2 columns with 3 entries 1. Then $N \subset S_7$ (with column 6), or by changing of rows or columns we obtain $M = S_8$.

LEMMA 10. *If S_9 or $\bar{S}_9 \subset B$, then N or $\bar{N} \subset B$.*

Proof. If we find S_9 in the first rows and columns of B, then by Lemma 2 we can assume $b_{3,j} = b_{2,j+5} = 1$ for $j = 1, 2, 3, 4$. Let M_1 be determined by rows 4 to 13 and columns 1 to 9. Either $N \subset B$, or in columns 1 to 5, and in columns 5 to 9 of M_1, respectively, there are at least 30 entries 1. Every column $\neq 5$ of M_1 has at most 6 entries 1, or we can use Lemma 3. Thus column 5 must have at least 6 entries 1. Since Lemma 5 yields N or $\bar{N} \subset B$, column 5 has at most 8 entries 1.

(i) $b_{i,5} = 1$ for all i with $8 \leqq i \leqq 13$, $b_{i,5} = 0$ otherwise: The elements in rows 4 to 7 of B and in columns 1 to 4, 6 to 9, 10 to 14 are denoted by M_2, M_3, M_4, respectively. Every row of M_2 and of M_3 has at least 3 entries 1, otherwise $N \subset B$. Then every pair of rows of M_2 and of M_3 has at least 2 columns with both entries 1. Together with rows 2 or 3, and M_4, we find $\bar{N} \subset B$, or every column of M_4 has at least 3 entries 0. Then 2 columns of M_4 together with column 5 of B yield $N \subset B$.

(ii) $b_{i,5} = 1$ for all i with $7 \leqq i \leqq 13$, $b_{i,5} = 0$ otherwise: Let M_5 and M_6 denote the elements of B in rows 4 to 6, and in columns 1 to 4, and 6 to 9, respectively. Rows 7 to 13 and columns 1 to 4 determine M_7. Every row of M_5 and of M_6 has at least 3 entries 1 (otherwise $N \subset B$), and then at most 3 entries 1 (otherwise $\bar{N} \subset B$). Every column of M_5 and of M_6 has at most one entry 0, otherwise $\bar{N} \subset B$. Thus we can assume $b_{4,1} = b_{5,2} = b_{6,3} = b_{4,6} = b_{5,7} = b_{6,8} = 0$, and 1 for all other elements of M_5 and of M_6. As in every pair of columns of M_7 at most 2 pairs 11 occur (otherwise together with column 5 of B we obtain $\bar{N} \subset B$), there is a row in M_7 with 11 in columns 1 and 4, 2 and 4, or 3 and 4, and we can assume $b_{7,3} = b_{7,4} = 1$. At least 2 elements of $b_{7,6}$ to $b_{7,9}$ are 1, otherwise $N \subset B$. These have to be $b_{7,6}$ and $b_{7,7}$, since rows 4, 5, 7 and columns 3, 4, 8, 9 yield $\bar{N} \subset B$ or $b_{7,8} = b_{7,9} = 0$. Then rows 5, 6, 7 and columns 1, 4, 6 imply $\bar{N} \subset B$ or $b_{7,1} = 0$, and rows 4, 6, 7 and columns 2, 4, 7 imply $\bar{N} \subset B$ or $b_{7,2} = 0$.

At least 14 entries 1 occur in M_7. Column 4 of M_7 contains at most 3 entries 1, since otherwise we can use Lemma 3. No column of M_7 has more than 4 entries 1, and thus two of columns 1 to 3 of M_7 contain exactly 4 entries 1. After possibly changing columns 1 and 2, and rows 4 and 5 we can assume $b_{8,1}=b_{9,1}=b_{10,1}=b_{11,1}=1$, and $b_{12,1}=b_{13,1}=0$. Now Lemma 9 guarantees N or $\bar{N}\subset B$, or we can assume S_8 in rows 5, 6, 8, 9, 10, 11 and columns 6, 9, 5, 7, 8, in these sequences.

Then rows 5, 6 and 8, 9, 10 or 11 and columns 1, 4 and 6, 6, 9 or 9, respectively, yield $\bar{N}\subset B$ or $b_{8,4}=b_{9,4}=b_{10,4}=b_{11,4}=0$. Rows 4, 5, 10 and columns 3, 8, 9 yield $\bar{N}\subset B$ or $b_{10,3}=0$. Rows 4, 6, 11 and columns 2, 7, 9 yield $\bar{N}\subset B$ or $b_{11,2}=0$. Then rows 1, 8, 10 and columns 3, 4, 7, rows 1, 7, 9 and columns 2, 8, 9, rows 1, 2, 10 and columns 2, 3, 4, rows 1, 2, 11 and columns 2, 3, 4 yield $N\subset B$ or $b_{8,3}=b_{9,2}=b_{10,2}=b_{11,3}=1$, respectively. Furthermore $\bar{N}\subset B$ or $b_{8,2}=b_{9,3}=0$ follow from rows 8, 9, 10 and columns 1, 2, 5, and from rows 8, 9, 11 and columns 1, 3, 5.

If $b_{12,3}=b_{13,3}=1$, then Lemma 4 can be used. Therefore we can assume $b_{12,3}=0$ after possibly changing rows 12 and 13. Then rows 1, 2, 12 and columns 1 to 4 imply $N\subset B$ or $b_{12,2}=b_{12,4}=1$. Rows 4, 6, 12 and columns 2, 4, 7, and columns 2, 4, 9 yield $\bar{N}\subset B$ or $b_{12,7}=b_{12,9}=0$. Then rows 1, 3, 12 and columns 6 to 9 yield $N\subset B$ or $b_{12,6}=b_{12,8}=1$. It follows $b_{13,6}=0$, or Lemma 4 can be used. If $b_{13,2}=0$, then rows 1, 2, 13 and columns 1 to 4 yield $N\subset B$ or $b_{13,3}=b_{13,4}=1$, rows 1, 7, 13 and columns 1, 2, 8 yield $N\subset B$ or $b_{13,8}=1$, and then we find $\bar{N}$ in rows 4, 5, 13 and columns 3, 4, 8. If, however, $b_{13,2}=1$, then rows 10, 12, 13 and columns 2, 5, 8 yield $\bar{N}\subset B$ or $b_{13,8}=0$, rows 1, 3, 13 and columns 6 to 9 yield $N\subset B$ or $b_{13,7}=b_{13,9}=1$, and then we find $\bar{N}$ in rows 4, 6, 13 and columns 2, 7, 9.

(iii) $b_{i,5}=1$ for all i with $6\leqq i\leqq 13$, $b_{i,5}=0$ otherwise: At least 2 of the elements $b_{3,10}$ to $b_{3,13}$ of B (say $b_{3,10}$ and $b_{3,11}$) are 0, otherwise $\bar{N}\subset B$ (with rows 1 and 2). At least one of the elements $b_{4,10}$, $b_{4,11}$, $b_{5,10}$, $b_{5,11}$ (say $b_{4,10}$) is 1, otherwise $N\subset B$ (with column 5). Columns 6 to 9, rows 4 to 5 and rows 6 to 13 of B determine M_8 and M_9, respectively. Both rows of M_8 have at least 3 entries 1 (otherwise $N\subset B$), and then at most 3 entries 1 (otherwise $\bar{N}\subset B$). If one column of M_8 has both entries 0, then $\bar{N}\subset B$. Thus we can assume $b_{4,9}=b_{5,8}=0$, and all other elements of M_8 are 1.

In every column of M_9 there are at most 4 entries 1, or we can use Lemma 4 (with column 5). At least 30 entries 1 are in columns 5 to 9 of M_1, or $N\subset B$. Thus exactly 4 entries 1 occur in every column of M_9, and exactly 2 entries 1 in every row of M_9. We can assume $b_{6,9}=b_{7,9}=b_{8,9}=b_{9,9}=0$. If in 2 of rows 1 to 4 of M_9, and in columns 1 and 2 of M_9 there are 4 entries 1, then we have found $\bar{S}_3$, and Lemma 4 can be used. Otherwise we can assume $b_{6,6}=b_{6,8}=b_{7,6}=b_{7,8}=1$, and $b_{6,7}=b_{7,7}=0$ (after possibly changing columns 6 and 7). Then in these sequences the elements of rows 3, 6, 7, 2, 4 and of columns 6, 8, 7, 9, 10 of B represent S_6, and Lemma 8 completes the proof.

LEMMA 11. *If S_{10} or $\bar{S}_{10}\subset B$, then N or $\bar{N}\subset B$.*

Proof. Let S_{10} be in the first rows and columns of B. Then $b_{2,6}=b_{2,7}=b_{2,8}=b_{3,1}=b_{3,2}=b_{3,3}=1$, or $N\subset B$. The elements of B in rows 4 to 13 and columns 1 to 3, 4 to 5, and 6 to 8 are denoted by M_1, M_2 and M_3, respectively. If we observe Lemma 4, it suffices to discuss 3 cases: M_2 has (i) one row 00, (ii) 4 rows 11, or (iii) 4 rows 01.

(i) $b_{4,4}=b_{4,5}=0$. Then $N\subset B$, or $b_{4,1}=b_{4,2}=b_{4,3}=b_{4,6}=b_{4,7}=b_{4,8}=1$. By Lemma 4 we can assume one entry 0 in each of rows 2 to 5 of M_2. The corresponding rows of M_1 and M_3 have exactly 2 entries 1, since otherwise N or $\bar{N}\subset B$. Then we can assume 2 equal rows in M_1 (say $b_{5,1}=b_{6,1}=b_{5,2}=b_{6,2}=1$ and $b_{5,3}=b_{6,3}=0$). As M_3 has at least one column with both entries 1 in rows 2 and 3 (say $b_{5,6}=b_{6,6}=1$), we find $\bar{N}$ in rows 4, 5, 6 and columns 1, 2, 6.

(ii) $b_{i,4}=b_{i,5}=1$ for all i with $4\leqq i\leqq 7$. In M_1 and M_3 rows 4 to 7 have at least one, and rows 8 to 13 at least 2 entries 1 (otherwise $N\subset B$). Together there are at least 16 entries 1, which guarantee at least one column with 6 entries 1 in M_1, and in M_3, since by Lemma 3 we can assume at most 6 entries 1 in every column. At most 2 entries 1 exist in the first 4 rows of every column of M_1 and M_3 (otherwise $\bar{N}\subset B$). If a column with 6 entries 1 in M_1 or M_3 has exactly one entry 1 in the first 4 rows, then N or $\bar{N}\subset B$ by Lemma 9, since the existence of S_8 would force a row 11 in rows 8 to 13 of M_2, and then Lemma 4 can be used. As M_1 and M_3 cannot both have a column with 4 entries 0 in the first 4 rows (otherwise we use Lemma 4), we can assume that in M_1 a column exists, which has 6 entries 1, and 2 of them in the first 4 rows. Thus we may choose $b_{i,3}=1$ for $i=6,7,\dots,11$ and $b_{4,3}=b_{5,3}=b_{12,3}=b_{13,3}=0$. Then by Lemma 9 we can assume S_8 in rows 6 to 11 and columns 4 to 8 of B. After possibly changing rows 12 and 13 we have $b_{12,4}=b_{13,5}=1$, and $b_{12,5}=b_{13,4}=0$ (otherwise we have cases (i) or (iii), or $\bar{S}_3$, and Lemma 4 can be used). Together with rows 1 and 2 of B we get $N\subset B$, or $b_{12,1}=b_{12,2}=b_{13,1}=b_{13,2}=1$.

We next prove, that 6 entries 1 in columns 2 or 3 of M_3 yield N or $\bar{N}\subset B$. After possibly changing rows 6 and 7, 8 and 9, 10 and 11, so as columns 7 and 8 of B we can assume $b_{12,8}=b_{13,8}=b_{5,8}=1$, and $b_{4,8}=0$ (rows 4 and 5 can be changed). By Lemma 9 we find N or $\bar{N}\subset B$, or S_8 in rows 5, 6, 12, 8, 13, 10 and columns 4, 5, 1, 3, 2 after possibly changing columns 1 and 2 of B. Thus we assume $b_{5,1}=b_{6,1}=b_{6,2}=b_{8,2}=b_{10,2}=0$, $b_{8,1}=b_{10,1}=b_{5,2}=1$. Then rows 8, 10, 12, 13 and columns 1, 6, 8 of B yield $N\subset B$, or $b_{12,6}=b_{13,6}=0$. Rows 12 and 13 together with rows 1 and 3 of B yield $N\subset B$, or $b_{12,7}=b_{13,7}=1$. Rows 8, 9, 10, 11 and columns 1, 3, 6 of B yield $\bar{N}\subset B$, or $b_{9,1}=b_{11,1}=0$. Together with rows 1 and 2 it follows $b_{9,2}=b_{11,2}=1$ (otherwise $N\subset B$). Rows 7, 9, 11 and columns 2, 3, 7 yield $\bar{N}\subset B$, or $b_{7,2}=0$. Rows 1, 6, 7 and columns 1, 2, 6 force $N\subset B$, or $b_{7,1}=1$. Rows 5, 12, 13 and columns 2, 7, 8 yield $\bar{N}\subset B$, or $b_{5,7}=0$. Rows 1, 5, 6 and columns 1, 6, 7 force $N\subset B$, or $b_{5,6}=1$. If now $b_{4,1}=1$, then 6 entries 1 are in column 1 of M_1, and by Lemma 9 it remains $b_{4,6}=1$, $b_{4,7}=0$ (S_8 is in rows 4, 7, 8, 12, 10, 13 and columns 4, 5, 8, 7, 6 of B). Then in case $b_{4,2}=0$ we find S_3 in columns 2, 7 and rows 1, 4, 6, 8, 10 of B, and we use Lemma 4, and in case $b_{4,2}=1$ rows 4, 5, 9 and columns 2, 4, 6 yield $\bar{N}\subset B$. If otherwise $b_{4,1}=0$, then rows 1, 2, 4 and columns 1, 2, 3 yield $N\subset B$, or $b_{4,2}=1$. Then $b_{4,7}=0$ forces $N\subset B$ (rows 1, 4, 5 and columns 1, 3, 7), and $b_{4,7}=1$ gives $\bar{S}_3\subset B$ (rows 4, 9, 11, 12, 13, columns 2, 7), and we use Lemma 4.

In the following we can assume that at most 5 entries 1 exist in columns 2 and 3 of M_3, that is, 6 entries 1 occur in column 1 of M_3. By Lemma 9, this is possible only in two

https://doi.org/10.1017/S0017089500004171 Published online by Cambridge University Press

cases, either $b_{4,6}=b_{5,6}=1$ and $b_{12,6}=b_{13,6}=0$, or $b_{4,6}=b_{5,6}=0$ and $b_{12,6}=b_{13,6}=1$. In the first case we find N in rows 1, 3, 12, 13 and columns 4 to 8, or $b_{12,7}=b_{12,8}=b_{13,7}=b_{13,8}=1$. Then more than 5 entries 1 occur in column 2 or 3 of M_3, or $N\subset B$ (rows 1, 4, 5, columns 3, 7, 8). In the second case we use Lemma 9, and we can assume S_8 in columns 3, 2, 1, 4, 5 of B, and (j) in rows 11, 9, 10, 8, 13, 12, or (jj) in rows 11, 8, 10, 9, 13, 12 of B, after possibly changing columns 1 and 2. In other words, we can assume $b_{10,2}=b_{11,1}=0$, $b_{10,1}=b_{11,2}=1$ and (j) $b_{8,2}=b_{9,1}=0$, $b_{8,1}=b_{9,2}=1$, or (jj) $b_{8,2}=b_{9,1}=1$, $b_{8,1}=b_{9,2}=0$.

In case (j) rows 9, 11, 12, 13 and columns 2, 6, 7 yield $\bar{N}\subset B$ or $b_{12,7}=b_{13,7}=0$, rows 8, 10, 12, 13 and columns 1, 6, 8 yield $\bar{N}\subset B$ or $b_{12,8}=b_{13,8}=0$, and then we have N in rows 1, 12, 13 and columns 3, 7, 8.

In case (jj) $b_{4,7}=b_{5,7}=0$ or 1 yield N in rows 1, 4, 5 and columns 3, 6, 7, or $\bar{N}$ in rows 4, 5, 7 and columns 4, 5, 7, respectively, and thus we can assume $b_{4,7}=0$, $b_{5,7}=1$, after possibly changing rows 4 and 5. Then rows 1, 3, 4 and columns 6, 7, 8 imply $N\subset B$ or $b_{4,8}=1$, and rows 4, 5, 6 and columns 4, 5, 8 imply $\bar{N}\subset B$ or $b_{5,8}=0$. If $b_{4,2}=0$, then $b_{5,2}=1$ or N is in rows 1, 4, 5 and columns 2, 3, 6. Rows 5, 11, 13 and columns 2, 5, 7 yield $\bar{N}\subset B$ or $b_{13,7}=0$. Then rows 1, 3, 13 and columns 4, 7, 8 imply $N\subset B$ or $b_{13,8}=1$. Rows 4, 10, 13 and columns 1, 5, 8 then yield $\bar{N}\subset B$ or $b_{4,1}=0$, and we find N in rows 1, 2, 4 and columns 1, 2, 3. If, otherwise, $b_{4,2}=1$, then $b_{6,2}=0$ or $\bar{N}$ is in rows 4, 6, 8 and columns 2, 4, 8, and $b_{12,8}=0$ or $\bar{N}$ is in rows 4, 8, 12 and columns 2, 4, 8. Rows 1, 3, 12 and columns 5, 7, 8 yield $N\subset B$ or $b_{12,7}=1$. Then rows 5, 9, 12 and columns 1, 4, 7 yield $\bar{N}\subset B$ or $b_{5,1}=0$. Rows 11, 12, 13 and columns 2, 6, 7 yield $\bar{N}\subset B$ or $b_{13,7}=0$. Then rows 1, 3, 13 and columns 4, 7, 8 imply $N\subset B$ or $b_{13,8}=1$. At last rows 4, 10, 13 and columns 1, 5, 8 yield $\bar{N}\subset B$ or $b_{4,1}=0$, and we find N in rows 1, 4, 5 and columns 1, 3, 6.

(iii) $b_{i,4}=0$, $b_{i,5}=1$ for all i with $4\leqq i\leqq 7$. There are at most 13 entries 1 in M_2 (otherwise (ii)). At least 3 entries 1 are in every row of rows 4 to 13 in columns 1 to 5, or in 4 to 8 (otherwise $N\subset B$). Thus at least 17 entries 1 exist in M_1, and in M_3, and either we use Lemma 3, or at least 2 columns of M_1, and at least 2 columns of M_3 have exactly 6 entries 1.

Every column in the first 4 rows of M_1 or of M_3 has at least 2 entries 1 (otherwise $S_3\subset B$, and we use Lemma 4). In the first 4 rows of M_1 and M_3 there exists at most one column with 4 entries 1 (otherwise $\bar{N}\subset B$), which we can assume not to be in M_1. Each of the first 4 rows of M_1 and of M_3 has at least 2 entries 1 (otherwise $N\subset B$). Altogether the first 4 rows of M_1 contain at least 8 entries 1, so that 2 columns have exactly 3 entries 1, and we can assume $b_{4,3}=b_{5,3}=b_{6,3}=b_{4,2}=b_{5,2}=b_{7,2}=1$, $b_{6,2}=b_{7,3}=0$ (otherwise $\bar{N}\subset B$). Then $b_{6,1}=b_{7,1}=1$, or $N\subset B$.

After possibly changing rows 6 and 7, and columns 2 and 3 of B we can assume column 3 to be one of the 2 columns of M_1 with 6 entries 1, that is, $b_{8,3}=b_{9,3}=b_{10,3}=1$. Then Lemma 9 can be used, and we find S_8 in rows 4, 8, 5, 9, 6, 10 and columns 8, 7, 6, 4, 5 of B (columns 4 and 5 of S_8 have to be in M_2), that is,

$$b_{4,7}=b_{4,8}=b_{5,6}=b_{5,8}=b_{6,6}=b_{6,7}=b_{8,4}=b_{8,7}=b_{8,8}$$

$$=b_{9,4}=b_{9,6}=b_{9,8}=b_{10,4}=b_{10,6}=b_{10,7}=1,$$

https://doi.org/10.1017/S0017089500004171 Published online by Cambridge University Press

and

$$b_{4,6}=b_{5,7}=b_{6,8}=b_{8,5}=b_{8,6}=b_{9,5}=b_{9,7}=b_{10,5}=b_{10,8}=0.$$

In rows 4, 5, 7 and columns 2, 5, 8 of B we find $\bar{N}$, or $b_{7,8}=0$, and then $b_{7,6}=b_{7,7}=1$ (otherwise $N\subset B$). Rows 4, 5, 8, 9 and columns 2, 3, 8 of B yield $\bar{N}\subset B$, or $b_{8,2}=b_{9,2}=0$.

If 6 entries 1 are in column 1 of M_1, then N or $\bar{N}\subset B$ by Lemma 9 (S_8 is impossible, since rows 6 and 7 are identical in columns 4 to 8). Thus 6 entries 1 exist in column 2 of M_1. Then Lemma 9 can be used, and columns 4 and 5 of S_8 have to be in M_2. Then at most 11 entries 1 exist in M_2, and therefore M_1 has more than 18 entries 1, that means, at least one column of M_1 has 7 entries 1, and Lemma 3 can be used.

Lemma 12. *Let Z_i denote the number of pairs of rows of B with i equal columns. Then either N or $\bar{N}\subset B$, or*

$$Z_9+Z_{10}\geqq 12+\sum_{i\leqq 6} Z_i. \tag{6}$$

Proof. If $Z_i>0$ for $i>10$, then by Lemma 2 we have N or $\bar{N}\subset B$. Otherwise, if s_i denotes the number of columns with i entries 0, then

$$\sum_{i\leqq 10} Z_i=\binom{13}{2}=78, \quad \text{and} \quad \sum_{i\geqq 0} s_i=17, \tag{7}$$

$$\sum_{i\leqq 10} iZ_i=\sum_{i\geqq 0}\left(\binom{i}{2}+\binom{13-i}{2}\right)s_i\geqq 36\sum_{i\geqq 0} s_i=612. \tag{8}$$

At most 2 pairs of rows in every triple of rows of B have an even number of equal columns (Lemma 1, $c=17$). Again the Theorem of Turán ([**2**], p. 17) implies then

$$Z_0+Z_2+Z_4+Z_6+Z_8+Z_{10}\leqq[13^2/4]=42. \tag{9}$$

With (7) and Z_8+Z_{10} from (9) we obtain

$$\begin{aligned}\sum_{i\leqq 10} iZ_i&=2(Z_9+Z_{10})+Z_8+Z_{10}+7\sum_{i\leqq 10} Z_i-\sum_{i\leqq 6}(7-i)Z_i\\ &\leqq 2(Z_9+Z_{10})+588-2\sum_{i\leqq 6} Z_i. \end{aligned}\tag{10}$$

From (8) and (10) we obtain (6).

Lemma 13. *If there are three rows a, b, c in B, so that the number of equal columns is at least 9 in rows a and b, and in rows a and c, then N or $\bar{N}\subset B$.*

Proof. Let a, b, c be rows 1, 2, 3 of B. If more than 5 columns with both entries 0 (or both entries 1) occur in one pair of rows we use Lemma 2. If then the pairs of rows 1 and 2, and of rows 1 and 3 both contain 5 columns with both entries 0, or both contain 5 columns with both entries 1, then N or $\bar{N}\subset B$ follows directly, or by use of Lemmas 7, 10, or 11. Thus in the following we can assume 5 pairs with both entries 0 in the first 5 columns of rows 1 and 2, 5 pairs with both entries 1 in columns 6 to 10 of rows 1 and 3, and exactly 4

https://doi.org/10.1017/S0017089500004171 Published online by Cambridge University Press

columns with both entries 1 in rows 1 and 2, and exactly 4 columns with both entries 0 in rows 1 and 3.

If in rows 2 and 3 there are less than 7 equal columns, then by Lemma 12 we have $Z_9+Z_{10}\geqq 13$ in B. Then the pigeonhole principle guarantess that one row of B exists which has together with each of two other rows at least 9 equal columns, among which in both pairs of rows occur 5 columns with both entries 0, or 5 columns with both entries 1. Thus we can use Lemmas 7, 10 or 11 to get N or $\bar{N}\subset B$.

In columns 11 to 17 occur at least 2 columns with both entries 0 in rows 1 and 3, and at least 2 columns with both entries 1 in rows 1 and 2. We can assume $b_{1,11}=b_{1,12}=b_{3,11}=b_{3,12}=0$ and $b_{1,13}=b_{1,14}=b_{2,13}=b_{2,14}=1$. Then S_1 or $\bar{S}_1\subset B$, or $b_{2,11}=b_{2,12}=1$ and $b_{3,13}=b_{3,14}=0$. Now at least 7 equal columns in rows 2 and 3 are possible only if at least 4 of them occur in columns 1 to 10. More than 4 columns, however, yield N or $\bar{N}\subset B$. Thus we can choose $b_{3,1}=b_{3,2}=b_{2,8}=b_{2,9}=b_{2,10}=0$, $b_{3,3}=b_{3,4}=b_{3,5}=b_{2,6}=b_{2,7}=1$, and without loss of generality $b_{2,15}=b_{2,16}=b_{3,15}=b_{3,16}=1$. Then $\bar{N}\subset B$, or $b_{1,15}=b_{1,16}=0$.

If column 15 in rows 4 to 13 contains more than 6 entries 0, or more than 5 entries 1, then S_2 or $\bar{S}_2\subset B$, and we can use Lemma 3. It remains to consider that column 15 in rows 4 to 13 contains (i) 5 entries 1, and (ii) 6 entries 0.

(i) $b_{i,15}=1$ for all i with $4\leqq i\leqq 8$. By Lemma 9 we can assume S_8 in rows 3 to 8 and columns 1 to 5. There exist 3 rows in this S_8 which together with row 3 of B have 2 columns with both entries 1. If one of these rows contains more than one entry 1 in columns 6 to 10, then $\bar{S}_{10}\subset B$, and we use Lemma 11. Otherwise in columns 6 to 10 we find $\bar{N}$, or we have 3 rows with at least 4 entries 0, and 8 rows with at least 3 entries 0. Then any distribution of $3\binom{4}{2}+8\binom{3}{2}=42$ pairs 00 among the columns 6 to 10 guarantees $S_3\subset B$ $\left(\text{since } 4\binom{5}{2}<42\right)$, and we use Lemma 4.

(ii) $b_{i,15}=0$ for all i with $4\leqq i\leqq 9$. By Lemma 9 we can assume $\bar{S}_8$ in rows 4 to 9 and columns 6 to 10. If column 16 has at least 4 entries 0 in rows 4 to 9, then $S_3\subset B$ (together with row 1 of B), and Lemma 4 can be used. Thus we can assume $b_{4,16}=b_{5,16}=b_{6,16}=1$. If then in columns 3 to 5 in one of rows 4 to 6 there are 2 entries 1, then $\bar{S}_{10}\subset B$, and we use Lemma 11. Otherwise $N\subset B$, or in rows 4 to 6 and columns 1 and 2 occur entries 1 only. Then, however, we find $\bar{N}$ in rows 4 to 6 and columns 1, 2 and 16, and Lemma 13 is proved.

The proof of Theorem 2 is complete, since Lemma 12 for any B guarantees the existence of three rows which enable us to apply Lemma 13.

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