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LOCALLY BOUNDED TOPOLOGIES ON THE RING OF INTEGERS OF A GLOBAL FIELD

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1. Introduction and basic definitions. A subset A of a topological ring R is *bounded* if given any neighborhood U of zero, there exists a neighborhood V of 0 such that $AV \subseteq U$ and $VA \subseteq U$. The topology on R is *locally bounded* if there exists a bounded neighborhood of 0.

We recall that a *seminorm* $\|\cdot\cdot\|$ on a ring R is a function from R into the non-negative real numbers satisfying $\|x\| = 0$ if $x = 0$, $\|-x\| = \|x\|$, $\|x + y\| \leqq \|x\| + \|y\|$ and $\|xy\| \leqq \|x\|\,\|y\|$ for all x, y in R. A seminorm $\|\cdot\cdot\|$ on R is a *norm* on R if $\|x\| = 0$ implies $x = 0$. We note that a seminorm $\|\cdot\cdot\|$ on R defines a locally bounded topology $T_{\widehat{\|\cdot\cdot\|}}$ on R, and a norm on R defines a Hausdorff, locally bounded topology on R. Two norms on R are said to be *equivalent* if they define the same topology on R.

Let D be a Dedekind domain that is not a field, let F be the quotient field of D and let P be the set of nonzero prime ideals of D. Every nonzero fractional ideal A of F has the unique factorization $A = \prod_{p \in P} p^{n_p(A)}$ where $n_p(A) \in Z$ and $n_p(A) = 0$ for all but finitely many $p \in P$. If A and B are nonzero fractional ideals, then $A \subseteq B$ if and only if $n_p(A) \geqq n_p(B)$ for all $p \in P$, $n_p(A + B) = \min\{n_p(A), n_p(B)\}$ and $n_p(AB) = n_p(A) + n_p(B)$ for all $p \in P$ [**2**, pp. 25-26]. For each $p \in P$, let v_p be the p-adic valuation on F defined by, $v_p(0) = \infty$ and $v_p(x) = n_p(Dx)$ for $x \neq 0$. Let $|\cdot\cdot|_p$ be the absolute value on F defined by $|a|_p = 2^{-v_p(a)}$ for all $a \in F$. Then $|\cdot\cdot|_p$ restricted to D defines a non-discrete, Hausdorff, locally bounded ring topology on D which will usually be denoted by T_p. If $p \in P$ and n is a nonnegative integer, then the set $\{p^n\}$ is a fundamental system of neighborhoods of zero for a nondiscrete, locally bounded ring topology on D. Henceforth, we will denote this topology by T'_{p^n} and we note that T'_{p^n} is associated with the seminorm $\|\cdot\cdot\|'_{p^n}$ defined on D by

$$\|d\|'_{p^n} = \begin{cases} 0 \text{ if } d \in p^n \\ 1 \text{ otherwise.} \end{cases}$$

If K is an algebraic number field, that is, a finite dimensional extension of the rational numbers $\mathbf{Q}$, let R be the integral closure of the rational integers $\mathbf{Z}$ in K, let P be the set of nonzero prime ideals of R and let P_∞

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be a set of proper archimedean absolute values on K such that each proper archimedean absolute value on K is equivalent to exactly one element of P_∞. Mahler proved that if $\|\cdot\cdot\|$ is a norm on R, then there is a finite subset P_1 of P, a finite subset P_2 of $P\backslash P_1$, positive integers $n(p)$ for $p \in P_2$ and a finite subset P_3 of P_∞ such that $\|\cdot\cdot\|$ is equivalent to

$$\sup\ (\sup_{p\in P_1}|\cdot\cdot|_p,\quad \sup_{p\in P_2}\|\cdot\cdot\|'_{p^{n(p)}},\quad \sup_{|..|\in P_3}|\cdot\cdot|)\quad [\mathbf{6}, \text{p. } 328].$$

In Section 2, we consider Hausdorff locally bounded topologies on a Dedekind domain R for which there exists a nonzero topological nilpotent (that is, a nonzero element c in R such that $c^n \to 0$) and we characterize all such topologies on the ring of integers of a separable, finite dimensional extension of $F_q(x)$ where F_q is a finite field having q elements and x is transcendental over F_q. We thereby obtain the analogue of Mahler's results for these rings (Corollary 2, Theorem 2). Then in Section 3, we generalize Mahler's Theorem and completely describe all Hausdorff, nondiscrete, locally bounded topologies on the ring of integers of an algebraic number field for which the set of topological nilpotents is a neighborhood of zero.

2. Locally bounded topologies with nonzero topological nilpotents.

THEOREM 1. *Let T be a Hausdorff, locally bounded ring topology on a Dedekind domain D. Suppose that the open ideals of D form a fundamental system of neighborhoods of zero for T and that there exists a nonzero topological nilpotent for T. Let $J = \{d \in D: d^n \to 0\}$.*

1. *J is a nonzero proper ideal of D and $J = p_1 \ldots p_r$ for some sequence of distinct nonzero prime ideals of D.*

2. *There exists a nonempty subset I of $[1, r]$ and positive integers n_j, $j \in [1, r]\backslash I$, such that*

$$T = \sup\ (\sup_{i\in I} T_{p_i}, \sup_{j\in[1,r]\backslash I} T'_{p_j^{n_j}}).$$

Proof. 1. Let $r_1, r_2 \in J$ and let $a_1, a_2 \in D$. Let U be a T-open ideal and let $n > 0$ be such that $r_1^n, r_2^n \in U$. Then for all $m \geqq 2n$,

$$\begin{aligned}(a_1r_1 + a_2r_2)^m &= (a_1r_1 + a_2r_2)^{m-2n} \sum_{i=0}^{2n} \binom{2n}{i} a_1^i r_1^i\, a_2^{2n-i} r_2^{2n-i} \\ &= (a_1r_1 + a_2r_2)^{m-2n}\Bigg[r_2^n \sum_{i=0}^{n} \binom{2n}{i} a_1^i r_1^i\, a_2^{2n-i} r_2^{n-i} \\ &\qquad + r_1^n \sum_{i=n+1}^{2n} \binom{2n}{i} a_1^i r_1^{i-n}\, a_2^{2n-i} r_2^{2n-i}\Bigg] \\ &\in D(UD + UD) \subseteq U.\end{aligned}$$

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So $a_1r_1 + a_2r_2 \in J$ and hence J is a nonzero ideal of D. As T is Hausdorff, $1 \notin J$ and so J is a proper ideal of D. Therefore there exist distinct nonzero prime ideals $p_1, \ldots, p_r$ of D and positive integers $\alpha_1, \ldots, \alpha_r$ such that

$$J = \prod_{i=1}^{r} p_i^{\alpha_i}.$$

Let $n = \max\{\alpha_i : 1 \leqq i \leqq r\}$. For each i, $1 \leqq i \leqq r$, let $x_i \in p_i$. Then $(x_1 \ldots x_r)^n \in J$, that is, $(x_1 \ldots x_r)^n$ is a topological nilpotent. Consequently, $x_1 \ldots x_r$ is a topological nilpotent and hence an element of J. As J is an ideal,

$$p_1 \ldots p_r \subseteq J = \prod_{i=1}^{r} p_i^{\alpha_i}.$$

So $\alpha_i = 1$ for $i = 1, 2, \ldots, r$.

2. We first show that if W is a T-open ideal of D, then $W = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ where $\alpha_i \geqq 0$ for $i \in [1, r]$. As J is an ideal of D, there exist h_1 and h_2 in J such that $J = Dh_1 + Dh_2$ [**9**, Theorem *I*, p. 110]. Then

$$p_1 \ldots p_r = J = Dh_1 + Dh_2 = \prod_{p \in P} p^{\min\{v_p(h_1), v_p(h_2)\}}.$$

So for all p in $P \backslash \{p_1, \ldots, p_r\}$, either $v_p(h_1) = 0$ or $v_p(h_2) = 0$. Note also that for all $n > 0$,

$$Dh_1^n = \prod_{p \in P} p^{nv_p(h_1)} \quad \text{and} \quad Dh_2^n = \prod_{p \in P} p^{nv_p(h_2)}.$$

Let $m > 0$ be such that $h_1^m, h_2^m \in W$. Then $Dh_1^m, Dh_2^m \subseteq W$ and so for all $p \in P$, $n_p(W) \leqq mv_p(h_1)$ and $n_p(W) \leqq mv_p(h_2)$. Therefore for all $p \in P \backslash \{p_1, \ldots, p_r\}$, $n_p(W) = 0$ and consequently, $W = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ where $\alpha_i \geqq 0$ for $i \in [1, r]$.

For each $i \in [1, r]$, let $n_i = \sup\{n_{p_i}(W) : W \text{ is a } T\text{-open ideal}\}$. Suppose there exists an i with $n_i = 0$. Without loss of generality, assume $i = 1$. Let $x_i \in p_i$ for $i = 2, \ldots, r$, let $h = x_2 \ldots x_r$ and let W be a T-open ideal. As $n_1 = 0$, $W = p_2^{\alpha_2} \ldots p_r^{\alpha_r}$. Let $n = \max\{\alpha_i : 2 \leqq i \leqq r\}$. Then for all $m \geqq n$, $h^m \in W$ and so $h \in J$. Consequently, as J is an ideal, $p_2 \ldots p_r \subseteq J = p_1 \ldots p_r$. This is a contradiction. So $n_i > 0$ for $i = 1, 2, \ldots, r$.

Define I by $I = \{i \in [1, r] : n_i = \infty\}$. Since T is Hausdorff and since the intersection of finitely many nonzero ideals of a Dedekind domain is a nonzero ideal, there exist infinitely many open ideals $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$. Hence there exists an i such that $n_i = \infty$ and so $I \neq \emptyset$.

Let $(\bigcap_{i \in I} p_i^{\alpha_i}) \cap (\bigcap_{j \in [1,r] \backslash I} p_j^{n_j})$ be a neighborhood of 0 for

$$\sup\,(\sup_{i \in I} T_{p_i}, \sup_{j \in [1,r] \backslash I} T'_{p_j n_j}).$$

If $i \in I$, then as $n_i = \infty$, there exists a T-open ideal W_i such that

$n_{p_i}(W_i) \geqq \alpha_i$. So $W_i \subseteq p_i^{\alpha_i}$. For each $j \in [1, r]\backslash I$, let W_j be a T-open ideal such that $n_{p_j}(W_j) = n_j$. Then $W_j \subseteq p_j^{n_j}$. So, the T-open ideal $\bigcap_{i=1}^r W_i$ is contained in

$$(\textstyle\bigcap_{i \in I}^r p_i^{\alpha_i}) \cap (\bigcap_{j \in [1,r]\backslash I} p_j^{n_j}).$$

Therefore, the supremum topology is weaker than T. Finally, as every T-open ideal is of the form $\bigcap_{i=1}^r p_i^{\alpha_i}$ where $\alpha_j \leqq n_j$ for all $j \in [1, r]\backslash I$, T is clearly weaker than the supremum topology.

COROLLARY 1. *Let T be a Hausdorff, locally bounded ring topology on D for which the open ideals form a fundamental system of neighborhoods of* 0. *If there exists a nonzero topological nilpotent for T, then T is normable.*

Proof. T is defined by $\|\cdot\cdot\|$ where

$$\|d\| = \sup\,(\sup_{i \in I}|d|_{p_i}, \sup_{j \in [1,r]\backslash I}\|d\|'_{p_j^{n_j}})$$

for all $d \in D$. As $I \neq \emptyset$, $\|\cdot\cdot\|$ is a norm on D.

If K is a finite dimensional extension of the rational numbers $\mathbf{Q}$, define R, P and P_∞ as in Section 1. Then R is a Dedekind domain properly contained in K and P_∞ is a finite set [**7**, Theorem 1]. In [**4**], we proved that if P_∞ has exactly one element then every Hausdorff, nondiscrete, locally bounded topology on R has a fundamental system of neighborhoods of zero consisting of ideals (Theorem 2).

COROLLARY 2. *Let K be a finite extension of $\mathbf{Q}$ such that P_∞ has exactly one element and let R be the integral closure of $\mathbf{Z}$ in K. Suppose that T is a Hausdorff, locally bounded ring topology on R for which there exists a nonzero topological nilpotent. Then there exists a nonempty, finite set $\{p_1, \ldots, p_r\}$ of P, a finite set $\{q_1, \ldots, q_s\}$ of $P\backslash\{p_1, \ldots, p_r\}$ and positive integers n_j for $j = 1, 2, \ldots, s$ such that*

$$T = \sup\,(\sup_{1 \leqq i \leqq r} T_{p_i}, \sup_{1 \leqq j \leqq s} T'_{q_j^{n_j}}).$$

In particular, T is normable.

Note. Mahler's characterization of a norm on R for the special case when P_∞ has exactly one element follows easily from Corollary 2.

COROLLARY 3. *If R is $\mathbf{Z}$ or the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{-m})$ where m is a positive, square free integer, then every Hausdorff, locally bounded ring topology on R for which there exists a nonzero topological nilpotent is normable.*

Proof. As P_∞ has exactly one element (see the proof of Corollary 3 to Theorem 3 of [**7**]), Corollary 3 follows from Corollary 2.

We recall that an *almost order* of a field F is subset H of F such that 1) $0, 1 \in H$, 2), $HH \subseteq H$, 3) $-H \subseteq H$, 4) there exists a nonzero h in H

such that $h(H + H) \subseteq H$ and 5) for each nonzero a in F there exist nonzero elements r and s in H such that $a = rs^{-1}$. If T is a Hausdorff, nondiscrete, locally bounded ring topology on F, then there exists an almost order H of F which is a T-bounded neighborhood of zero. Conversely, if H is a proper almost order of F, then there exists a unique Hausdorff, nondiscrete, locally bounded topology T_H on F for which H is a bounded neighborhood of 0 [**5**, Theorems 5 and 6].

Let K be a finite, separable extension of $F_q(x)$ where F_q is a finite field having q elements and x is transcendental over F_q. Let R be the integral closure of $F_q[x]$ in K, let v_∞ be the valuation on $F_q(x)$ defined by $v_\infty(f/g) = \deg g - \deg f$ and let P_∞ be a complete set of extensions of v_∞ to K [**1**, Definition 3, p. 140 and Theorem 1, p. 143]. For each $v \in P_\infty$, let $|\cdot\cdot|_v$ be the absolute value defined on K by $|r|_v = 2^{-v(r)}$. Then P_∞ is a finite set, R is a Dedekind domain properly contained in K, K is the quotient field of R and every proper valuation on K is equivalent to either v_p for some nonzero prime ideal p of R or to v_i for some $v_i \in P_\infty$ [**7**, Theorem 2].

In [**7**], we proved that if T is a Hausdorff, nondiscrete, locally bounded topology on K, then there exists a nonempty, proper subset S of $P \cup P_\infty$ such that the set $O(S)$ defined by,

$$O(S) = \{z \in K: v(z) \geqq 0 \text{ for all } v \in S\},$$

is an almost order of K and $T = T_{O(S)}$ (Theorem 4). We note that if S is a nonempty, proper subset of $P \cup P_\infty$, then $O(S)$ is a Dedekind domain with quotient field K [**8**, Theorem 21.3, p. 43 and Theorem 33.12, p. 78].

LEMMA 1 [**4**, Lemma 2]. *Let I be an integral domain properly contained in its quotient field F. If T is a Hausdorff, nondiscrete, locally bounded ring topology on I, then there exists a bounded neighborhood H of 0 for T such that H is a proper almost order of F.*

LEMMA 2. *Let K be a finite, separable extension of $F_q(x)$ and define R, P and P_∞ as before. Let T be a nondiscrete, Hausdorff, locally bounded ring topology on R. Then there exists a Dedekind domain R' contained in R such that K is the quotient field of R', R' is open for T and the open $R'-$ideals form a fundamental system of neighborhoods of 0 for T. Moreover, there exists a proper subset S of P_∞ such that*

$$R' = \{z \in R: v(z) \geqq 0 \textit{ for all } v \in S\}.$$

Proof. Let H be a bounded neighborhood of 0 for T such that H is a proper almost order of K. Let $\hat{T}$ be the unique Hausdorff, nondiscrete, locally bounded topology on K for which H is a bounded neighborhood of 0. We note that $\hat{T}|_R \supseteq T$.

By [**7**, Theorem 4], there exists a nonempty proper subset S' of $P \cup P_\infty$ such that $O(S')$ is a bounded neighborhood of 0 for $\hat{T}$. Let $h \in H\backslash\{0\}$ be such that $hO(S') \subseteq H \subseteq R$. Then $O(S') \subseteq R$. Indeed, if $O(S') \nsubseteq R$, let $z \in O(S')\backslash R$ and let $p \in P$ be such that $v_p(z) < 0$. Choose $m > 0$ such that $v_p(h) + mv_p(z) < 0$. Then $hz^m \in hO(S') \subseteq R$ but $v_p(hz^m) < 0$. This is a contradiction. So $O(S') \subseteq R = O(P)$. Therefore, $P \subseteq S'$. Let $S = S' \cap P_\infty$. Then $S \subset P_\infty$ as S' is a proper subset of $P \cup P_\infty$. Let $R' = O(P \cup S)$. Then R' is a Dedekind domain with quotient field K [**8**, Theorem 21.3, p. 43 and Theorem 33.12, p. 78] and $R' \subseteq R$. As R' is open for $\hat{T}$, there exists a nonzero element h_1 in H such that $h_1H \subseteq R'$. Arguing as before we find that $H \subseteq R'$. So R' is open for T.

Now let U be any closed T-neighborhood of 0. We may assume that $U \subseteq R' \subseteq R$. As R is a $\hat{T}$-neighborhood of 0 and as $\hat{T}|_R \supseteq T$, U is a $\hat{T}$-neighborhood of 0. Then, since R' is a $\hat{T}$-bounded neighborhood of 0, there exists $r \in R'\backslash\{0\}$ such that $R'r \subseteq U$. Therefore, U contains a nonzero R'-ideal I which we may assume is closed for $T|_{R'}$. Then R'/I, equipped with the topology induced by $T|_{R'}$, is a Hausdorff ring [**3**, Proposition 18, p. 25]. Also, R'/I is finite [**8**, Theorem 33.2, p. 67] and hence discrete. Therefore, as the canonical epimorphism from R' to R'/I is continuous, I is open for $T|_{R'}$ and hence for T as well.

THEOREM 2. *Let K be a finite, separable extension of $F_q(x)$ and let R be the integral closure of $F_q[x]$ in K. Let T be a Hausdorff, locally bounded ring topology on R for which there exists a nonzero topological nilpotent. Then there exist disjoint, finite subsets P_1 and P_2 of P, a proper subset S of P_∞ and positive integers $n(p)$ for each $p \in P$ such that $P_1 \cup S \neq \emptyset$ and*

$$T = \sup\,(\sup_{p \in P_1} T_p, \sup_{s \in S} T_s, \sup_{p \in P_2} T'_{p^{n(p)}}).$$

In particular, T is normable.

Proof. Let R' be a Dedekind domain contained in R such that R' is open and the open R'-ideals form a fundamental system of neighborhoods of 0 for T. Let $S' \subset P_\infty$ be such that $R' = O(P \cup S')$. If $S' = \emptyset$, then $R = R'$ and the result follows from Theorem 1. So we may assume that $S' \neq \emptyset$ and hence P_∞ has two or more elements.

Let P' be the set of nonzero prime ideals of R'. As R' is T-open, R' contains a nonzero topological nilpotent. So, by Theorem 1, there exists a nonempty, finite subset P'_1 of P', a finite subset P'_2 of $P'\backslash P'_1$ and positive integers $m(p')$, $p' \in P_2'$, such that

$$T|_{R'} = \sup\,(\sup_{p' \in P'_1} T_{p'}, \sup_{p' \in P'_2} T'_{p'^{m(p')}}),$$

where for each $p' \in P'_1$ and each $p' \in P'_2$, $T_{p'}$ and $T'_{p'^{m(p')}}$ are the appropriate topologies defined on R'.

Each $p' \in P'_1$ defines a proper valuation $v_{p'}$ on K as K is the quotient field of R' and hence $v_{p'}$ is equivalent to v_p for some $p \in P$ or to v_i for some $v_i \in P_\infty$. Therefore, the topology induced on R' by $v_{p'}$ is the same as that induced on R' by v_p for some $p \in P$ or by v_i for some $v_i \in P_\infty$.

Let $p' \in P'_2$. If $v_{p'}$ is equivalent to v_i for some $v_i \in P_\infty$, then there exists a positive integer n such that

$$p'^{m(p')} = \{z \in R' : v_{p'}(z) \geqq m(p')\} = \{z \in R' : v_i(z) \geqq n\}.$$

Hence $\{z \in R' : v_i(z) \geqq n\}$ is a neighborhood of 0 for $T|_{R'}$ and therefore for T as well. By the definition of $m(p')$, if $t > m(p')$, then $(p')^t$ is not a neighborhood of 0 for $T|_{R'}$. Therefore, $\{z \in R' : v_i(z) \geqq L\}$ is not a neighborhood of 0 for $T|_{R'}$ for any $L > n$ and thus $\{z \in R' : v_i(z) \geqq L\}$ is not a T-neighborhood of 0 if $L > n$. We will show that this leads to a contradiction and therefore conclude that for each $p' \in P'_2$, $v_{p'}$ is equivalent to v_p for some $p \in P$. As P_∞ has more than one element, $P_\infty \backslash \{v_i\} \neq \emptyset$. Consequently, by the Very Strong Approximation Theorem [**8**, Theorem 33.11, p. 77], for each $m > 0$, there exists $h \in R\backslash\{0\}$ such that $v(h) > m$ for all $v \in P_\infty \backslash \{v_i\}$. Therefore, by the Product Theorem [**8**, Theorem 33.1, p. 66], there exists h in R such that $v_i(h) \leqq -1$. As $\{z \in R' : v_i(z) \geqq n\}$ is a T-neighborhood of 0 and as the mapping $z \to hz$ from R to R is continuous at 0 when R is given the T-topology, there exists a T-neighborhood U of 0 such that $hU \subseteq \{z \in R' : v_i(z) \geqq n\}$. Then for each $u \in U$,

$$v_i(h) + v_i(u) = v_i(hu) \geqq n$$

and so

$$v_i(u) \geqq n - v_i(h) \geqq n + 1.$$

Therefore,

$$U \cap R' \subseteq \{z \in R' : v_i(z) \geqq n + 1\}$$

and so $\{z \in R : v_i(z) \geqq n + 1\}$ is a T-neighborhood of 0. This is a contradiction. Hence if $p' \in P'_2$, then $v_{p'}$ is equivalent to v_p for some $p \in P$.

Let $P_1 = \{p \in P : v_p$ is equivalent to $v_{p'}$ for some $p' \in P'_1\}$. Let $S'' = \{v_i : v_i$ is equivalent to $v_{p'}$ for some $p' \in P'_1\}$ and denote $S' \cup S''$ by S. As $P'_1 \neq \emptyset$, $P_1 \cup S \neq \emptyset$. Let $P_2 = \{p \in P : v_p$ is equivalent to $v_{p'}$ for some $p' \in P'_2\}$. For each $p \in P_2$, let $n(p) > 0$ be such that

$$\{z \in R' : v_p(z) \geqq n(p)\} = \{z \in R' : v_{p'}(z) \geqq m(p')\}.$$

Note that $S \neq P_\infty$. Indeed, if $v \in S$, then

$$\{x \in R : v(x) > 0\} \supseteq R'$$

and hence is a T-neighborhood of zero. So if $S = P_\infty$, then $\{z \in R:$

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$v(z) \geqq 0$ for all $v \in P_\infty\}$ is a T-neighborhood of 0. But $\{z \in R: v(z) \geqq 0$ for all $v \in P_\infty\}$ is a finite set [**8**, Theorem 33.4, p. 69], contradicting the hypothesis that T has a nonzero topological nilpotent and is therefore nondiscrete. So $S \subset P_\infty$. We note further that if $v \in S'$, then the topology induced on R' by v is weaker than $T|_{R'}$. The proof follows the same argument as before and the fact that $\{z \in R: v(z) \geqq 0\}$ is T-open since $R' \subseteq \{z \in R: v(z) \geqq 0\}$.

Consider

$$\sup\,(\sup_{p \in P_1} T_p, \sup_{v \in S} T_v, \sup_{p \in P_2} T'_{p^{n(p)}})$$

where each T_p, T_v and $T'_{p^{n(p)}}$ is the appropriate topology defined on R. As

$$R' \supseteq \bigcap_{v \in S} \{z \in R: v(z) \geqq 0\},$$

R' is a neighborhood of 0 for the supremum topology. It follows from the definitions of P_1, S, P_2 and $n(p)$ that T induces the same topology on R' as the supremum topology. Since R' is open for both topologies,

$$T = \sup\,(\sup_{p \in P_1} T_p, \sup_{v \in S} T_v, \sup_{p \in P_2} T'_{p^{n(p)}}).$$

Finally, as $P_1 \cup S \neq \emptyset$, the seminorm $\|\cdot\cdot\|$ defined on R by

$$\|r\| = \sup\,(\sup_{p \in P_1} |r|_p, \sup_{v \in S} |r|_v, \sup_{p \in P_2} \|r\|'_{p^{n(p)}})$$

is a norm on R which clearly defines T.

COROLLARY 1. *If T is a Hausdorff, locally bounded ring topology on $F_q[x]$ for which there exists a nonzero topological nilpotent, then there exist prime polynomials $p_1, \ldots, p_r, q_1, \ldots, q_s$ in $F_q[x]$ with $r \geqq 1$ and positive integers n_i for $i = 1, 2, \ldots s$ such that*

$$T = \sup\,(\sup_{1 \leqq i \leqq r} T_{p_i}, \sup_{1 \leqq i \leqq s} T'_{q_i^{n_i}}).$$

COROLLARY 2. *A Hausdorff, nondiscrete, locally bounded ring topology T on the integral closure R of $F_q[x]$ in K is defined by a nontrivial norm if and only if there exists a nonzero topological nilpotent for T. Furthermore, every nontrivial norm on R is equivalent to the supremum of finitely many p-adic absolute values on R, finitely many absolute values on R associated with the valuations v_i extending v_∞ to K, and finitely many seminorms on R defined by p^n where p is nonzero prime ideal of R and n is a positive integer.*

3. Locally bounded topologies on the ring of integers of an algebraic number field. Throughout this section we assume that K is a finite extension of $\mathbf{Q}$ and that R, P and P_∞ are defined as in Section 1. In Section 2, we considered the case when P_∞ has exactly one element. Henceforth, we assume that the cardinality of P_∞ is greater than 1.

LEMMA 3. *Let T be a nondiscrete, Hausdorff, locally bounded ring topology on R for which the set N_1 of topological nilpotents is a neighborhood of* 0. *For each $a \in N_1 \setminus \{0\}$, $T \subseteq T_a$ where*

$$T_a = \sup\left(\sup_{p \in P_1(a)} T_p, \sup_{|\cdot\cdot| \in P_2} T_{|\cdot\cdot|}\right),$$
$$P_1(a) = \{p \in P: v_p(a) > 0\} \quad \text{and}$$
$$P_2 = \{|\cdot\cdot| \in P_\infty: T_{|\cdot\cdot|} \subseteq T\}.$$

Proof. Let U be a T-bounded neighborhood of zero such that $1 \in U$, $-U = U$ and $UU \subseteq U$. Let $n_0 > 0$ be such that $a^n(U + U) \subseteq U$ for all $n \geqq n_0$ and let $b = a^{n_0}$. As a is a topological nilpotent, b is as well and furthermore $v_p(a) > 0$ if and only if $v_p(b) > 0$ for $p \in P$. Clearly, $(b^n U)_{n \geqq 0}$ is a decreasing sequence of subsets of R such that

$$0 \in b^n U, \quad b^{n+1}U + b^{n+1}U \subseteq b^n U, \; -(b^n U) = b^n U \quad \text{and}$$
$$(b^n U)\,(b^n U) \subseteq b^n U \quad \text{for all } n \geqq 0.$$

Also, if $x \in R$, then as the mapping $y \to yx$ is continuous at 0 and as b is a topological nilpotent, there exists $m > 0$ such that $b^m x \in U$. Then $b^{m+n}Ux \subseteq b^n UU \subseteq b^n U$.

Therefore, $(b^n U)_{n \geqq 0}$ is a fundamental system of neighborhoods of 0 for a ring topology T_b on R. Obviously, b is a topological nilpotent for T_b, U is a T_b-bounded neighborhood of 0 and the mapping $x \to bx$ from R to R is an open mapping when R is equipped with the T_b topology. Furthermore, $T_b \supseteq T$; indeed, if W is a T-neighborhood of 0, then as U is T-bounded and as b is a T-topological nilpotent, there exists $n > 0$ such that $b^n U \subseteq W$. Consequently $T_b \supseteq T$. T_b is therefore Hausdorff and every T_b-topological nilpotent is a T-topological nilpotent. By [**10**, Theorem 4], there exists a norm N_b on R which defines T_b. By Mahler's Theorem [**6**, p. 328], there exist disjoint finite subsets P_1 and P_3 of P, positive integers $n(p)$, $p \in P_3$, and a finite subset P_2 of P_∞ such that N_b is equivalent to

$$\sup\left(\sup_{p \in P_1} |\cdot\cdot|_p, \sup_{|\cdot\cdot| \in P_2} |\cdot\cdot|, \sup_{p \in P_3} \|\cdot\cdot\|'_{p^{n(p)}}\right).$$

Since b is a T_b-topological nilpotent, $|b| < 1$ for all $|\cdot\cdot| \in P_2$ and $v_p(b) > 0$ for all $p \in P_1 \cup P_3$.

We first show that $P_3 = \emptyset$. If $P_3 \neq \emptyset$, let $m > 0$ be such that

$$b^m U \subseteq \prod_{p \in P_3} p^{n(p)}.$$

So $v_p(u) \geqq n(p) - mv_p(b)$ for all $u \in U$, $p \in P_3$. Given $t > 0$, let $l > 0$ be such that

$$n(p) - mv_p(b) + lv_p(b) > t \quad \text{for all } p \in P_3.$$

Then $b^l U \subseteq \prod_{p \in P_3} p^t$ and so p^t is open for T_b for all $p \in P_3$, $t > 0$, a contradiction. Therefore, $P_3 = \emptyset$ and so N_b is equivalent to sup $(\sup_{p\in P_1} |\cdot\cdot|_p, \sup_{|..|\in P_2} |\cdot\cdot|)$ for some finite subset P_1 of P and some finite subset P_2 of P_∞.

We show that $P_2 = \{|\cdot\cdot| \in P_\infty: T_{|..|} \subseteq T\}$. If $|\cdot\cdot| \in P_2$, let $m > 0$ be such that

$$b^m U \subseteq \{x \in R: |x| < 1\}.$$

Then $|u| < 1/|b|^m$ for all $u \in U$. Hence $\{x \in R: |x| < 1/|b|^m\}$ is a T-neighborhood of 0. Let $\epsilon > 0$. By the Very Strong Approximation Theorem and Product Theorem [**8**, Theorem 33.11, p. 77 and Theorem 33.1, p. 66], there exists $r \in R$ such that $|r| > 1/\epsilon|b|^m$. As $x \to rx$ is continuous at 0 when R is given the T-topology, there exists a T-neighborhood V of 0 such that

$$rV \subseteq \{x \in R: |x| < 1/|b|^m\}.$$

Then $|v| < \epsilon$ for all $v \in V$ and hence $\{x \in R: |x| < \epsilon\}$ is a T-neighborhood of 0. Consequently, if $|\cdot\cdot| \in P_2$, then $T_{|..|} \subseteq T$. Suppose $|\cdot\cdot| \in P_\infty$ is such that $T_{|..|} \subseteq T$. If $|\cdot\cdot| \notin P_2$, then there exists a c in R such that $v_p(c) > 0$ for all $p \in P_1$, $|c|_i < 1$ for all $|\cdot\cdot|_i \in P_2$ and $|c| > 1$ [**8**, Theorem 33.11, p. 77 and Theorem 33.1, p. 66]. So c is a T_b-topological nilpotent. As $T_b \supseteq T \supseteq T_{|..|}$, c is a topological nilpotent for $T_{|..|}$. But $|c| > 1$, a contradiction. So $P_2 = \{|\cdot\cdot| \in P_\infty: T_{|..|} \subseteq T\}$.

Since $v_p(b) > 0$ for all $p \in P_1$, we may replace P_1 with the set $P_1(b)$ defined by

$$P_1(b) = \{p \in P: v_p(b) > 0\},$$

and the normable topology thus defined is stronger than T. Finally as $v_p(a) > 0$ if and only if $v_p(b) > 0$,

$$P_1(b) = \{p \in P: v_p(a) > 0\} = P_1(a)$$

and so $T_a \supseteq T$.

LEMMA 4. *There exists a nonzero topological nilpotent a such that the set N_1 of T-topological nilpotents is contained in*

$$\left(\prod_{p\in P_1(a)} p\right) \cap \{x \in R: |x| < 1 \text{ for all } |\cdot\cdot| \in P_2\}.$$

Proof. If there exists a unit u of R such that $u \in N_1$, then $P_1(u) = \emptyset$ and so

$$T \subseteq \sup_{|..|\in P_2} T_{|..|}.$$

Hence

$$T = \sup_{|..|\in P_2} T_{|..|}.$$

Therefore we may assume that $P_1(a) \neq \emptyset$ for all $a \in N_1\backslash\{0\}$. Let

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$a \in N_1 \backslash \{0\}$ be such that

$$Ra = \prod_{i=1}^{n} p_i^{\alpha_i}, \quad \alpha_i \geqq 1 \quad \text{for } i = 1, 2, \ldots, n,$$

but there is no element c in $N_1\backslash\{0\}$ with

$$Rc = \prod_{j \in J} p_j^{\beta_j}, \quad \beta_j \geqq 1, j \in J,$$

for some proper subset J of $[1, n]$. Let $b \in N_1\backslash\{0\}$ and let $p_1, \ldots, p_r$, $q_1, \ldots, q_s$ be distinct nonzero prime ideals of R such that

$$Rb = \prod_{j=1}^{r} p_j^{\beta_j} \prod_{i=1}^{s} q_i^{\gamma_i}$$

where $\beta_j \geqq 1$, $\gamma_i \geqq 1$ for all i and j. Suppose $r < n$. As N_1 is a neighborhood of zero, there exists a neighborhood W of zero such that $W + W \subseteq N_1$. Since T_b and T_a are stronger than T, there exists $n_0 > 0$ and $\epsilon > 0$ such that

$$\left(\prod_{j=1}^{n} p_j^{n_0} \prod_{i=1}^{s} q_i^{n_0}\right) \cap B_\epsilon \quad \text{and} \quad \left(\prod_{i=1}^{n} p_i^{n_0}\right) \cap B_\epsilon$$

are contained in W where

$$B_\epsilon = \{x \in R: |x| < \epsilon \text{ for all } |\cdot\cdot| \in P_2\}.$$

Then

$$(Ra^{n_0} \cap B_\epsilon) + (Rb^{n_0} \cap B_\epsilon) \subseteq \left[\left(\prod_{i=1}^{n} p_i^{n_0}\right) \cap B_\epsilon\right] + \left[\left(\prod_{j=1}^{r} p_j^{n_0} \prod_{i=1}^{s} q_i^{n_0}\right) \cap B_\epsilon\right] \subseteq N_1.$$

Let $m > 0$ be such that $(\mathrm{Ra}^{n_0} + Rb^{n_0})^m$ is principal [**8**, Theorem 33.13, p. 79]. Let $c \in R$ be such that $Rc = (Ra^{n_0} + Rb^{n_0})^m$ Then

$$Rc = \prod_{i=1}^{r} p_i^{mn_0 \min\{\alpha_i, \beta_i\}} = Ra^{mn_0} + Rb^{mn_0}.$$

So, there exist r_1 and r_2 in R such that

$$c = r_1 a^{mn_0} + r_2 b^{mn_0}.$$

As T is nondiscrete and $T \supseteq T_{|\cdot\cdot|}$ for all $|\cdot\cdot| \in P_2$, $P_2 \subset P_\infty$ since otherwise, $R \cap O(P_\infty)$, a finite set, is a T-neighborhood of zero. Since P_∞ has more than one element, there exists a u in R such that $v_p(u) = 0$ for all p in P and $|u| < 1$ for all $|\cdot\cdot| \in P_2$ [**8**, Theorem 33.8, p. 74]. Let $t > 0$ be such that

$$|u|^t|r_2 b^{mn_0}|, |u|^t|r_1 a^{mn_0}| < \epsilon \quad \text{for all } |\cdot\cdot| \in P_2.$$

Then

$$cu^t \in (Ra^{mn_0} \cap B_\epsilon) + (Rb^{mn_0} \cap B_\epsilon)$$

and so $cu^t \in N_1$. But

$$v_p(cu^t) = v_p(c) + tv_p(u) = 0 \quad \text{for all } p \text{ in } P\backslash\{p_1, \ldots, p_r\}.$$

Therefore, cu^t is a nonzero topological nilpotent for T such that

$$Rcu^t = \prod_{j=1}^{r} p_j^{mn_0 \min\{\alpha_j, \beta_j\}},$$

contradicting the choice of a. Hence $r \geqq n$ and therefore

$$N_1 \subseteq p_1 \ldots p_n = \prod_{p \in P(a)} p.$$

Finally, as $T \supseteq T_{|..|}$ for all $|\cdot\cdot| \in P_2$, if $r \in N_1$, then r is a topological nilpotent for $T_{|..|}$. Consequently,

$$N_1 \subseteq p_1 \ldots p_n \cap \{x \in R: |x| < 1 \text{ for all } |\cdot\cdot| \in P_2\}.$$

THEOREM 3. *Let K be a finite extension of $\mathbf{Q}$ and let R be the integral closure of $\mathbf{Z}$ in K. Let T be a nondiscrete, Hausdorff, locally bounded ring topology on R for which the set of topological nilpotents N_1 is a neighborhood* of 0. *Then there exists a proper subset P_2 of P_∞, finite disjoint subsets P_1 and P_3 of P, and positive integers $n(p)$ for $p \in P_3$ such that $P_1 \cup P_2 \neq \emptyset$ and*

$$T = \sup\left(\sup_{p \in P_1} T_p, \sup_{|..| \in P_2} T_{|..|}, \sup_{p \in P_3} T'_{p^{n(p)}}\right).$$

In particular, T is normable.

Proof. If there exists a unit r in $N_1\backslash\{0\}$, then as in Lemma 4,

$$T = \sup_{|..| \in P_2} T_{|..|}.$$

So we may assume that each element of $N_1\backslash\{0\}$ is a nonunit of R. Choose an element a in $N_1\backslash\{0\}$ such that

$$N_1 \subseteq \left(\prod_{p \in P_1(a)} p\right) \cap \{x \in R: |x| < 1 \text{ for all } |\cdot\cdot| \in P_2\}.$$

Since $T \subseteq T_a$,

$$N_1 = \left(\prod_{p \in P_1(a)} p\right) \cap \{x \in R: |x| < 1 \text{ for all } |\cdot\cdot| \in P_2\}.$$

So p is a T-neighborhood of 0 for all $p \in P_1(a)$. For each $p \in P_1(a)$ define $n(p)$ by

$$n(p) = \sup\{n \in Z: p^n \text{ is a } T\text{-neighborhood of } 0\}.$$

By the above observation, $n(p) \geqq 1$ for all p in $P_1(a)$. Let

$$P_1 = \{p \in P_1(a): n(p) = \infty\}$$

and let

$$P_3 = \{p \in P_1(a) \colon n(p) < \infty\}.$$

Clearly,

$$\sup\,(\sup_{p \in P_1} T_p, \sup_{|..| \in P_2} T_{|..|}, \sup_{p \in P_3} T'_{p^{n(p)}}) \subseteq T.$$

Let U be a closed neighborhood of 0 for T. As $T \subseteq T_a$, there exists $\epsilon > 0, \alpha_1, \ldots, \alpha_r \geqq 0$ such that

$$(p_1^{\alpha_1} \ldots p_r^{\alpha_r}) \cap \{x \in R \colon |x| < \epsilon \text{ for all } |\cdot\cdot| \in P_2\} \subseteq U.$$

Let W be the closure of $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ in T. Then W is an ideal of R containing $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$. So

$$W = p_1^{\beta_1} \ldots p_r^{\beta_r}$$

with $0 \leqq \beta_i \leqq \alpha_i$ for $i = 1, 2, \ldots, r$.

Since W is T-closed, the topology induced on R/W by T is Hausdorff. As R/W is a finite ring [**8**, Theorem 33.2, p. 67], R/W is discrete. So W is open for T. Therefore $\beta_i \leqq n(p_i)$ for all $p_i \in P_3$ and so $p_i^{n(p_i)} \subseteq p_i^{\beta_i}$.

Consider the set W_1 defined by

$$W_1 = W \cap \{x \in R \colon |x| < \epsilon/2 \text{ for all } |\cdot\cdot| \in P_2\}.$$

Let $y \in W_1$ and let V be a T-neighborhood of 0. As $\{x \in R \colon |x| < \epsilon/2$ for all $|\cdot\cdot| \in P_2\}$ is a T-neighborhood of 0, we may assume that $|v| < \epsilon/2$ for all $v \in V$, $|\cdot\cdot| \in P_2$. Since y is in the T-closure of $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, there exists v in V such that

$$y + v \in p_1^{\alpha_1} \ldots p_r^{\alpha_r}.$$

Furthermore,

$$|y + v| \leqq |y| + |v| < \epsilon \quad \text{for all } |\cdot\cdot| \in P_2.$$

Hence

$$y + v \in p_1^{\alpha_1} \ldots p_r^{\alpha_r} \cap \{x \in R \colon |x| < \epsilon \text{ for all } |\cdot\cdot| \in P_2\} \subseteq U.$$

Therefore, y is in the T-closure of U, that is, y is in U. Then $W_1 \subseteq U$. As $p_i^{n(p_i)} \subseteq p_i^{\beta_i}$ for all $p_i \in P_3$,

$$(\textstyle\bigcap_{p_i \in P_1} p_i^{\beta_i}) \cap (\textstyle\bigcap_{p_i \in P_3} p_i^{n(p_i)}) \cap \{x \in R \colon |x| < \epsilon/2 \text{ for all } |\cdot\cdot| \in P_2\} \subseteq U.$$

Therefore,

$$T \subseteq \sup\,(\sup_{p \in P_1} T_p, \sup_{|..| \in P_2} T_{|..|}, \sup_{p \in P_3} T'_{p^{n(p)}}).$$

COROLLARY. *If T is a Hausdorff, locally bounded ring topology on R, then T is normable if and only if the set of topological nilpotents is a neighborhood of* 0.

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