F-EQUATION FOURIER TRANSFORMS

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Introduction. For solutions of the differential difference 'F-equation'

$$\frac{d}{dz}F(z,\alpha) = F(z,\alpha+1), \tag{1}$$

there are representations as a generating series [**5**],

$$F(z,\alpha) = \sum_{n=0}^{\infty} f(\alpha+n)\frac{z^n}{n!}, \tag{2}$$

and as a contour integral [**3**],

$$F(z,\alpha) = \int_C \exp(\alpha s + ze^s)g(s)ds. \tag{3}$$

Integrals of the type (3) have certain formal properties which are reflected in identities satisfied by solutions of (1). In this note are given some relations which hold when the integral is taken to be a complex Fourier transform with respect to the variable α. The special properties of Fourier integrals lead to some additional results in this case.

The interest of the equation (1) lies in the fact, which was pointed out by Truesdell [**5**], that many useful special functions can be expressed as solutions of the equation. Working with the F-equation, Truesdell was able to classify many of the formulae satisfied by such functions, to generalize some of them, and to discover additional relations.

1. Fourier integral representations. If a solution $F(z,\alpha)$ of (1) is represented in the forms (2) and (3), we may call $f(\alpha)$ the coefficient function, and $g(s)$ the spectrum function, of the given solution. For the case in hand, these two functions are a Fourier pair. That is, the formulae

$$f(\alpha) = \int_{-\infty}^{\infty} e^{2\pi i\alpha s}g(s)ds, \quad g(s) = \int_{-\infty}^{\infty} e^{-2\pi i\alpha s}f(\alpha)d\alpha, \tag{4}$$

hold in some sense. The factor 2π has been written in the exponentials for convenience. We consider, therefore, solutions with the representations

$$F(z,\alpha) = \int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi is})g(s)ds = \sum_{n=0}^{\infty} f(\alpha+n)\frac{z^n}{n!}. \tag{5}$$

Since the relation between $f(\alpha)$ and $g(s)$ is skew reciprocal, we may, with the alteration of a sign, form a second solution in which their roles are interchanged. In this way a conjugate solution

$$G(z,\alpha) = \int_{-\infty}^{\infty} \exp(-2\pi i\alpha s + ze^{-2\pi is})f(s)ds = \sum_{n=0}^{\infty} g(\alpha+n)\frac{z^n}{n!} \tag{6}$$

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is constructed, whose coefficient function is the spectrum function of $F(z, \alpha)$. The relation between $F(z, \alpha)$ and $G(z, \alpha)$ is skew reciprocal with respect to α.

Example 1. Let

$$f(\alpha) = \begin{cases} \dfrac{1}{\Gamma(\mu)}\alpha^{\mu-1}, & \alpha > 0 \\ 0 & , \alpha < 0 \end{cases} \qquad 0 < R(\mu) < 1.$$

Then [2, § 521],

$$g(s) = (2\pi i s)^{-\mu} e^{-\pi i \mu},$$

and we have

$$F(z,\alpha) = \frac{1}{\Gamma(\mu)} \sum_{n=0}^{\infty} (\alpha + n)^{\mu-1} \frac{z^n}{n!},$$

$$G(z,\alpha) = (2\pi i)^{-\mu} e^{-\pi i \mu} \sum_{n=0}^{\infty} \frac{z^n}{(\alpha + n)^{\mu} n!}.$$

These functions are related to the generalized zeta function and to Spence's transcedent [5, p. 172].

THEOREM I. *The functions*

$$\exp(te^{-2\pi i\alpha})F(z,\alpha),\ \exp(ze^{2\pi i\alpha})G(t,\alpha), \tag{7}$$

are solutions of (1) *in the variables* z, α *and* t, α, *respectively. For all* z, t, *they are a Fourier pair in* α.

Proof. The exponential factors are periodic in α of period 1, which clearly justifies the first statement. To demonstrate the transform property, we note that the transform of $g(s + n)$ is $f(\alpha)e^{-2\pi i n\alpha}$, so that if n is an integer

$$\int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi i\alpha s})g(s + n)ds = \int_{-\infty}^{\infty} \exp(2\pi i\alpha(t - n) + ze^{2\pi i t})g(t)dt$$
$$= e^{-2\pi i\alpha n}F(z,\alpha).$$

Multiply by $t^n/n!$ and sum. Using (2) and (3) we find that

$$\int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi i s})G(t,s)ds = \exp(te^{-2\pi i\alpha})F(z,\alpha), \tag{8}$$

which expresses the result.

We may therefore introduce parameters into our formulae by replacing any pair $f(\alpha)$, $g(s)$ by the pair (7). We shall see below that any valid formula containing the pair $f(\alpha)$, $g(s)$ remains valid if this substitution is made.

We investigate the sense in which the above formulae hold. Since we shall have $f(\alpha)$ and its transform $g(s)$ tending ultimately to zero except in sets of small measure, it is clear that the series in (5) and (6) converge for all z like the exponential function, and define entire analytic functions of z, for almost all α. Since s is real, we have

$$|\exp(ze^{2\pi i s})| \leqslant e^{|z|}. \tag{9}$$

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Hence, if we suppose that f and g are $L(-\infty, \infty)$, or $L^2(-\infty, \infty)$, the integrals (5) and (6) converge, or converge in mean, uniformly with respect to z. Thus, if (4) hold in either sense, then (5), (6), and (7) hold in the same sense for all z.

Let $f(\alpha)$ be $L^p(-\infty, \infty)$, then except in a set of arbitrarily small measure, the members of the sequence $|f(\alpha + n)|$ $(n = 0, 1, 2, \ldots)$ are less than some function $M(\alpha)$. Hence, except in this set, the series (5) converges like the exponential function and we have (see also [**5**, Corollary 11.6]).

THEOREM II. *If $f(\alpha)$ is $L^p(-\infty, \infty)$, $p > 0$, the function $F(z, \alpha)$ given by* (5) *satisfies*

$$|F(z,\alpha)| \leqslant M(\alpha)e^{|z|}, \tag{10}$$

where $M(\alpha)$ is defined for almost all α.

2. Product formulae. The results of this section concern solutions of (1) whose coefficient or spectrum functions are products of the corresponding functions for known solutions. We use the formulae for Fourier transforms given in [**4**, Chap. 2].

THEOREM III. *Let $F_1(z, \alpha)$ and $F_2(z, \alpha)$ have coefficient functions $f_1(\alpha)$, $f_2(\alpha)$, and spectrum functions $g_1(s)$, $g_2(s)$, respectively. Let $g_1(s)$ and $g_2(s)$ be $L(-\infty, \infty)$. Then the unique solution $F(z, \alpha)$ of* (1) *with coefficient function $f_1(\alpha)f_2(\alpha)$ is given by*

$$\begin{aligned} F(z,\alpha) &= F_1^* F_2(z,\alpha) \\ &= \int_{-\infty}^{\infty} e^{2\pi i\alpha s} g_1(s) F_2(ze^{2\pi is},\alpha)ds \\ &= \int_{-\infty}^{\infty} e^{2\pi i\alpha s} g_2(s) F_1(ze^{2\pi is},\alpha)ds \\ &= \int_{-\infty}^{\infty} \exp(2\pi i\alpha s + ze^{2\pi is}) g(s)ds \\ &= \sum_{n=0}^{\infty} f_1(\alpha+n) f_2(\alpha+n)\frac{z^n}{n!}, \end{aligned} \tag{11}$$

where

$$g(s) = \int_{-\infty}^{\infty} g_1(s-t)g_2(t)dt \tag{12}$$

is the transform of the product $f_1(\alpha)f_2(\alpha)$.

Proof. Under these hypotheses, the first two integrals converge absolutely, since the exponential factors are bounded and the F factors bounded and periodic in s. The convergence is uniform with respect to z and α, provided that the absolute values of these variables are bounded. The integrals define, therefore, analytic functions of z. Since [**4**, Theorem 41] $g(s)$ is $L(-\infty, \infty)$, the third integral also converges absolutely and uniformly. Each expression is a solution of (1), as is easily verified. Finally, the coefficient function of each solution is

$f_1(\alpha)f_2(\alpha)$, since each expression reduces to this value when $z = 0$. It is clear that the coefficient function determines the solution uniquely, and the theorem follows.

This theorem is an adaptation to the present case of a result given in [**3**].

Example. The product solution of the pair given in Example 1 is

$$F^*G(z,\alpha) = \frac{(2\pi i)^{-\mu}e^{-\pi i\mu}}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{z^n}{(\alpha+n)n!}$$
$$= \frac{(2\pi i)^{-\mu}e^{-\pi i\mu}}{\Gamma(\mu)\alpha}\,{}_1F_1(\alpha;\alpha+1;z),$$

where ${}_1F_1$ is the confluent hypergeometric function.

THEOREM IV. *Let $f_1(\alpha)$, $f_2(\alpha)$ and their transforms be $L^2(-\infty,\infty)$. Then*

$$\int_{-\infty}^{\infty}F_1^*G_2(z,\alpha)d\alpha = \int_{-\infty}^{\infty}F_2^*G_1(z,\alpha)d\alpha = Ce^z. \tag{13}$$

Proof. Substitute the series expansion for the solution on the left. We find

$$\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}f_1(\alpha+n)g_2(\alpha+n)\frac{z^n}{n!}d\alpha = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{-\infty}^{\infty}f_1(\alpha+n)g_2(\alpha+n)d\alpha$$
$$= e^z\int_{-\infty}^{\infty}f_1(\alpha)g_2(\alpha)d\alpha = Ce^z,$$

the last step holding since the integral is independent of n. In view of the Parseval theorem [**4**, p. 69], the integral corresponding to the second expression has the same value C. This proves (13).

The next result is a sort of dual to Theorem III, since it involves solutions whose spectrum functions are products. The conditions are somewhat different however.

THEOREM V. *Let $f_1(\alpha)$ and $f_2(\alpha)$ be $L^2(-\infty,\infty)$, and let their convolution be denoted by $f(\alpha)$. Then the solution of* (1) *with coefficient function $f(\alpha)$ is given by*

$$H(z,\alpha) = F_1^0F_2(z,\alpha)$$
$$= \int_{-\infty}^{\infty}F_1(z,\alpha-s)f_2(s)ds$$
$$= \int_{-\infty}^{\infty}F_2(z,\alpha-s)f_1(s)ds \tag{14}$$
$$= \int_{-\infty}^{\infty}\exp(2\pi i\alpha s + ze^{2\pi is})g_1(s)g_2(s)ds$$
$$= \sum_{n=0}^{\infty}f(\alpha+n)\frac{z^n}{n!}.$$

Proof. The conditions stated ensure that $f(\alpha)$ exists, that $g_1(s)$ and $g_2(s)$ are

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$L^2(-\infty,\infty)$, and therefore that $g_1(s)\, g_2(s)$ is $L(-\infty,\infty)$. The third integral is therefore absolutely and uniformly convergent. Since F_1 and F_2 are also $L^2(-\infty,\infty)$, uniformly with respect to z and α, in their dependence upon s, the first two integrals also converge absolutely and uniformly. The four expressions therefore define analytic functions of z which are entire. As in the proof of Theorem III, it is clear that each expression is a solution of (1) and that each reduces to the same coefficient function $f(\alpha)$ when $z = 0$. This completes the proof.

THEOREM VI. *$H(z, \alpha)$ being defined as in Theorem* V, *we have*

$$\int_{-\infty}^{\infty} F_1(x,\alpha - s)F_2(y,\beta + s)ds = H(x + y, \alpha + \beta). \tag{15}$$

Proof. From (5) it is clear that $F_1(x, \alpha + s)$ is the transform of $\exp(2\pi i\alpha s + xe^{2\pi is})g_1(s)$ and $F_2(y,\beta+s)$ the transform of $\exp(2\pi i\beta s+ye^{2\pi is})g_2(s)$. The integral (15) is the convolution of these two solutions and therefore its transform is the product of their transforms. But the third integral of (14) shows that this transform is precisely $H(x + y, \alpha + \beta)$.

THEOREM VII. *Let $F_i(z, \alpha)$ and $G_i(z, \alpha)$ be pairs of conjugate solutions for $i = 1, 2$. Then*

$$F_1^0 F_2(z,\alpha), \quad G_1^* G_2(z,\alpha) \tag{16}$$

are a pair of conjugate solutions.

Proof. The coefficient functions of the pair (16) are the Fourier pair

$$\int_{-\infty}^{\infty} f_1(\alpha - s)f_2(s)ds, \quad g_1(\alpha)g_2(\alpha).$$

We shall now give some examples of these formulae. If $f(\alpha)$ is a rational function of α, expressible by quotients of gamma functions, the series (5) takes the form of a generalized hypergeometric function. The transform $g(s)$ is a sum of rational functions multiplied by exponentials, and the conjugate series (6) can also be expressed in terms of hypergeometric series in this case.

Example 2. [**2**, § 438]. Let

$$f_\beta(\alpha) = \begin{cases} e^{-\beta\alpha}, & \alpha > 0, \\ 0 \quad , & \alpha < 0. \end{cases}$$

Then

$$g_\beta(s) = (\beta - 2\pi is)^{-1}, \qquad R(\beta) > 0,$$

$$F_\beta(z,\alpha) = \exp(-\beta\alpha + ze^{-\beta}), \qquad \alpha > 0,$$

$$G_\beta(z,\alpha) = (\beta - 2\pi i\alpha)^{-1}{}_1F_1(\alpha - \beta/2\pi i; \alpha + 1 - \beta/2\pi i; z).$$

We have

$$F_\beta^* F_\gamma = F_{\beta+\gamma}(z,\alpha), \qquad R(\beta) > 0, R(\gamma) > 0,$$

$$G_\beta^* G_\gamma(z,a) = (\beta - 2\pi i a)^{-1}(\gamma - 2\pi i a)^{-1}$$
$$\times\ {}_2F_2(a - \beta/2\pi i, a - \gamma/2\pi i; a + 1 - \beta/2\pi i; a + 1 - \gamma/2\pi i; z),$$

$$F_\beta^0 F_\gamma = (\gamma - \beta)^{-1}[\exp(-\beta a + ze^{-\beta}) - \exp(-\gamma a + ze^{-\gamma})],$$

using [**2**, § 448], and

$$G_\beta^0 G_\gamma = G_{\beta+\gamma}.$$

From Theorem IV we have [**2**, p. 30]

$$\int_{-\infty}^{\infty} F_\beta^* G_\gamma(z,a)da = \frac{\exp(z - \beta\gamma/2\pi i)}{2\pi i}\mathrm{Ei}\left(\frac{\beta\gamma}{2\pi i}\right).$$

Example 3. Let [**2**, § 708],

$$f_\beta(a) = \exp(-\pi a^2/\beta), \qquad R(\beta) > 0,$$

$$g_\beta(s) = \sqrt{\beta}\exp(-\pi\beta s^2) = \sqrt{\beta}\, f_{1/\beta}(s).$$

We then have

$$F_\beta(z,a) = \sum_{n=0}^{\infty}\exp(-\pi(a+n)^2/\beta)\frac{z^n}{n!},$$

$$G_\beta(z,a) = \sqrt{\beta}F_{1/\beta}(z,a).$$

By Theorem I, the functions

$$\sum_{n=0}^{\infty}\exp(-\pi(a+n)^2/\beta + te^{-2\pi i a})\frac{z^n}{n!}, \qquad \sqrt{\beta}\sum_{n=0}^{\infty}\exp(-\pi\beta(a+n)^2 + ze^{2\pi i a})\frac{t^n}{n!}$$

are a Fourier pair. Also

$$F_\beta^* F_\gamma = F_{\beta+\gamma}; \quad F_\beta^0 F_\gamma = \sqrt{\beta+\gamma}F_{\beta\gamma/\beta+\gamma}.$$

By Theorem IV,

$$\int_{-\infty}^{-\infty} F_\beta^* F_\gamma(z,a)da = \sqrt{\beta\gamma/(1+\beta\gamma)}e^z,$$

and by Theorem VI,

$$\int_{-\infty}^{\infty} F_\beta(x, a+s)F_\gamma(y, b-s)ds = \sqrt{\beta+\gamma}F_{\beta\gamma/(\beta+\gamma)}(x+y, a+b).$$

3. Generating series. In this section we shall use the Poisson formula for Fourier transforms to evaluate certain bilateral generating series for conjugate solutions. The Poisson formula [**1**, p. 33]

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} g(n), \tag{17}$$

holds if, for instance, $f(a)$ is $L(-\infty, \infty)$, of bounded variation, and tends monotonically to zero at infinity.

THEOREM VIII. *Let* $F(z, a)$ *and* $G(z, a)$ *be a pair of conjugate solutions of* (1). *Let the Fourier pair determining them be such that* (17) *holds. Then*

$$\sum_{n=-\infty}^{\infty} F(z,a+n)e^{2\pi in\theta} = \exp(ze^{-2\pi i\theta})K(\theta,a), \tag{18}$$

$$\sum_{n=-\infty}^{\infty} G(z,a+n)e^{2\pi in\theta} = \exp(ze^{-2\pi i\theta})L(\theta,a),$$

where $K(\theta, a)$ *and* $L(\theta, a)$ *are periodic of period* 1 *in* a. *Furthermore,*

$$L(\theta,a) = e^{-2\pi ia\theta}K(-a,\theta). \tag{19}$$

Proof. We assume that θ and a are real, though the result may hold more generally. Under the conditions stated, and in view of Theorem II, the generating series (18) define analytic functions of z regular for $z = 0$. Let $S = S(z, a, \theta)$ denote the first of these series. Then

$$\frac{dS}{dz} = \sum_{-\infty}^{\infty} F(z,a+n+1)e^{2\pi in\theta} = e^{-2\pi i\theta}S,$$

using (1) and changing the summation index from n to $n + 1$. Integrating this equation, we find that S is of the form (18). The periodicity of K and L is evident. Setting $z = 0$, we have to prove (19). Now the transform of $f(s + a)$ is $g(s)e^{2\pi isa}$, and the transform of $g(s + a)$ is $f(s)e^{-2\pi isa}$. Inserting these in (17) and taking account of (18), we find precisely (19). The formula taking account of the translations holds under the same conditions as (17). This proves the theorem. Two particular cases of this result may be mentioned.

(a) If $F(z, a)$ and $G(z, a)$ are conjugate and (17) holds, then

$$\sum_{n=-\infty}^{\infty} F(z,n) = \sum_{n=-\infty}^{\infty} G(z,n) = Ke^{z}. \tag{20}$$

Example 4. Taking $f_\beta(a) = \exp(-\pi a^2/\beta)$ as in Example 3, we find that

$$\beta^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty} F_\beta(z,n) = \sum_{n=-\infty}^{\infty} F_{1/\beta}(z,n) = \mathfrak{D}(\beta)e^{z},$$

where [1, p. 152]

$$\mathfrak{D}(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

is related to the theta functions.

(b) If $F(z, a)$ is any solution of (1) such that the indicated series converge,

$$\sum_{n=-\infty}^{\infty} F(z,a+n)e^{2\pi in\theta} = \exp(ze^{-2\pi i\theta})\sum_{n=-\infty}^{\infty} f(a+n)e^{2\pi in\theta}. \tag{21}$$

Example 5 [2, § 444].

$$f(a) = (a^2+b^2)^{-1}; \quad g(s) = 2\pi b^{-1}e^{-2\pi b|s|}.$$

Then

$$F(z,a) = (a^2+b^2)^{-1}{}_2F_2(a+ib, a-ib; a+ib+1, a-ib+1; z),$$

$$G(z,\alpha) = 2\pi b^{-1}\exp(-2\pi\alpha b + ze^{-2\pi b}), \qquad \alpha > 0.$$

To evaluate the sum $K(\theta, \alpha)$ on the right-hand side of (21), we use (17), since the sum over $g(s)$ is more tractable. Letting $[x]$ denote the greatest integer contained in x, we find

$$K(\theta,\alpha) = \frac{e^{-2\pi i\alpha\theta}}{4\pi b}\left\{\frac{\exp(-2\pi b\{\theta+1-[\theta]\} + 2\pi i[\theta+1]\alpha)}{1-\exp(-2\pi b - 2\pi i\alpha)} + \frac{\exp(-2\pi b\{\theta-[\theta]\} + 2\pi i\theta\alpha)}{1-\exp(-2\pi b+2\pi i\alpha)}\right\}.$$

As another example of some of the formulae of this paper, we include

Example 6. For $n = 1, 2, \ldots$, let [4, p. 76]

$$f_n(\alpha) = \frac{1}{(2^{n-1}n!)^{\frac{1}{2}}} e^{-\pi\alpha^2}H_n(2\sqrt{\pi}\alpha), \; g_n(s) = i^{-n}f_n(s).$$

Here $H_n(x)$ is the nth Hermite polynomial. The $f_n(\alpha)$ form a complete orthonormal set in $L^2(-\infty,\infty)$, hence the solutions $F_n(z,\alpha)$ form a base for the solutions of (1) whose coefficient functions are L^2. We have

$$G_n(z,\alpha) = i^{-n}F_n(z,\alpha),$$

$$F_n^*F_m(z,\alpha) = \sum_{n=0}^{\infty} a(m,n,p)F_p(z,\alpha),$$

$$F_n^0F_m(z,\alpha) = i^{m+n}\sum_{p=0}^{\infty} i^{-p}a(m,n,p)F_p(z,\alpha),$$

where

$$a(m,n,p) = \int_{-\infty}^{\infty} f_m(s)f_n(s)f_p(s)ds$$

is symmetric in m, n, and p, and vanishes if $m+n+p$ is odd. Some further expansions in terms of these coefficients are

$$\int_{-\infty}^{\infty} e^{2\pi i\alpha s}F_m^*F_p(z,s)ds = \exp(ze^{-2\pi i\alpha})\sum_{r=0}^{\infty} i^r a(m,p,r)f_r(\alpha),$$

$$\int_{-\infty}^{\infty} e^{2\pi i\alpha s}F_m^0F_p(z,s)ds = \exp(ze^{-2\pi i\alpha})i^{m+p}\sum_{r=0}^{\infty} a(m,p,r)f_r(\alpha),$$

and

$$\int_{-\infty}^{\infty} F_m(x,\alpha+s)F_p(y,\beta-s)ds = i^{m+p}\sum_{r=0}^{\infty} i^r a(m,p,r)F_r(x+y,\alpha+\beta).$$

Also (18) hold for the pair $F_m(z, \alpha)$, $G_m(z, \alpha)$, with

$$K_m(\theta,\alpha) = i^m L_m(\theta,\alpha) = i^m e^{-2\pi i\alpha\theta} K_m(-\alpha,\theta).$$

The function $K_0(\theta, p)$, p an integer, is a theta function.

Although solutions $F(z, \alpha)$ of (1) of the Fourier type possess convenient formal properties, their behaviour as functions of z is rather restricted, as Theorem II shows. However, many of these properties carry over to integrals of Laplace transform type which represent many of the functions to which the equation (1) may be applied. It may be noted that the Fourier solutions exhibit markedly different behaviour as functions of z and of α, since they are analytic in z but may only be measurable as functions of α.

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References

1. S. Bochner, *Vorlesungen über Fourierintegrale* (Leipzig, 1932).
2. G. A. Campbell and R. M. Foster, *Fourier integrals for practical application* (New York, 1931).
3. G. F. D. Duff, *A composition theorem for Truesdell's equation*, Comptes Rendus, vol. 229 (1949), 1195–1197.
4. E. C. Titchmarsh, *Theory of Fourier integrals* (Oxford, 1937).
5. C. A. Truesdell, *A unified theory of special functions*, Ann. Math. Studies, no. 18 (Princeton, 1948).

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