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Functions of asymptotic expansions

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The paper extends a theorem given by Entringer on the asymptotic expansion of a composite function.

In a paper on asymptotic expansions, Entringer [2] considered the problem: Given $f(x) \to \infty$ and $f(x) \sim g(x)$ as $x \to \infty$, for what functions h can we say that $h\{f(x)\} \sim h\{g(x)\}$ as $x \to \infty$? His general conclusion was that $h\{f(x)\} \sim h\{g(x)\}$ if $h(x)$ behaves like x^r for large x but not if $h(x)$ behaves like e^x. More precisely, he proves in the first part of his Theorem 1:

If $f(x) \to \infty$ *and* $f(x) \sim g(x)$ *as* $x \to \infty$, *and if* $h(x)$ *is monotonic and* $h'(x)/h(x) = O(1/x)$ *for all sufficiently large* x, *then* $h\{f(x)\} \sim h\{g(x)\}$ *as* $x \to \infty$.

As it stands, this theorem does not cover the situation typified by the following example. From Copson [1, pp. 51-53], we have the relation (as $x \to \infty$)

$$\log\Gamma(x) \sim x(-1+\log x) + \frac{1}{2}\log(2\pi/x) + \sum_{1}^{\infty} a_m x^{1-2m} , \tag{1}$$

where the coefficients a_m are known. (Following Entringer, we take the variable to be real and use x instead of Copson's p; also we have corrected a misprint where Copson has p^{-2m} instead of p^{1-2m}.) The question arises whether an asymptotic relation for $\Gamma(x)$ can be obtained from (1) by taking the exponential of suitable terms on the right hand side. In fact this can be done; if we neglect terms of order x^{-2N-1} in

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the power series in (1), we obtain a valid relation of the form

$$(2) \qquad \Gamma(x) \sim (2\pi/x)^{1/2}(x/e)^{x}\left\{1 + \sum_{1}^{2N} b_m x^{-m} + O\left(x^{-2N-1}\right)\right\},$$

where the coefficients b_m can be found from the known values of $a_1, \ldots, a_N$. For instance, using $a_1 = 1/12$, $a_2 = -1/360$ and neglecting the $O\left(x^{-5}\right)$ terms in (1), we expand $\exp\{(a_1/x)+(a_2/x^3)\}$ as far as x^{-4} terms and get

$$b_1 = a_1 = 1/12 ,$$

$$b_2 = (1/2)a_1^2 = 1/288 ,$$

$$b_3 = a_2 + \left(a_1^3/6\right) = -139/51840 ,$$

$$b_4 = a_1a_2 + \left(a_1^4/24\right) = -571/(10 \times 12^5) .$$

Thus b_1, b_2, b_3, b_4 are easily evaluated and in this respect the above method compares favourably with other ways of evaluating these coefficients. (Compare, for example, the method given by Copson in the next section of his book.)

This result does not conflict with Entringer's Theorem, which gives a sufficient rather than a necessary condition. However, Entringer's restriction on $h(x)$ can be relaxed when the relationship between $f(x)$ and $g(x)$ as $x \to \infty$ is of the form $f(x) = g(x) + O\left(x^{-m}\right)$, where $m > 0$. It will be seen that a relationship of this type arises from (1) if we take $f(x)$ as $\log\Gamma(x)$ and $g(x)$ as $x(-1+\log x) + \frac{1}{2}\log(2\pi/x)$. (For greater accuracy, we can include one or more terms from the power series in $g(x)$.) We might expect this type of relationship between f and g to be appropriate whenever logarithms are involved. With this interpretation of $f(x) \sim g(x)$, we prove the following theorem:

THEOREM. *If* $f(x) \to \infty$ *and* $f(x) = g(x) + O\left(x^{-m}\right)$ *as* $x \to \infty$, *where* $m > 0$, *and if* $h(x)$ *is monotonic and* $h'(x) = O\{h(x)\}$ *as* $x \to \infty$, *then*

$$h\{f(x)\} = [h\{g(x)\}]\left[1+O\left(x^{-m}\right)\right] \quad \text{as} \quad x \to \infty .$$

Proof. We note that if the theorem holds for $h(x)$ then it holds for

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$h_1(x) = -h(x)$ and that one of these functions must be monotonic increasing. So we can take $h'(x) \geq 0$ for $x > x_0$ without introducing any effective restriction. Also we can find positive constants A and B and numbers x_1, x_2 such that

$$(3) \qquad 0 \leq h'(x) < A|h(x)| \quad \text{for} \quad x > x_1 ,$$

$$(4) \qquad 0 \leq |f(x)-g(x)| < Bx^{-m} \quad \text{for} \quad x > x_2 .$$

Since $f(x) \to \infty$ as $x \to \infty$, the second of these inequalities shows that $g(x) \to \infty$ also as $x \to \infty$. Hence if $h(x) = \text{const.}$ for $x > x_3$, $h\{f(x)\} = h\{g(x)\}$ for sufficiently large x and the theorem is trivial. In particular, this covers the case where $h(x) = 0$ for $x > x_3$. In all other cases we can find x_4 such that $h(x) \neq 0$ for $x > x_4$.

Let us suppose first of all that $h(x) > 0$ for $x > x_4$. By taking x large enough, say $x > X_1 = \max(x_1, x_2, x_4)$, we can ensure that $h(x) > 0$ and that inequalities (3) and (4) are valid simultaneously. For $x > X_1$, we now have

$$(5) \qquad 0 \leq h'(x)/h(x) < A$$

and integrating from x_5 to x_6, with x_5 and $x_6 > X_1$,

$$(6) \qquad 0 \leq |\log\{h(x_6)/h(x_5)\}| < A|x_6-x_5| .$$

Now by taking x large enough, say $x > X_2$, we can ensure that $f(x) > X_1$ and $g(x) > X_1$. If we replace x_5 by $g(x)$ and x_6 by $f(x)$, we get

$$0 \leq |\log\{h(f(x))/h(g(x))\}| < A|f(x)-g(x)| < ABx^{-m} ,$$

for $x > X_0 = \max(X_1, X_2)$. Hence for $x > X_0$

$$\exp(-ABx^{-m}) < h\{f(x)\}/h\{g(x)\} < \exp(ABx^{-m}) .$$

Thus as $x \to \infty$,

$$h\{f(x)\}/h\{g(x)\} = 1 + O(x^{-m}) .$$

In the case where $h(x) < 0$ for $x > x_4$ the proof follows the same lines. In (5), $h(x)$ must be replaced by $-h(x)$ but relation (6) still holds and the remainder of the proof need not be modified.

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This completes the proof of the theorem and it will be clear that it covers the derivation of (2) from the expansion of $\log\Gamma(x)$. By taking $h(x) = e^{-x}$, the asymptotic expansion for $1/\Gamma(x)$ can also be derived from (1).

References

[1] E.T. Copson, *Asymptotic expansions* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 55, Cambridge University Press, Cambridge, 1965).

[2] R.C. Entringer, "Functions and inverses of asymptotic functions", *Amer. Math. Monthly* 74 (1967), 1095-1097.

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