Correspondence

Dear Editor,

Diagonal Problem Conjecture

Is it possible to prove that for integer values $m \geqslant 0$ and $n \geqslant 0$, the following expression involving the Factorial and Gamma functions, viz.

$$\sum_{q=0}^{2m+1} \frac{(-1)^{1-q}}{q!} \frac{(q+n)!}{(q+2n+1)!} \frac{(2m+2n+q)!}{(2m+1-q)!} \frac{1}{\Gamma(n+q+\frac{1}{2})}$$

or equivalently

$$\frac{1}{\sqrt{n}} \sum_{q=0}^{2m+1} \frac{(-1)^{1-q}}{q!} \frac{((q+n)!)^2}{(q+2n+1)!} \frac{(2m+2n+q)!}{(2m+1-q)!} \frac{2^{2n+2q}}{(2n+2q)!}$$

is identically zero?

For initial values of m ($m = 0, 1, 2$) it is a straightforward but increasingly tedious exercise to show that the expression vanishes for arbitrary n-values, but it would be nice to see a general proof of the conjecture. For other randomly specified input values of m and n, it is also a straightforward matter to undertake a validation exercise using a spreadsheet to appreciate that the expression does appear to vanish, at least within the limits of accuracy associated with the evaluation of spreadsheet functions.

The above conjecture arose out of work carried out by the author many years ago, associated with the problem of determining the radiation pattern due to an electric field distribution that varied cosinusoidally across a circular aperture. It is also of relevance to the development of coefficients required in a Fourier-Bessel series expansion for the cosine function. The work involved matrix multiplication that resulted in a product matrix with various alternate diagonal terms vanishing, specifically those in cells where the $(2m+1+n)$th row intersected the nth column – hence the description "Diagonal Problem". In the event, only the above initial values for m were required by the author because of the rapid decay of other, associated multiplicative terms.

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Dear Editor,

Recently, while surfing the web, I came across the website [1] and became intrigued by the following problem, number 425, on page 161:

There was a diagram consisting of a pentagon labelled *ABCDEA* together with the five diagonals, and the problem was "A man started in a car from the town *A*, and wished to make a complete tour of these roads, going along every one of them once, and once only. How many different routes are there from which he can select? It is puzzling unless you can devise some ingenious method. Every route must end at the town *A*, from

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which you start, and you must go straight from town to town – never turning off at crossroads."

The answer on page 369 simply said: "The number of different routes is 264. It is quite a difficult puzzle, and consideration of space does not admit of my showing the best method of making the count."

This sounded like a challenge. Since the graph is a complete graph on five vertices I decided to start by counting all the paths whose first visits to each town occur in the order *ABCDE*. Any of the 23 non-trivial permutations of B, C, D and E should then give a different valid path. Now the only possible ways to start are *ABCAD*, *ABCDA*, *ABCDB* and *ABCDE*. In each case, on removal of the used arcs and untangling the remainder to remove crossings it is easy to list and count the possible continuations, getting 6, 4, 6 and 6, respectively. This gives a total of 22, and multiplying by 24 gives 528. I have no idea why Dudeney got his answer. The number of different Eulerian cycles is indeed 264, but each should be counted twice, since it passes through A on two occasions. It is also interesting that (the late) Martin Gardner, although he corrected many of the errors in the original, has missed this one.

It was certainly a lot easier than the proof of Fermat's Last Theorem!

Reference

1. Henry-Ernest-Dudeney-536-Puzzles-and-Curious-Problems-Scribner-1970.pdf (xn--webducation-dbb.com)

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Feedback

On 105.28: Peter Giblin writes: In his interesting Note Clive Johnson establishes conditions for a sequence of the form $x_{m+1} = \dfrac{ax_m + b}{cx_m + d}$, $m = 0, 1, 2, \ldots$, where $ad \neq bc$ and x_0 is a given real initial term, to be periodic with a given period n. (Traditionally if $cx_m + d = 0$ then x_{m+1} is declared to be ∞ and the next term x_{m+2} is a/c. That is the real line is 'compactified with a single point at infinity'.) In what follows I shall assume the sequence is real, that is a, b, c, d and x_0 are real numbers, but this is not strictly necessary. Summarising his statements:

> The above sequence has period $n > 1$, that is $x_n = x_0$ for all (real) choices of initial term x_0 and this does not hold for smaller values of n, if, and only if, the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the property that M_n is a scalar matrix (the identity matrix multiplied by a real number), and this is not true for any smaller value of n. Furthermore this holds if,