THE SATELLITE CASE OF THE THREE-BODY PROBLEM

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1. EQUATIONS OF MOTION

In the lunar or satellite theory, the disturbing function is developed in the form in which each term consists of two factors. The first factor depends on the position of the sun, and the second one that of the satellite. The main term of the disturbing function is

$$F_2 = \nu^2 a^2 \left(\frac{a_1}{r_1}\right)^3 [C_A(A+1+\frac{3}{2}e^2) + C_B(B+\frac{15}{2}e^2) + C_C \cdot C] \quad , \tag{1}$$

where

$$C_A = -\frac{1}{8} + \frac{3}{8}c^2 + \frac{3}{8}s^2\cos(2f_1-2h), \quad C_B = \frac{1}{8}s^2\cos 2g + \frac{1}{16}(1+c)^2\cos(2f_1-2g-2h)$$

$$+\frac{1}{16}(1-c)^2\cos(2f_1+2g-2h), \quad C_C = \frac{1}{8}s^2\sin 2g - \frac{1}{16}(1+c)^2\sin(2f_1-2g-2h) \tag{2}$$

$$+\frac{1}{16}(1-c)^2\sin(2f_1+2g-2h) \quad .$$

These, together with the factor $(a_1/r_1)^3$, give the solar position. The subscript 1 is referred to the sun. While, $A+1+(3/2)e^2$, $B+(15/2)e^2$, and C, where

$$A = -e^2 - 2e\cos u + \frac{1}{2}e^2\cos 2u, \quad B = -3e^2 - 6e\cos u + (3-\frac{3}{2}e^2)\cos 2u,$$

$$C = \eta(6e\sin u - 3\sin 2u) \quad , \tag{3}$$

give the position of the satellite. In equations (2) and (3),

$$c = \cos i, \quad s = \sin i, \quad g = \omega, \quad h = \Omega - \omega_1, \quad \eta = \sqrt{1-e^2} \quad , \tag{4}$$

R. L. Duncombe (ed.), Dynamics of the Solar System, 77–83.

and f_1 is the true anomaly of the sun, u the eccentric anomaly of the satellite, ν in equation (1) being the mean motion of the sun.

In terms of Delaunay variables, the equations of motion are

$$\frac{d}{dt}(L,G,H,K) = \frac{\partial F}{\partial(\ell,g,h,k)}, \quad \frac{d}{dt}(\ell,g,h,k) = -\frac{\partial F}{\partial(L,G,H,K)}, \tag{5}$$

with the Hamiltonian

$$F = \frac{\mu^2}{2L^2} - \nu K + F_2 \quad , \tag{6}$$

where k is the mean anomaly of the sun.

2. ELIMINATION OF ℓ

Applying canonical perturbation theory (Hori 1966), the first canonical transformation results in the new Hamiltonian

$$F^* = \frac{\mu^2}{2L'^2} - \nu K' + \nu^2 a^2 (\frac{a_1}{r_1})^3 [C_A(1+\frac{3}{2}e^2) + C_B\frac{15}{2}e^2] \quad , \tag{7}$$

and the equation for the determining function, of the form

$$n\frac{\partial S_2}{\partial \ell'} + \nu\frac{\partial S_2}{\partial k} = \nu^2 a^2 (\frac{a_1}{r_1})^3 \sum_{X=A,B,C} C_X \cdot X \quad . \tag{8}$$

This equation can be integrated by the successive application of the integration by parts to find S_2 in power series in ν/n, and in the closed form both in e and e_1.

$$S_2 = \frac{\nu^2 a^2}{n}(\frac{a_1}{r_1})^3 \Sigma C_X \cdot {}^{(1)}X - \frac{\nu^3 a^2}{n^2}\Sigma \{C_X(\frac{a_1}{r_1})^3\}^{(1)} \cdot {}^{(2)}X + O(\frac{\nu^4 a^2}{n^3}) \quad , \tag{9}$$

where

$$\begin{aligned} &{}^{(1)}A = (-2e + \frac{3}{4}e^3)\sin u + \frac{3}{4}e^2\sin 2u - \frac{1}{12}e^3\sin 3u, \quad {}^{(1)}B = (-\frac{15}{2}e + \frac{15}{4}e^3)\sin u \\ &+ (\frac{3}{2} + \frac{3}{4}e^2)\sin 2u + (-\frac{1}{2}e + \frac{1}{4}e^3)\sin 3u, \quad {}^{(1)}C = \eta[-\frac{15}{4}e^2 - \frac{15}{2}e\cos u + (\frac{3}{2} + \frac{3}{2}e^2)\times \\ &\times\cos 2u - \frac{1}{2}e\cos 3u], \quad {}^{(2)}A = e^2 - \frac{3}{16}e^4 + (2e - \frac{3}{8}e^3)\cos u + (-\frac{7}{8}e^2 + \frac{1}{6}e^4)\cos 2u \\ &+ \frac{11}{72}e^3\cos 3u - \frac{1}{96}e^4\cos 4u, \quad {}^{(2)}B = \frac{33}{8}e^2 - \frac{27}{16}e^4 + (\frac{33}{4}e - \frac{27}{8}e^3)\cos u \end{aligned} \tag{10}$$

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$$+(-\frac{3}{4}-\frac{19}{8}e^2+e^4)\cos 2u+(\frac{5}{12}e+\frac{1}{24}e^3)\cos 3u+(-\frac{1}{16}e^2+\frac{1}{32}e^4)\cos 4u,$$

$$^{(2)}C=\eta[(-\frac{33}{4}e+3e^3)\sin u+(\frac{3}{4}+\frac{11}{4}e^2)\sin 2u+(-\frac{5}{12}e-\frac{1}{4}e^3)\sin 3u+\frac{1}{16}e^2\sin 4u],$$

and

$$\{C_A(\frac{a_1}{r_1})^3\}^{(1)}=(-\frac{1}{8}+\frac{3}{8}c^2)(-3\frac{e_1}{\eta_1})(\frac{a_1}{r_1})^4\sin f_1-\frac{3}{8}s^2\frac{1}{\eta_1}(\frac{a_1}{r_1})^4[-\frac{1}{2}e_1\sin(f_1-2h)$$

$$+2\sin(2f_1-2h)+\frac{5}{2}e_1\sin(3f_1-2h)],\quad \{C_B(\frac{a_1}{r_1})^3\}^{(1)}=\frac{1}{8}s^2(-\frac{3}{2}\frac{e_1}{\eta_1})(\frac{a_1}{r_1})^4\times$$

$$\times[\sin(f_1-2g)+\sin(f_1+2g)]-\frac{1}{16}(1+c)^2\frac{1}{\eta_1}(\frac{a_1}{r_1})^4[-\frac{1}{2}e_1\sin(f_1-2g-2h)$$

$$+2\sin(2f_1-2g-2h)+\frac{5}{2}e_1\sin(3f_1-2g-2h)]-\frac{1}{16}(1-c)^2\frac{1}{\eta_1}(\frac{a_1}{r_1})^4\times$$

$$\times[-\frac{1}{2}e_1\sin(f_1+2g-2h)+2\sin(2f_1+2g-2h)+\frac{5}{2}e_1\sin(3f_1+2g-2h)]\quad,\qquad(11)$$

$$\{C_C(\frac{a_1}{r_1})^3\}^{(1)}=\frac{1}{8}s^2(-\frac{3}{2}\frac{e_1}{\eta_1})(\frac{a_1}{r_1})^4[\cos(f_1-2g)-\cos(f_1+2g)]-\frac{1}{16}(1+c)^2\frac{1}{\eta_1}(\frac{a_1}{r_1})^4\times$$

$$\times[-\frac{1}{2}e_1\cos(f_1-2g-2h)+2\cos(2f_1-2g-2h)+\frac{5}{2}e_1\cos(3f_1-2g-2h)]$$

$$-\frac{1}{16}(1-c)^2\frac{1}{\eta_1}(\frac{a_1}{r_1})^4[-\frac{1}{2}e_1\cos(f_1+2g-2h)+2\cos(2f_1+2g-2h)+\frac{5}{2}e_1\times$$

$$\times\cos(3f_1+2g-2h)]\quad.$$

If the partial derivative with respect to e is denoted by the subscript e, we have

$$A_e=-e-2\cos u,\quad B_e=\frac{a}{r}[(-9+9e^2)\cos u-3e\cos 2u+3\cos 3u]\quad,$$

$$C_e=\frac{a}{r\eta}[(9-\frac{33}{2}e^2)\sin u+(3e+3e^3)\sin 2u+(-3+\frac{3}{2}e^2)\sin 3u]\quad,$$

$$^{(1)}A_e=(-2+e^2)\sin u+\frac{1}{2}e\sin 2u,\quad ^{(1)}B_e=(-9+9e^2)\sin u-\frac{3}{2}e\sin 2u+\sin 3u\quad,$$

$$^{(1)}C_e=\frac{1}{\eta}[-\frac{9}{2}e+\frac{33}{4}e^3+(-9+\frac{33}{2}e^2)\cos u+(-\frac{3}{2}e-\frac{3}{2}e^2)\cos 2u+(1-\frac{1}{2}e^2)\cos 3u]\quad,$$

$$^{(2)}A_e=e-\frac{3}{8}e^3+(2-\frac{3}{4}e^2)\cos u+(-\frac{3}{4}e+\frac{1}{4}e^3)\cos 2u+\frac{1}{12}e^2\cos 3u\quad,\qquad(12)$$

$$^{(2)}B_e = \frac{9}{2}e - \frac{39}{8}e^3 + \left(9 - \frac{39}{4}e^2\right)\cos u + \left(-\frac{5}{4}e + \frac{9}{4}e^3\right)\cos 2u + \left(-\frac{1}{3} - \frac{1}{4}e^2\right)\cos 3u$$

$$+\frac{1}{8}e\cos 4u, \quad ^{(2)}C_e = \frac{1}{\eta}\left[\left(-9 + \frac{87}{4}e^2 - \frac{15}{2}e^4\right)\sin u + \left(\frac{5}{4}e - \frac{19}{4}e^3\right)\sin 2u\right.$$

$$\left. + \left(\frac{1}{3} + \frac{1}{12}e^2 + \frac{1}{4}e^4\right)\sin 3u + \left(-\frac{1}{8}e + \frac{1}{16}e^3\right)\sin 4u\right] \quad .$$

Equations (12) are of use in the derivation of short-period perturbation in ℓ and g.

3. ELIMINATION OF k

After the mean anomaly of the satellite is eliminated, the new equations of motion are

$$\frac{d}{dt}(L',G',H',K') = \frac{\partial F^*}{\partial(g',h',k)} \ , \quad \frac{d}{dt}(\ell',g',h',k) = -\frac{\partial F^*}{\partial(L',G',H',K')} \ , \quad (13)$$

with the Hamiltonian (7) or

$$F^* = \frac{\mu^2}{2L'^2} - \nu K' + \nu^2 a^2 \left[C_A^*\left(A_1 + \frac{1}{\eta_1{}^3}\right) + C_B^* B_1 + C_C^* C_1\right] \quad , \quad (14)$$

where

$$A_1 = \left(\frac{a_1}{r_1}\right)^3 - \frac{1}{\eta_1{}^3} \ , \quad B_1 = \left(\frac{a_1}{r_1}\right)^3 \cos 2f_1 \ , \quad C_1 = \left(\frac{a_1}{r_1}\right)^3 \sin 2f_1 \ ,$$

$$C_A^* = \sum_{j=0}^{1} c_{0,2j}\cos 2jg' \ , \quad C_B^* = \sum_{j=-1}^{1} c_{2,2j}\cos(2jg'-2h') \ , \quad (15)$$

$$C_C^* = -\sum_{j=-1}^{1} c_{2,2j}\sin(2jg'-2h') \quad ,$$

and

$$c_{00} = \left(-\frac{1}{8} + \frac{3}{8}c^2\right)\left(1 + \frac{3}{2}e^2\right) \ , \quad c_{02} = \frac{15}{16}s^2e^2 \ , \quad c_{20} = \frac{3}{8}s^2\left(1 + \frac{3}{2}e^2\right) \ ,$$

$$c_{2\mp 2} = \frac{15}{32}(1 \pm c)^2 e^2 \quad . \quad (16)$$

The second canonical transformation, L', G', H', K', ℓ', g', h', k→L", G", H", K", ℓ'', g", h", k, which eliminates k from the new Hamiltonian F**, leads to the result:

$$F^{**} = \frac{\mu^2}{2L'^2} - \nu K'' + \frac{\nu^2 a^2}{\eta_1{}^3}C_A^* + \nu^3 a^4\left[\{C_A^*, C_C^*\}\langle {}^{(1)}C_1A_1\rangle + \{C_B^*, C_C^*\}\langle {}^{(1)}C_1B_1\rangle\right] \ , \quad (17)$$

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and

$$S_1^* + S_2^* = \nu a^2 \sum_{X=A,B,C} C_X^* \cdot {}^{(1)}X_1 + \nu^2 a^4 [\{C_A^*, C_C^*\} \cdot {}^{(1)}D_1$$
$$+ \{C_A^*, C_C^*\} \cdot {}^{(1)}E_1 + \{C_B^*, C^*_C\} \cdot {}^{(1)}F_1] \quad , \tag{18}$$

where

$${}^{(1)}A_1 = \frac{1}{\eta_1{}^3}(f_1 - k + e_1 \sin f_1), \quad {}^{(1)}B_1 = \frac{1}{\eta_1{}^3}(\frac{e_1}{2}\sin f_1 + \frac{1}{2}\sin 2f_1 + \frac{e_1}{6}\sin 3f_1),$$
$${}^{(1)}C_1 = \frac{1}{\eta_1{}^3}[-\frac{1+2\eta_1}{6(1+\eta_1)^2}e_1{}^2 - \frac{e_1}{2}\cos f_1 - \frac{1}{2}\cos 2f_1 - \frac{e_1}{6}\cos 3f_1] \quad , \tag{19}$$

and

$$\{C_A^*, C_B^*\} = \sum_{j=-1}^{1} c_{2,2j}^* \sin(2jg''-2h''), \quad \{C_A^*, C_C^*\} = \sum_{j=-1}^{1} c_{2,2j}^* \times$$
$$\times \cos(2jg''-2h''), \quad \{C_B^*, C_C^*\} = -\sum_{j=0}^{1} c_{0,2j}^* \cos 2jg'' \quad , \tag{20}$$

with

$$c_{00}^* = \frac{9}{32}\frac{\eta c}{na^2}[2s^2 + (33+17c^2)e^2], \quad c_{02}^* = \frac{135}{32}\frac{\eta c s^2}{na^2}e^2 \quad ,$$
$$c_{20}^* = \frac{9}{32}\frac{\eta c s^2}{na^2}(2-17e^2), \quad c_{2\mp 2}^* = \pm\frac{45}{64}\frac{\eta}{na^2}(1\pm c)^2(2\mp 3c)e^2 \quad , \tag{21}$$

and finally,

$$\langle {}^{(1)}C_1 A_1 \rangle = -[\frac{1}{4} + \frac{1+2\eta_1}{6(1+\eta_1)^2}]\frac{e_1{}^2}{\eta_1{}^6}, \quad \langle {}^{(1)}C_1 B_1 \rangle = -(\frac{1}{4} + \frac{1}{6}e_1{}^2)\frac{1}{\eta_1{}^6},$$
$$\langle {}^{(1)}A_1 \cdot {}^{(1)}B_1 \rangle = -[\frac{1-3\eta_1}{6(1+\eta_1)} + \frac{2}{3e_1{}^2}\ln\frac{1+\eta_1}{2\eta_1}]\frac{1}{\eta_1{}^3}, \quad {}^{(1)}D_1 = -\frac{1}{2}{}^{(1)}A_1 \cdot {}^{(1)}B_1$$
$$+\frac{1}{2}\langle {}^{(1)}A_1 \cdot {}^{(1)}B_1 \rangle + \frac{1}{\eta_1{}^6}[-\frac{e_1{}^2}{48(1+\eta_1)^2}(\frac{65}{3} + \frac{130}{3}\eta_1 + 31\eta_1{}^2 + 12\eta_1{}^3) - \frac{3}{4}e_1\cos f_1$$
$$-(\frac{1}{4} + \frac{1}{6}e_1{}^2)\cos 2f_1 - \frac{5}{36}e_1\cos 3f_1 - \frac{1}{48}e_1{}^2\cos 4f_1], \quad {}^{(1)}E_1 = -\frac{1}{2}{}^{(1)}A_1 \cdot {}^{(1)}C_1$$
$$+\frac{1}{\eta_1{}^6}[-\{\frac{1+2\eta_1}{6(1+\eta_1)^2}e_1{}^2 + \frac{3}{4}\}e_1\sin f_1 - (\frac{1}{4} + \frac{1}{6}e_1{}^2)\sin 2f_1 - \frac{5}{36}e_1\sin 3f_1 \tag{22}$$
$$-\frac{1}{48}e_1{}^2\sin 4f_1], \quad {}^{(1)}F_1 = -\frac{1}{2}{}^{(1)}B_1 \cdot {}^{(1)}C_1 + \frac{1}{\eta_1{}^6}[-\{\frac{1+2\eta_1}{12(1+\eta_1)^2} + \frac{7}{12}\}e_1\sin f_1$$

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$$-\{\frac{1+2\eta_1}{12(1+\eta_1)^2}+\frac{7}{48}\}e_1{}^2\sin2f_1-\{\frac{1+2\eta_1}{36(1+\eta_1)^2}+\frac{1}{8}\}e_1\sin3f_1-(\frac{1}{16}+\frac{1}{24}e_1{}^2)\sin4f_1$$

$$-\frac{1}{24}e_1\sin5f_1-\frac{1}{144}e_1{}^2\sin6f_1] \quad .$$

In my previous work on the same subject (Hori 1963), i and e_1 were assumed to be vanishingly small. In that case equation (17) is reduced to

$$F^{**}=\frac{\mu^2}{2L'^2}-\nu K''+\frac{1}{4}\nu^2a^2(1+\frac{3}{2}e^2)+\frac{225}{64}\frac{\nu^3a^2}{n}\eta e^2 \quad , \tag{23}$$

and the new equations are immediately integrated. In the present general case, F** still depends on g" and h", and another canonical transformation is required for the complete solution.

Let the transformation be L', G", H", K", ℓ", g", h", k→L', G"', H"', K", ℓ"', g"', h"', k. By a proper choice of a determining function, we may have an integral

$$\frac{\mu^2}{2L'^2}-\nu K''+\nu^2a^2[(-\frac{1}{8}+\frac{3}{8}c^2)(1+\frac{3}{2}e^2)+\frac{15}{16}s^2e^2\cos2g''']=\text{const.} \quad , \tag{24}$$

where

$$c=\cos i=\frac{H'''}{G'''} \quad , \quad \eta=\frac{G'''}{L'} \quad . \tag{25}$$

Such a determining function is $O(\nu/n)$, and the use of elliptic functions is inevitable if no restriction is put on the size of i and e_1.

When $\alpha=H'''/L'$ is smaller than $(3/5)^{1/2}$, libration in g"' occurs. The center of libration is at $g'''=\pi/2$ and $\eta=(5\alpha^2/3)^{1/4}$. If η is smaller than $(5\alpha^2/3)^{1/2}$ when g"' is $\pi/2$, g"' has circulation even when $\alpha<(3/5)^{1/2}$.

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DISCUSSION

Marchal: Do you obtain some cases in which the eccentricity goes until one or at least reaches large values?

Hori: I indeed can obtain large variations of the eccentricity; however, I cannot say that I reach one since my theory is not valid for collision orbits.

Message: Does the equilibrium point shown in your last figure represent a periodic solution of the third sort?

Hori: Yes, in triply primed variables.

Garfinkel: In your final result, what has become of the periodic terms involving the argument h?

Hori: Being small terms of the third order in ν, the calculation of their effect appears in the next order of the perturbation.

Garfinkel: With the use of your integral, the Hamiltonian reduces to that of the Ideal Resonance Problem, exhibiting both the libration and the circulation regimes in the argument g of the perihelion.

Hori: Yes. Indeed, it is in the form of your Ideal Resonance Problem in the variables G''' and g'''.

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