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An Elementary Proof of a Weak Exceptional Zero Conjecture

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Abstract. In this paper we extend Darmon's theory of "integration on $\mathcal{H}_p \times \mathcal{H}$" to cusp forms f of higher even weight. This enables us to prove a "weak exceptional zero conjecture": that when the p-adic L-function of f has an exceptional zero at the central point, the $\mathcal{L}$-invariant arising is independent of a twist by certain Dirichlet characters.

1 Introduction

Let f be a cusp form for $\Gamma_1(N)$ of weight k and character ϵ which is an eigenform for the Hecke operator T_p. In their paper [MTT], Mazur, Tate and Teitelbaum define, using modular symbols, a p-adic L-function $L_p(f, \chi, s)$. Here χ is a Dirichlet character, and $s \in \mathbb{Z}_p$. Their p-adic L-function interpolates the usual complex L-function: to be precise, we have an equation (in a suitable $\mathbb{C}_p$-vector space V_f)

$$L_p(f, \omega^j \chi, j) = e_p(\alpha, \chi, j) K(\chi, j) L(f_{\bar{\chi}}, j+1) \tag{1}$$

for $0 \leq j \leq k-2$, where ω is the Teichmüller character, χ is a Dirichlet character, $K(\chi, j)$ is a nonzero complex number and $e_p(\alpha, \chi, j) \in \bar{\mathbb{Q}}$ is the p-adic multiplier. Here α is an 'allowable' root of the equation $X^2 - a_p X + \epsilon(p) p^{k-1} = 0$, where $T_p f = a_p f$.

L_p is said to have an exceptional zero when the p-adic multiplier is zero.

In particular, suppose f is a newform for $\Gamma_0(N)$ of even weight k and level N where $p || N$, and suppose $T_p f = w p^{\frac{k-2}{2}} f$ for some $w = \pm 1$ (where $T_p = U_p$ is the Hecke operator at p). Now the only allowable root is $a_p = w p^{\frac{k-2}{2}}$. Then there is an exceptional zero at the central point $j = \frac{k-2}{2}$ for any Dirichlet character χ satisfying $\chi(p) = w$.

Mazur, Tate and Teitelbaum conjectured that the exceptional zero is "of local type", meaning that there is an equation

$$L'_p(f, \omega^{\frac{k-2}{2}} \chi, t)\big|_{t=\frac{k-2}{2}} = \mathcal{L}_p(f, \chi) K(\chi, j) L(f_{\bar{\chi}}, k/2) \tag{2}$$

where the L-invariant $\mathcal{L}_p(f) = \mathcal{L}_p(f, \chi)$ is independent of the choice of χ. It was hoped that the L-invariant could be defined explicitly using only the p-adic Galois representation $V_p(f)$ as a representation of the local Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

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Such an explicit definition was later made by Fontaine and Mazur. There were also definitions of $\mathcal{L}_p(f)$ made by Teitelbaum and Coleman. It has now been shown that these values agree when they are all defined.

Then the full exceptional zero conjecture/theorem states, for such an explicitly defined $\mathcal{L}_p(f)$:

***Theorem* (Kato, Kurihara, Tsuji/Stevens)** *Equation* (2) *is satisfied with* $\mathcal{L}_p(f,\chi) = \mathcal{L}_p(f)$ *for any Dirichlet character* χ *of conductor prime to N satisfying* $\chi(p) = w$.

This has been proved by Kato, Kurihara and Tsuji, and independently by Stevens, using deep methods from arithmetic geometry.

The aim of this paper is to give a more elementary proof of the following weaker statement:

Proposition *There exist constants* $\mathcal{L}_p^{w_\infty}(f) \in \mathbb{C}_p$ *for* $w_\infty = \pm 1$ *such that* (2) *holds with* $\mathcal{L}_p(f,\chi) = \mathcal{L}_p^{w_\infty}(f)$ *for any Dirichlet character* χ *of conductor prime to N satisfying* $\chi(p) = w$ *and* $\chi(-1) = w_\infty$.

This has been done in the first part of Darmon's paper [Dar] for the case $k = 2$ (see the remark in Section 3.2 of [Dar]), and his proof is extended here to the higher weight case.

The idea of this method is to construct two cohomology classes lc_f and oc_f in the group $H^1(\Gamma, \mathcal{M}_{k-2})$ where $\mathcal{M}_{k-2}$ is a space of $\mathbb{C}_p$-valued modular symbols, and $\Gamma \subset \mathrm{PSL}_2(\mathbb{Q})$. The class lc_f will interpolate values of L_p' and the class oc_f will interpolate values of L_∞. By showing that the two classes are contained in the same one-dimensional Hecke eigenspace, an equation like (2) will be obtained.

These cohomology classes will be obtained by interpreting cusp forms for $\Gamma_0(N)$ which are new at p as cusp forms on $\mathcal{E}(\mathcal{T}) \times \mathcal{H}$ for the group $\Gamma \subset \mathrm{PSL}_2(\mathbb{Q})$, where $\mathcal{H}$ is the complex upper half plane and $\mathcal{T}$ is the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

To be more precise:

Let $N = Mp$ where M and p are coprime.

$$R := \{\gamma \in M_2(\mathbb{Z}[1/p]) : c \equiv 0 \bmod M\}$$

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{Q})$ be the image of the set of elements of R of determinant 1.

We can interpret cusp forms for $\Gamma_0(N)$ which are new at p as cusp forms on $\mathcal{E}(\mathcal{T}) \times \mathcal{H}$ for Γ. Such a form consists of a set of forms f_e, one for each (oriented) edge of $\mathcal{T}$, such that f_e is a cusp form for $\Gamma_e := \mathrm{Stab}_\Gamma(e)$, related by a Γ-invariance property. f will be taken to correspond to a newform f_0.

We define a modular symbol κ_f lying in $\mathrm{C}^{\mathrm{har}}\big(\mathrm{Hom}(\mathbb{D}_0, \mathrm{V}_{k-2}(\mathbb{C}))\big)^\Gamma$, where $V_{k-2}(\mathbb{C}) = \mathrm{Hom}(\mathcal{P}_{k-2}, \mathbb{C})$, $\mathcal{P}_{k-2}$ is the space of polynomials of degree $\leq k-2$ with coefficients in $\mathbb{Z}$, $\mathbb{D}_0$ is the space of divisors of degree zero on $\mathbb{P}^1(\mathbb{Q})$ and $\mathrm{C}^{\mathrm{har}}(\mathrm{C})$ denotes the space of harmonic cocycles with values in C.

This can be written in terms of modular symbols as in [MTT], and by symmetrizing or antisymmetrizing the modular symbols and dividing by a period, you can get a symbol with algebraic values, written $\kappa_f^{w_\infty}\{x \to y\}(e)(P)$ for $x, y \in \mathbb{P}^1(\mathbb{Q})$, $e \in \mathcal{E}(\mathcal{T})$ and $P \in \mathcal{P}_{k-2}$.

It is harmonic in e, with sufficiently bounded growth to define a distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ with

$$\int_{U(e)} P(z)\, d\mu_f\{x \to y\}(z) = \kappa_f\{x \to y\}(e)(P).$$

In Darmon's case $\kappa_f^{w_\infty}$ takes integer values, so this distribution is in fact a measure, and he can also define a multiplicative integral.

Now Darmon defines a double multiplicative integral,

$$⨰_{z_1}^{z_2} \int_x^y \omega := ⨰_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t - z_2}{t - z_1}\right) d\mu_{f,\mathrm{Dar}}\{x \to y\}(t)$$

where $z_i \in \mathcal{H}_p$ and $x, y \in \mathbb{P}^1(\mathbb{Q})$.

In the general even weight case, $\mu_f\{x \to y\}$ is only a tempered distribution of order $\frac{k-2}{2}$ (in the sense of [Col]). So we no longer have a multiplicative double integral, but choosing a branch of the p-adic logarithm such that $\log_p(p) = 0$ we can still define an additive double integral:

$$\int_{z_1}^{z_2} \int_x^y (P)\omega := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p\left(\frac{t - z_2}{t - z_1}\right) P(t)\, d\mu_f^{w_\infty}\{x \to y\}(t).$$

We choose an embedding $\Psi\colon K = \mathbb{Q} \times \mathbb{Q} \to M_2(\mathbb{Q})$, γ_Ψ a generator of $\bar{\Psi}(K^*) \cap \Gamma$, $x_\Psi, y_\Psi \in \mathbb{P}^1(\mathbb{Q})$ the fixed points of $\bar{\Psi}(K^*)$.

Darmon defines a 'period',

$$I_\Psi := ⨰_z^{\gamma_\Psi z} \int_{x_\Psi}^{y_\Psi} \omega$$

It can be shown that $I_\Psi \in \mathbb{Q}_p^*$, $\mathrm{ord}_p(I_\Psi)$ is related to values of complex L-functions at 1, and $\log_p(I_\Psi)$ is related to values of the derivative of the p-adic L-function.

In our situation we no longer have I_Ψ, but we still have values LI_Ψ and W_Ψ corresponding to $\log_p(I_\Psi)$ and $\mathrm{ord}_p(I_\Psi)$ respectively and their values are still related to the complex and p-adic L-functions.

The period I_Ψ was a special value of a cocycle c_f in $H^1(\Gamma, \mathcal{M}(\mathbb{C}_p^*))$, where the space of modular symbols is defined by $\mathcal{M}(C) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{D}_0, C)$. By showing that $\log_p c_f$ and $\mathrm{ord}_p\, c_f$ belong to the same one-dimensional $\mathbb{C}_p$-subspace of $H^1(\Gamma, \mathcal{M}(\mathbb{C}_p))$, the result is obtained for $k = 2$.

Similarly, for $k > 2$, we still have cocycles lc_f and oc_f corresponding to $\log_p c_f$ and $\mathrm{ord}_p\, c_f$ which are in the same one-dimensional Hecke eigenspace of $H^1(\Gamma, \mathcal{M}_{k-2})$ where $\mathcal{M}_{k-2} = \mathcal{M}(V_{k-2}(\mathbb{C}_p))$.

2 Cusp Forms on the Tree

2.1 Definitions

Let k be an even positive integer, and $N = Mp$ where p is a prime not dividing M. Let

$$R := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[1/p]) : c \equiv 0 \bmod M \right\}$$

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{Q})$ be the image of the set R_1^* of elements of R of determinant 1.

Let $\tilde{\Gamma} \subset \mathrm{PGL}_2(\mathbb{Q})$ be the image of the set R_+^* of invertible elements of R with positive determinant.

$\mathcal{T}$ will be the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, with a fixed vertex v_* corresponding to the class of the lattice $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and a fixed edge e_* with source v_* and target corresponding to the class of $\mathbb{Z}_p \oplus p\mathbb{Z}_p$. Vertices will be called *odd* or *even* according to the parity of their distance from v_* (so v_* is even). Edges will be called odd or even according to the parity of their source vertex (so e_* is even). Then Γ acts on the tree, with $\mathrm{Stab}_\Gamma(e_*) = \Gamma_0(N)$ and $\mathrm{Stab}_\Gamma(v_*) = \Gamma_0(M)$.

Define the set of cusp forms on the tree $\mathcal{T}$ to be the set $S_k(\mathcal{T}, \Gamma)$ of all

$$f : \mathcal{E}(\mathcal{T}) \times \mathcal{H} \to \mathbb{C}$$

satisfying

(i) $f(\gamma e, \gamma z) = (cz + d)^k f(e, z) \quad \forall \gamma \in \Gamma$,
(ii) f is harmonic in e, *i.e.*, we have $f(\bar{e}, z) = -f(e, z)$ for $e \in \mathcal{E}(\mathcal{T})$ and $\sum_{s(e)=v} f(e, z) = 0$ for $v \in \mathcal{V}(\mathcal{T})$,
(iii) $f_e = f(e, \cdot\,)$ is a cusp form of weight k for $\Gamma_e = \mathrm{Stab}_\Gamma(e)$.

The action of Γ is transitive on the unoriented edges of $\mathcal{T}$, and preserves the parity of edges. The harmonicity condition relates values on e and $\bar{e}$, so the restriction

$$\rho_T \colon S_k(\mathcal{T}, \Gamma) \to S_k\big(\Gamma_0(N)\big), \quad f \mapsto f_{e_*}$$

is injective.

Further, define cusp forms on the edges or vertices of the tree for the group $\tilde{\Gamma}$ as follows:

Let $S_k(\mathcal{E}(\mathcal{T}), \tilde{\Gamma})$ be the set of all $f : \mathcal{E}(\mathcal{T}) \times \mathcal{H} \to \mathbb{C}$ satisfying

(i) $f(\gamma e, \gamma z) = \dfrac{(cz + d)^k}{(\det \gamma)^{k/2}} f(e, z) \quad \forall \gamma \in \tilde{\Gamma}$ and
(iii) $f_e = f(e, \cdot\,)$ is a cusp form of weight k for $\tilde{\Gamma}_e = \mathrm{Stab}_{\tilde{\Gamma}}(e)$.

Let $S_k(\mathcal{V}(\mathcal{T}), \tilde{\Gamma})$ be the set of all $f : \mathcal{V}(\mathcal{T}) \times \mathcal{H} \to \mathbb{C}$ satisfying

(i) $f(\gamma v, \gamma z) = \dfrac{(cz + d)^k}{(\det \gamma)^{k/2}} f(v, z) \quad \forall \gamma \in \tilde{\Gamma}$ and
(iii) $f_v = f(v, \cdot\,)$ is a cusp form of weight k for $\tilde{\Gamma}_v = \mathrm{Stab}_{\tilde{\Gamma}}(v)$.

Because $\tilde{\Gamma}$ acts transitively on the edges and vertices of $\mathcal{T}$, the restrictions

$$\rho_E \colon S_k\big(\mathcal{E}(\mathcal{T}), \tilde{\Gamma}\big) \to S_k\big(\Gamma_0(N)\big), \quad f \mapsto f_{e_*}$$

and

$$\rho_V \colon S_k\big(\mathcal{V}(\mathcal{T}), \tilde{\Gamma}\big) \to S_k\big(\Gamma_0(M)\big), \quad f \mapsto f_{v_*}$$

are injective.

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There are maps relating these spaces. Firstly, fix $\alpha \in \tilde{\Gamma} \setminus \Gamma$ in the normalizer of $\Gamma_0(N)$, *i.e.*, such that $\alpha e_* = \bar{e}_*$. For $f \in S_k(\mathcal{T}, \Gamma)$ and $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$, define

$$(f|\gamma)(e,z) := \frac{\det(\gamma)^{k/2}}{(cz+d)^k} f(\gamma e, \gamma z).$$

so the condition (i) in the definition of $S_k(\mathcal{T}, \Gamma)$ says $f|\gamma = f$ for all $\gamma \in \Gamma$.

Lemma 2.1 *We can define an injection $i\colon S_k(\mathcal{T},\Gamma) \to S_k(\mathcal{E}(\mathcal{T}), \tilde{\Gamma})$ by $f \mapsto \tilde{f}$ such that $\tilde{f}(e,z) = f(e,z)$ if e is an even edge, and $\tilde{f}(e,z) = (f|\alpha)(e,z)$ for e an odd edge.*

Proof This is similar to the weight 2 case given in [Dar]. ■

Lemma 2.2 *Defining maps π_s and $\pi_t\colon S_k(\mathcal{E}(\mathcal{T}), \tilde{\Gamma}) \to S_k(\mathcal{V}(\mathcal{T}), \tilde{\Gamma})$ by*

$$\pi_s(f)(v,z) = \sum_{s(e)=v} f(e,z) \quad \pi_t(f)(v,z) = \sum_{t(e)=v} f(e,z),$$

the following is exact:

$$0 \to S_k(\mathcal{T},\Gamma) \xrightarrow{i} S_k\big(\mathcal{E}(\mathcal{T}), \tilde{\Gamma}\big) \xrightarrow{\pi_s \oplus \pi_t} S_k\big(\mathcal{V}(\mathcal{T}), \tilde{\Gamma}\big) \oplus S_k\big(\mathcal{V}(\mathcal{T}), \tilde{\Gamma}\big).$$

Proof Again this works similarly to the case $k = 2$. ■

The maps defined above will correspond to the "degeneracy" maps for cusp forms, $\phi_s, \phi_t\colon S_k(\Gamma_0(N)) \to S_k(\Gamma_0(M))$. If $\Gamma_0(M) = \coprod_{j=1}^{p+1} \gamma_j \Gamma_0(N)$ then

$$\phi_s(f_0) = \sum_{j=1}^{p+1} f_0|\gamma_j^{-1}, \quad \phi_t(f_0) = \sum_{j=1}^{p+1} f_0|(\alpha\gamma_j^{-1}).$$

The space of p-new forms is defined to be the kernel of $\phi_s \oplus \phi_t$.

As in the case $k = 2$ we can show the following:

Lemma 2.3 *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc}
0 \longrightarrow & S_k(\mathcal{T},\Gamma) & \xrightarrow{i} & S_k\big(\mathcal{E}(\mathcal{T}),\tilde{\Gamma}\big) & \xrightarrow{\pi_s\oplus\pi_t} & S_k\big(\mathcal{V}(\mathcal{T}),\tilde{\Gamma}\big) \oplus S_k\big(\mathcal{V}(\mathcal{T}),\tilde{\Gamma}\big) \\
& \downarrow \rho_T & & \downarrow \rho_E & & \downarrow \rho_V \oplus \rho_V \\
0 \longrightarrow & S_k\big(\Gamma_0(N)\big)^{p-\mathrm{new}} & \longrightarrow & S_k\big(\Gamma_0(N)\big) & \xrightarrow{\phi_s\oplus\phi_t} & S_k\big(\Gamma_0(M)\big) \oplus S_k\big(\Gamma_0(M)\big)
\end{array}$$

in which the vertical arrows are isomorphisms.

Proof As mentioned above, the three restriction homomorphisms are injective. Further, any $f_0 \in S_k(\Gamma_0(N))$ can be lifted to $S_k(\mathcal{E}(\mathcal{T}), \tilde{\Gamma})$ by defining

$$f(\gamma e_*, z) := \frac{(cz+d)^k}{(\det \gamma)^{k/2}} f_0(\gamma^{-1} z),$$

so ρ_E is an isomorphism. Similarly, ρ_V is an isomorphism.

The second square of the diagram commutes because $\{\gamma_1 e_*, \dots, \gamma_{j+1} e_*\}$ is the set of edges with source v_*, and $\{\gamma_1 \alpha e_*, \dots, \gamma_{j+1} \alpha e_*\}$ is the set of edges with target v_*. ■

We will also need modular symbols. Mazur, Tate and Teitelbaum [MTT] use the symbol

$$\phi_f\{x \to y\}(P) := 2\pi i \int_x^y f_0(z) P(z)\, dz \quad \text{or} \quad \phi_f(P, y) := \phi_f\{\infty \to y\}(P)$$

which is in the space $\operatorname{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})^{\Gamma_0(N)}$, where $\mathbb{D}_0$ is the group of degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$ and $\mathcal{P}_{k-2}$ is the vector space of polynomials of degree $\leq k-2$ with coefficients in $\mathbb{Z}$.

We can define a modular symbol on the tree by

Definition 2.1

$$\kappa_f\{x \to y\}(e, P) := 2\pi i \int_x^y f_e(z) P(z)\, dz.$$

This will be shown to be in the space $C^{\text{har}}\big(\operatorname{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})\big)^{\Gamma}$ of harmonic cocycles on $\mathcal{E}(\mathcal{T})$ with values in $\operatorname{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})$, where the group action is described in the next section.

2.2 Note on Actions of $\mathrm{GL}_2(\mathbb{Q})$

The action of $\mathrm{GL}_2(\mathbb{Q})^+$ on cusp forms is given by

$$(f_0|\gamma)(z) := \frac{\det(\gamma)^{k/2}}{(cz+d)^k} f_0(\gamma z).$$

Let $\mathcal{P}_{k-2}(\mathbb{Q})$ be the space of polynomials with degree $\leq k-2$ and coefficients in $\mathbb{Q}$. This has an action of $\mathrm{GL}_2(\mathbb{Q})$ defined by

$$(P|\gamma)(z) := \frac{(cz+d)^{k-2}}{\det(\gamma)^{(k-2)/2}} P(\gamma z).$$

Then we get an action of $\mathrm{GL}_2(\mathbb{Q})$ on the space of symbols $\operatorname{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})$ by

$$(\phi|\gamma)\{x \to y\}(P) = \phi\{\gamma x \to \gamma y\}(P|\gamma^{-1}).$$

These satisfy the property that scalar multiples of the identity act trivially. Hence on $\mathcal{P}_{k-2}(\mathbb{Q})$ and even $\mathcal{P}_{k-2} := \mathcal{P}_{k-2}(\mathbb{Z})$ we have an action of $\mathrm{PGL}_2(\mathbb{Q})$.

(These differ slightly from the definitions used in [GS], where the scalar matrices no longer act trivially.)

The action on $S_k(\Gamma_0(N))$ was extended to $S_k(\mathcal{T}, \Gamma)$ by

$$(f|\gamma)(e, z) := \frac{\det(\gamma)^{k/2}}{(cz+d)^k} f(\gamma e, \gamma z).$$

We get an action on the space $\mathrm{C^{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})\big)$ similarly by

$$(\kappa|\gamma)\{x \to y\}(e, P) = \kappa\{\gamma x \to \gamma y\}(\gamma e, P|\gamma^{-1})$$

2.3 Action of Γ, W_p, $\tilde{\Gamma}$

Let $f \in S_k(\mathcal{T}, \Gamma)$, then

$$f(\gamma e, \gamma z) = (cz+d)^k f(e, z) \quad \forall \gamma \in \Gamma$$

Thus for $\gamma \in \Gamma$, and $P \in \mathcal{P}_{k-2}(\mathbb{Q})$,

$$f(\gamma e, \gamma z) P(\gamma z)\, d(\gamma z) = f(e, z)(P|\gamma)(z)\, dz \tag{3}$$

Lemma 2.4 **(Properties of κ_f)** *κ_f is harmonic in e, linear in P, additive in $\{x \to y\}$ and Γ-invariant in the sense that*

$$\kappa_f\{\gamma x \to \gamma y\}(\gamma e, P|\gamma^{-1}) = \kappa_f\{x \to y\}(e, P).$$

Proof The harmonicity follows immediately from the corresponding property of f. Linearity in P and additivity in $\{x \to y\}$ follow immediately from the definition.

Γ-invariance follows from the definition of κ_f and the Γ-invariance of f. ∎

This lemma shows that $\kappa_f \in \mathrm{C^{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})\big)^{\Gamma}$. So we have a commutative diagram:

$$\begin{array}{ccc} S_k(\mathcal{T}, \Gamma) & \xrightarrow{\kappa.} & \mathrm{C^{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})\big)^{\Gamma} \\ {\scriptstyle \rho_T}\big\downarrow \cong & & \big\downarrow {\scriptstyle \rho_M} \\ S_k(\Gamma_0(N))^{p-\mathrm{new}} & \xrightarrow{\phi.} & \mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})^{\Gamma_0(N)} \end{array} \tag{4}$$

where ρ_M also denotes restriction to the edge e_*, which is injective. (The image of ρ_M can be described as a 'p-new' part exactly as in Lemma 2.3.)

It is possible to define various operators on the space $S_k(\mathcal{T}, \Gamma)$ by their action on the corresponding cusp form for $\Gamma_0(N)$. For example the standard Atkin-Lehner involution at p is defined by:

The operator W_p is defined on $S_k(\Gamma_0(N))$ by $W_p f_0 = f_0|\gamma$ for any $\gamma \in \tilde{\Gamma} \setminus \Gamma$ normalizing $\Gamma_0(N)$. It is defined similarly on $\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})^{\Gamma_0(N)}$ by $W_p\phi = \phi|\gamma$.

In particular, in these definitions we can use the previously fixed α in the normalizer of $\Gamma_0(N)$ for the element γ.

Definition 2.2 Define an operator W_p on $S_k(\mathcal{T},\Gamma)$ by $W_p f = -f|\beta$ for any $\beta \in \tilde{\Gamma} \setminus \Gamma$.

On $C^{\mathrm{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2},\mathbb{C})\big)^{\Gamma}$, define W_p similarly by $W_p\kappa = -\kappa|\beta$.

Lemma 2.5 *These operators W_p are well defined, and W_p commutes with all the maps in (4).*

In the case where $f \in S_k(\mathcal{T},\Gamma)$ is associated to a newform $f_0 = f_{e_*}$ it is an eigenform for W_p with eigenvalue $-w$, say. Then the action of $\gamma \in \tilde{\Gamma}$ can be described by

$$f|\gamma = w^{|\gamma|} f \quad \text{and} \quad \kappa_f|\gamma = w^{|\gamma|}\kappa_f$$

where $|\gamma| := \mathrm{ord}_p \det\gamma$.

We have an explicit inverse to ρ_M on the part of its image where $W_p = -w$:

Lemma 2.6 *Suppose $\kappa \in C^{\mathrm{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2},\mathbb{C})\big)^{\Gamma}$ satisfies $W_p\kappa = -w\kappa$. If $\gamma \in \tilde{\Gamma}$ is such that $\gamma e = e_*$ then*

$$\kappa\{x \to y\}(e,P) = w^{|\gamma|}\rho_M(\kappa)\{\gamma x \to \gamma y\}(P|\gamma^{-1}).$$

2.4 Action of Hecke Operators

Let l be a prime not dividing N. We can pick δ_j such that

$$\Gamma_0(N)\begin{pmatrix}1 & 0\\ 0 & l\end{pmatrix}\Gamma_0(N) = \coprod_{j=0}^{l}\Gamma_0(N)\delta_j \quad \text{and} \quad \Gamma\begin{pmatrix}1 & 0\\ 0 & l\end{pmatrix}\Gamma = \coprod \Gamma\delta_j.$$

The action of the Hecke operator $T(l)$ on $S_k(\Gamma_0(N))$ is defined by

$$f_0|T(l) = l^{\frac{k-2}{2}}\sum_j f_0|\delta_j.$$

It is defined on $\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2},\mathbb{C})^{\Gamma_0(N)}$ by

$$\phi|T(l) = l^{\frac{k-2}{2}}\sum_j \phi|\delta_j.$$

These are the usual Hecke operators.

Definition 2.3 The action of $T(l)$ on $S_k(\mathcal{T},\Gamma)$ is defined by

$$f|T(l) := l^{\frac{k-2}{2}}\sum_j f|\delta_j,$$

and similarly on $C^{\mathrm{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2},\mathbb{C})\big)^{\Gamma}$,

$$\kappa|T(l) := l^{\frac{k-2}{2}}\sum_j \kappa|\delta_j.$$

Lemma 2.7 *The Hecke operators in Definition 2.3 are well defined. The operations $T(l)$ for l prime to N commute with all the maps in (4).*

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2.5 Action of W_∞

The "Atkin-Lehner involution at ∞", W_∞, will correspond to the action of $\alpha_\infty = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. This doesn't act on the cusp forms (as it has negative determinant), but does act on the modular symbols.

Definition 2.4 W_∞ is defined on $\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})^\Gamma$ by $W_\infty\phi := \phi|\alpha_\infty$.
It is defined on $C^{\mathrm{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})\big)^\Gamma$ by $W_\infty f := f|\alpha_\infty$.

Lemma 2.8 *W_∞ is well defined here and commutes with ρ_M.*

Now let f_0 be a newform, so in particular it is an eigenform for W_p with eigenvalue $-w$, say, and for each $T(l)$ with eigenvalue a_l. Then the associated $f \in S_k(\mathcal{T}, \Gamma)$ is also an eigenform, as are ϕ_f and κ_f.

Let $\phi_f^\pm$ be elements of the ±1-eigenspace for W_∞ in $\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})^{\Gamma_0(N)}$ satisfying

$$\phi_f = \Omega_f^+\phi_f^+ + \Omega_f^-\phi_f^-.$$

From Shimura's theorem (described in [GS]), we can choose the $\phi_f^\pm$ and complex periods $\Omega_f^\pm$ in such a way that $\phi_f^\pm$ applied to polynomials with integer coefficients take values in $\mathcal{O}_f$ (the ring of integers of the finite extension of $\mathbb{Q}$ generated by the Hecke eigenvalues of f). Write also $\phi_f^{w_\infty}$ for $\phi_f^\pm$ where $w_\infty = \pm1$.

These $\phi_f^{w_\infty}$ are in the same Hecke eigenspace as ϕ_f, in particular are still Hecke eigenfunctions for $T(l)$, l prime to N, and W_p, with the same eigenvalues as f_0.

Note that in the case $\frac{k-2}{2}$ odd, this $\phi_f^\pm$ corresponds to [GS]'s $\Phi_f^\mp$.

We have

$$\Omega_f^{w_\infty}\phi_f^{w_\infty} = \frac{1}{2}(\phi_f + w_\infty W_\infty \phi_f),$$

so $\phi_f^{w_\infty}$ is still contained in the image of ρ_M.

Define $\kappa_f^{w_\infty}$ to be the element of $C^{\mathrm{har}}\big(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C})\big)^\Gamma$ mapped to $\phi_f^{w_\infty}$ by ρ_M. By Lemma 2.6, if $\gamma e = e_*$, then we have

$$\kappa_f^{w_\infty}\{x \to y\}(e, P) = w^{|\gamma|}\phi_f^{w_\infty}\{\gamma x \to \gamma y\}(P|\gamma^{-1}). \tag{5}$$

Lemma 2.9 **(Properties of $\kappa_f^{w_\infty}$)**

1. *$\kappa_f^{w_\infty}\{x \to y\}(e, P)$ is harmonic in e, additive in x, y, and linear in P. For $\gamma \in \tilde{\Gamma}$,*

$$\kappa_f^{w_\infty}|\gamma = w^{|\gamma|}\kappa_f^{w_\infty}.$$

2. *$\kappa_f^{w_\infty}$ is a Hecke eigenfunction with the same eigenvalues as κ_f,* i.e.,

$$a_l\kappa_f^{w_\infty}\{x \to y\}(e, P) = l^{\frac{k-2}{2}} \sum_{j=0}^{l} \kappa_f^{w_\infty}\{\delta_j x \to \delta_j y\}(\delta_j e, P|\delta_j^{-1}).$$

3. *There is the additional relation* $W_\infty \kappa_f^{w_\infty} = w_\infty \kappa_f^{w_\infty}$, i.e.,

$$\kappa_f^{w_\infty}\{\alpha_\infty x \to \alpha_\infty y\}(\alpha_\infty e, P|\alpha_\infty) = w_\infty \kappa_f^{w_\infty}\{x \to y\}(e, P),$$

so most generally, for any $\gamma \in R^*$,

$$\kappa_f^{w_\infty}\{x \to y\}(e, P) = w^{|\gamma|} w_\infty^{\mathrm{sign}(\gamma)} \kappa_f^{w_\infty}\{\gamma x \to \gamma y\}(\gamma e, P|\gamma^{-1}),$$

where $\mathrm{sign}(\gamma) = 0, 1$ *if* $\det \gamma$ *is positive or negative respectively.*

Proof These follow from the properties of $\phi_f^{w_\infty}$.

1. These are all from the fact that $\kappa_f^{w_\infty}$ is an element of $\mathrm{C^{har}}(\mathrm{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C}))^\Gamma$ and that $W_p \kappa_f^{w_\infty} = -w\kappa_f^{w_\infty}$ (because W_p commutes with ρ_M).
2. This is immediate as $T(l)$ commutes with ρ_M and $\phi_f^{w_\infty}$ is a Hecke eigenfunction.
3. For $\gamma = \alpha_\infty$ this follows from $W_\infty \phi_f^{w_\infty} = w_\infty \phi_f^{w_\infty}$. The general result follows because $\tilde{\Gamma}$ is of index 2 in R^*. ■

In terms of modular symbols,

$$\lambda_f^{w_\infty}(P(z); -a, m) = (-1)^{\frac{k-2}{2}} w_\infty \lambda_f^{w_\infty}(P(-z); a, m),$$

and in particular for $P(z) = z^{\frac{k-2}{2}}$ we get

$$\lambda_f^{w_\infty}(z^{\frac{k-2}{2}}; -a, m) = w_\infty \lambda_f^{w_\infty}(z^{\frac{k-2}{2}}, a, m). \tag{6}$$

Note that for χ a primitive Dirichlet character of conductor c,

$$\begin{aligned}\Lambda^{w_\infty}(f, \chi, j) &:= \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \chi(a) \lambda_f^{w_\infty}(f, z^j; a, c) \\ &= \begin{cases} 0 & \text{if } \chi(-1) = -w_\infty \\ K(\chi, j) L(f_{\bar{\chi}}, j+1)/\Omega_f^{w_\infty} & \text{if } \chi(-1) = w_\infty \end{cases}\end{aligned} \tag{7}$$

where

$$K(\chi, j) = \frac{c^{j+1} j!}{(-2\pi i)^j \tau(\bar{\chi})} \in \mathbb{C}.$$

The "algebraic part" of the complex L-function is defined to be $\Lambda = \Lambda^+ + \Lambda^-$.

3 Defining a Distribution

3.1 The Mazur-Tate-Teitelbaum Distribution

In [MTT] a construction of Vishik and Amice-Velu is used to define a distribution on $\mathbb{Z}_{p,c}^* := \varprojlim(\mathbb{Z}/cp^n\mathbb{Z})^*$ with values in $\mathbb{C}_p$, on the set of locally analytic functions $\mathbb{Z}_{p,c}^* \to \mathbb{C}_p$. The result used is as follows.

Suppose we are given a distribution μ on polynomials of degree $\leq h$ satisfying the property

$$\int_{D(a,\nu)} (x-a)_p^n \, d\mu(x) \in p^{\nu(n-r)}\Omega \quad 0 \leq n \leq h \tag{8}$$

for some fixed r, $0 \leq r \leq h$, and a fixed $\mathcal{O}_p$-lattice Ω in a $\mathbb{C}_p$-vector space V, where $D(a,\nu) := a + cp^\nu\mathbb{Z}_{p,c} \subset \mathbb{Z}_{p,c}^*$.

Then μ can be extended uniquely to the space of locally analytic functions in such a way that (8) is satisfied for all $n \geq 0$, and if a function F has a convergent power series expansion on $D(a,\nu)$, say $F(x) = \sum_{n\geq 0} c_n(x-a)_p^n$, then

$$\int_{D(a,\nu)} F(x)\, d\mu(x) = \sum c_n \int_{D(a,\nu)} (x-a)_p^n \, d\mu(x).$$

To evaluate such an integral, we use truncations (*cf.* the proof of the existence of the distribution given in [MTT]). If on an open set U, the locally analytic function F has a convergent power series of the form $F(z) = \sum_0^\infty c_n(z-a)_p^n$ with $a \in U$ then write

$$\mathrm{Trunc}_{U,a}^N(F) = \sum_0^N c_n(z-a)^n.$$

By the uniqueness property, once r is fixed, the distributions defined by this construction for $h \geq r$ are the same, so we may as well assume $h = r$.

Then by definition

$$\int_U F(x)\, dx := \lim_{\nu\to\infty} \sum_{U=\cup D(a_i,\nu)} \int (\mathrm{Trunc}_{D(a_i,\nu),a_i}^h F)(x)\, d\mu(x).$$

In [MTT] this is applied to $\mu = \mu_{f,\mathrm{MTT}}$, constructed from modular symbols. Then $r = \frac{k-2}{2}$ and $\frac{k-2}{2} \leq h \leq k-2$. $V = V_f := \mathbb{C}_p \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}} L_f$ and Ω is a multiple of the $\mathcal{O}$ lattice generated by L_f, where L_f is the lattice in $\mathbb{C}$ generated by the values of ϕ_f on $\mathbb{D}_0 \otimes \mathcal{P}_{k-2}(\mathbb{Z})$.

In the case where f is a newform for $\Gamma_0(N)$ we have, for $0 \leq n \leq k-2$,

$$\int_{D(a,\nu)} P(x_p)\mu(x) = (wp^{\frac{k-2}{2}})^{-\nu}\lambda(f,P;a,p^\nu c).$$

Then the p-adic L-function is defined by:

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For a p-adic character $\psi\colon \mathbb{Z}_{p,c}^* \to \mathbb{C}_p$ and character $\langle\cdot\rangle$ defined by $\langle x\rangle = x_p\omega^{-1}(x)$ (the projection to $1+\mathbb{Z}_p$),

$$L_p(f,\psi,s) := \int_{\mathbb{Z}_{p,c}^*} \chi(x)\langle x\rangle^s \, d\mu_{f,\mathrm{MTT}}(x).$$

This defines a locally analytic function of $s \in \mathbb{Z}_p$.

By making a substitution of $\phi_f^{\pm}$ for ϕ_f in the definition of the modular symbol, we can define similarly $\mathbb{C}_p$-valued distributions $\mu_{f,\mathrm{MTT}}^{\pm}$, and p-adic L-functions $L_p^{\pm}$, now with values in $\bar{\mathbb{Q}} \otimes \mathbb{C}_p = \mathbb{C}_p$. Define the algebraic part of the p-adic L-function to be $\mathbf{L}_p = L_p^+ + L_p^-$ (*cf.* [Kit]).

In fact if $\chi(-1) = w_\infty(-1)^{\frac{k-2}{2}}$ then $L_p^{-w_\infty}(f,\chi,s) = 0$ and $\mathbf{L}_p(f,\chi,s) = L_p^{w_\infty}(f,\chi,s) = L_p(f,\chi,s)/\Omega_f^{w_\infty}$.

3.2 Generalization to $\mathbb{P}^1(\mathbb{Q}_p)$

As described in [Tei], this construction can be generalized to distributions on $\mathbb{P}^1(\mathbb{Q}_p)$.

This time, suppose we are given a distribution on the polynomials of degree $\le h$ on $\mathbb{P}^1(\mathbb{Q}_p)$, satisfying

$$(9)\quad \int_{U(e)} (z-a)^n \, d\mu(z) \in p^{\alpha(e)(n-r)}\Omega \quad \text{for } a \in U(e),\ \infty \notin U(e), \text{ and } 0 \le n \le h,$$

where $\alpha(e) = \inf_{u,v\in U(e)}\{v_p(u-v)\}$ for $\infty \notin U(e)$, and

$$(10)\quad \int_{U(e)} (z-a_0)^n d\mu(z) \in p^{\alpha(e)(r-n)}\Omega \quad \text{for } \infty \in U(e),\ a_0 \notin U(e), \text{ and } 0 \le n \le h,$$

with $\alpha(e) = -\inf_{u,v\notin U(e)}\{v_p(u-v)\}$ for $\infty \in U(e)$.

Again, Ω is some fixed lattice in a $\mathbb{C}_p$-vector space V, and r and h are fixed integers with $0 \le r \le h$. (*N.B.* if equation (10) holds for $a_0 = 0$ then it holds for any a_0)

Then the distribution can be extended uniquely to the space $\mathcal{C}_h$ of functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic on $\mathbb{Q}_p$ and may have a pole of order at most h at infinity. This is now subject to (9) and (10) holding for all $n \ge 0$ and all $n \le h$ respectively. The continuity condition becomes:

If $\infty \notin U(e)$, $a \in U(e)$, and F has a power series expansion

$$F(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

convergent on $U(e)$, then

$$\int_{U(e)} F(z)\, d\mu(z) = \sum_{n=0}^{\infty} c_n \int_{U(e)} (z-a)^n \, d\mu(z).$$

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If $\infty \in U(e)$, $a_0 \notin U(e)$, and F has a Laurent series expansion

$$F(z) = \sum_{n=-\infty}^{h} c_n (z - a_0)^n$$

convergent on $U(e)$, then

$$\int_{U(e)} F(z)\, d\mu(z) = \sum_{n=-\infty}^{h} c_n \int_{U(e)} (z - a_0)^n \, d\mu(z).$$

This implies that if $\infty \notin U(e)$ and we have a series of functions $F_n \in \mathcal{C}_h$ satisfying $F_n \to F$ pointwise in $U(e)$ that

$$\int_{U(e)} F_n(z)\, d\mu(z) \to \int_{U(e)} F(z)\, d\mu(z).$$

Now the definition of the distribution is

$$\int_U F(z)\, d\mu(z) := \lim_{\alpha(U_i)\to\infty} \sum_{U=\cup U_i} \int (\mathrm{Trunc}^h_{U_i,a_i} F)(z)\, d\mu(z).$$

For $\infty \in U$, if F has a Laurent series expansion

$$F(z) = \sum_{n=-\infty}^{h} c_n (z - a_0)^n$$

where $a_0 \notin U$, then the truncation is defined as

$$\mathrm{Trunc}^h_{U,a_0}(F) = \sum_{0}^{h} c_n (z - a_0)^n.$$

For $\infty \notin U$ the truncations are defined as in the previous section.

Now the uniqueness implies that once r is fixed, the distribution defined by $h = r$ is the restriction to $\mathcal{C}_r$ of the distribution defined by $h = h_0 > r$.

3.3 Application

Let $f \in S_k(\mathcal{T}, \Gamma)$ correspond to a new form f_0 for $\Gamma_0(N)$ as before. Because of the harmonicity of $\kappa_f^{w_\infty}\{x \to y\}(\ ,\ )$ we can use it to define a distribution $\mu_f^{w_\infty}\{x \to y\}$ on the space of functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally polynomial of degree $\leq k-2$ via:

$$\int_{U(e)} P(z)\, d\mu_f^{w_\infty}\{x \to y\}(z) := \kappa_f^{w_\infty}\{x \to y\}(e, P).$$

This can be used to define a distribution on $\mathcal{C}_{k-2}$ by applying the previous section with $r = \frac{k-2}{2}$, and $h = k - 2$.

Proposition 3.1 *The properties* (9) *and* (10) *are satisfied by any* $\mu_f^{w_\infty}\{x \to y\}$ *thus they can be extended to distributions on* $\mathcal{C}_{k-2}$.

Proof We use equation (5) and particular simple representatives for the edges of $\mathcal{T}$. To be precise, let

$$\gamma = \gamma_{r,s,a} = \begin{pmatrix} p^r & -a \\ 0 & p^s \end{pmatrix}$$
$$\gamma^{-1} = \begin{pmatrix} p^{-r} & ap^{-(r+s)} \\ 0 & p^{-s} \end{pmatrix}$$

where $0 \le a \le p^s - 1$, and a is prime to p^r.

Then $\gamma^{-1}e_*$ represents the edge e corresponding to the open set

$$U(e) = \{x \in \mathbb{Q}_p : p^r x \equiv a \bmod p^s\mathbb{Z}_p\}.$$

So $\gamma e = e_*$ and $\alpha(e) = s - r$.

It is easy to see that for $r + s$ fixed this gives all the edges at distance $r + s$ from v_*, with all except the one with $s = 0$ oriented away from v_*.

Now from equation (5) we have

$$\int_{U(e)} (z-b)^n \mu_f^{w_\infty}\{x \to y\}(z) = w^{|\gamma|}\phi_f^{w_\infty}\{\gamma x \to \gamma y\}(P|\gamma^{-1})$$

where $P(z) = (z - b)^n$ and $b \in U(e)$. Now,

$$(P|\gamma^{-1})(z) = (p^{r-s})^{\frac{k-2}{2}-n}\Big(z + \frac{a - p^r b}{p^s}\Big)^n$$

and $b \in U(e)$ implies that $\frac{a-p^r b}{p^s}$ is integral.

We know that for a polynomial with integral coefficients, $\phi_f^{w_\infty}$ takes values in $\mathcal{O}_f$, the integers of the field K_f generated over $\mathbb{Q}$ by the Hecke eigenvalues of f_0. It follows that for $\Omega := \mathcal{O}_p$,

$$\int_{U(e)} (z-b)^n \, d\mu_f^{w_\infty}\{x \to y\}(z) \in p^{\alpha(e)(-\frac{k-2}{2}+n)}\Omega.$$

The property for $\infty \in U(e)$ follows immediately from the case $s = 0$, as in [Tei], by harmonicity. ■

Remark Equation (9) now says precisely that when restricted to $\mathbb{Z}_p$ or $\mathbb{Q}_p$ the distribution is tempered of order $\frac{k-2}{2}$.

There is an operation of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathcal{C}_{k-2}$ given by

$$(f|\gamma)(z) = \frac{(cz+d)^{k-2}}{(\det\gamma)^{\frac{k-2}{2}}} f\Big(\frac{az+b}{cz+d}\Big)$$

which specializes to the normal action on $\mathcal{P}_{k-2}$.

We need to know how this action is related to the truncation operations. This is given by the following lemma:

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Lemma 3.1 *If F is a locally analytic function on U and $a_0 \in U$, $\infty \notin U$ and $\infty \notin \gamma U$ then*

$$\big(\mathrm{Trunc}^{k-2}_{U,a_0}(F)\big) \,\big|\, \gamma^{-1} = \mathrm{Trunc}^{k-2}_{\gamma U,\gamma a_0}(F|\gamma^{-1}).$$

The same relation holds if $a_0 \notin U$, $\infty \in U$ and $\gamma \cdot \infty = \infty$.

If $\infty \in U$, $\gamma \cdot \infty \notin U$ and $a_0 \notin U$ then

$$\big(\mathrm{Trunc}^{k-2}_{U,a_0}(F)\big) \,\big|\, \gamma^{-1} = \mathrm{Trunc}^{k-2}_{\gamma U,\gamma\cdot\infty}(F|\gamma^{-1}).$$

Note that this is sufficient for evaluating integrals, as if $\infty \in U$ and $\gamma \cdot \infty \neq \infty$ we can take a smaller U with $\infty \notin \gamma U$. The third relation also gives us the case $\infty \notin U$, $\gamma\infty \in U$ of course.

***Lemma 3.2* (Properties of the Distributions)** *The distributions defined as above have the following properties.*

1. *If $P \in \mathcal{P}_{k-2}$ then*

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P\mu_f^{w_\infty}\{x \to y\} = 0.$$

2. *If $\gamma \in \tilde{\Gamma}$ and $F \in \mathcal{C}_{k-2}$ then*

$$\int_{U(\gamma e)} (F|\gamma^{-1})\mu_f^{w_\infty}\{\gamma x \to \gamma y\} = w^{|\gamma|}\int_{U(e)} F\mu_f^{w_\infty}\{x \to y\}.$$

3. *If $F \in \mathcal{C}_{k-2}$ then*

$$a_l \int_{U(e)} F(z)\, d\mu_f^{w_\infty}\{x \to y\}(z) = l^{\frac{k-2}{2}} \sum_{j=0}^{l} \int_{U(\delta_j e)} (F|\delta_j^{-1})(z)\, d\mu_f^{w_\infty}\{\delta_j x \to \delta_j y\}(z).$$

4. *More generally than 2, for $\gamma \in R^*$ and $F \in \mathcal{C}_{k-2}$,*

$$\int_{U(\gamma e)} (F|\gamma^{-1})\mu_f^{w_\infty}\{\gamma x \to \gamma y\} = w_\infty^{\mathrm{sign}(\gamma)} w^{|\gamma|}\int_{U(e)} F\mu_f^{w_\infty}\{x \to y\}.$$

Proof The first property holds by the harmonicity of κ. The others hold for polynomials by the equivalent properties of $\kappa_f^{w_\infty}$ given in Lemma 2.9. They extend to locally analytic functions by Lemma 3.1. ■

4 Double Integrals

Now as in [Dar] we can define a double integral (though not a multiplicative integral in this case):

Definition 4.1

$$\int_{z_1}^{z_2}\int_x^y (P)\omega := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p\Big(\frac{t-z_2}{t-z_1}\Big) P(t)\, d\mu_f^{w_\infty}\{x \to y\}(t)$$

where $z_i \in \mathcal{H}_p$, $P \in \mathcal{P}_{k-2}(\mathbb{C}_p)$ and $x, y \in \mathbb{P}^1(\mathbb{Q})$.

***Lemma 4.1* (Properties of Double Integral)** *The double integral satisfies the following properties:*

1. *The double integral is additive in x, y and in z_1, z_2, and linear in P.*
2. *For $\gamma \in \tilde{\Gamma}$*
$$\int_{\gamma z_1}^{\gamma z_2} \int_{\gamma x}^{\gamma y} (F|\gamma^{-1})\omega = w^{|\gamma|} \int_{z_1}^{z_2} \int_{x}^{y} (F)\omega.$$
3. *For all $F \in \mathcal{C}_{k-2}$,*
$$a_l \int_{z_1}^{z_2} \int_{x}^{y} (F)\omega = l^{\frac{k-2}{2}} \sum_{j=0}^{l} \int_{\delta_j z_1}^{\delta_j z_2} \int_{\delta_j x}^{\delta_j y} (F|\delta_j^{-1})\omega.$$
4. *More generally than 2, for $\gamma \in R^*$*
$$\int_{z_1}^{z_2} \int_{x}^{y} (F)\omega = w^{|\gamma|} w_\infty^{\operatorname{sign}(\gamma)} \int_{\gamma z_1}^{\gamma z_2} \int_{\gamma x}^{\gamma y} (F|\gamma^{-1})\omega.$$

Proof Additivity in x, y follows from the equivalent property of κ_f. Additivity in z_1, z_2 follows from properties of $\log_p$. The remaining properties follow from the equivalent properties of the distributions in Lemma 3.2. ■

5 Cohomology Groups

5.1 Definition of Cocycles

We can't define a single cohomology class to correspond to Darmon's c_f, but we can define classes lc_f and oc_f corresponding to $\log_p c_f$ and $\operatorname{ord}_p c_f$ respectively. Let $\mathcal{M}_{k-2} := \mathcal{M} \otimes \mathcal{P}_{k-2}^\vee = \operatorname{Hom}(\mathbb{D}_0 \otimes \mathcal{P}_{k-2}, \mathbb{C}_p)$ with operation of $\mathrm{PGL}_2(\mathbb{Q})$ defined by

$$(\gamma \cdot \phi)\{x \to y\}(P) = \phi\{\gamma^{-1}x \to \gamma^{-1}y\}(P|\gamma)$$

where $x, y \in \mathbb{P}^1(\mathbb{Q})$ and $P \in \mathcal{P}_{k-2}$.

The classes to be defined will be elements of $H^1(\Gamma, \mathcal{M}_{k-2})$.

Definition 5.1

$$\widetilde{\mathrm{oc}}_{f,v}(\gamma)\{x \to y\}(P) := \sum_{e \in v \to \gamma v} \kappa_f^{w_\infty}\{x \to y\}(e, P)$$

Lemma 5.1 *$\widetilde{\mathrm{oc}}_{f,v}$ is a cocycle in $Z^1(\Gamma, \mathcal{M}_{k-2})$ and its class $\mathrm{oc}_f \in H^1(\Gamma, \mathcal{M}_{k-2})$ is independent of v.*

Proof This is straightforward using the properties of the modular symbol in Lemma 2.9. ■

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Definition 5.2

$$\widetilde{\mathrm{lc}}_{f,\tau}(\gamma)\{x \to y\}(P) := \int_{\tau}^{\gamma\tau}\int_{x}^{y}(P)\omega.$$

Lemma 5.2 *$\widetilde{\mathrm{lc}}_{f,\tau}$ is a cocycle in $Z^1(\Gamma, \mathcal{M}_{k-2})$ and its class $\mathrm{lc}_f \in H^1(\Gamma, \mathcal{M}_{k-2})$ is independent of τ.*

Proof This follows from the properties of the double integral in Lemma 4.1. ■

5.2 Action of Hecke Operators and W_∞

The action of the Hecke operators $T(l)$, for $l \nmid N$, on $H^1(\Gamma, \mathcal{M}_{k-2})$ is defined as in [Hi, Section 6.3]: Given a cohomology class c, pick a cocycle $\tilde{c}$ representing it. For $\gamma \in \Gamma$ we can choose $\gamma_j \in \Gamma$ such that $\delta_j\gamma = \gamma_j\delta_{i(j)}$, where $i(j)$ defines a permutation of the j.

Define a new cocycle by

$$\widetilde{T(l)(c)}(\gamma) = l^{\frac{k-2}{2}}\sum_{j=0}^{l}\delta_j^{-1}\cdot\tilde{c}(\gamma_j).$$

The operator W_∞ acts on $\mathcal{M}_{k-2}$ as the matrix α_∞, and on $H^1(\Gamma, \mathcal{M}_{k-2})$ via

$$\widetilde{W_\infty(c)}(\gamma) = \alpha_\infty \cdot \tilde{c}(\alpha_\infty\gamma\alpha_\infty).$$

Lemma 5.3 *We have*

$$T(l)\,\mathrm{oc}_f = a_l\cdot\mathrm{oc}_f \quad \textit{and} \quad T(l)\,\mathrm{lc}_f = a_l\cdot\mathrm{lc}_f,$$

and

$$W_\infty\,\mathrm{oc}_f = w_\infty\cdot\mathrm{oc}_f \quad \textit{and} \quad W_\infty\,\mathrm{lc}_f = w_\infty\cdot\mathrm{lc}_f\,.$$

Proof These also follow from the properties of the double integral in Lemma 4.1 and of the modular symbol in Lemma 2.9. ■

6 Embeddings and Special Values of Cocycles

6.1 Definitions

Now Ψ will be an embedding $\mathbb{Q}\times\mathbb{Q} = K \hookrightarrow M_2(\mathbb{Q})$ as in [Dar], γ_Ψ a generator of $\bar{\Psi}(K^*)\cap\Gamma$, $x_\Psi, y_\Psi \in \mathbb{P}^1(\mathbb{Q})$ the fixed points of $\bar{\Psi}(K^*)$, chosen such that for $t \in \mathcal{H}_p$

$$\gamma_\Psi^n(t) \to y_\Psi \text{ as } n \to \infty \quad \gamma_\Psi^n(t) \to x_\Psi \text{ as } n \to -\infty.$$

We will also need a polynomial fixed by γ_Ψ,

$$P_\Psi(z) = \left(\mathrm{Tr}\left(\gamma_\Psi\begin{pmatrix} z & -z^2 \\ 1 & -z\end{pmatrix}\right)\right)^{\frac{k-2}{2}}.$$

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M_Ψ will be a Möbius transform with coefficients in $\mathbb{Q}_p$ taking $x_\Psi \mapsto \infty$, and $y_\Psi \mapsto 0$ (this is well defined up to multiplication by a scalar).

Viewing the points x_Ψ, y_Ψ as elements of $\mathbb{P}^1(\mathbb{Q}_p)$ they define an infinite path in $\mathcal{T}$. Let v be a vertex on this path $(x_\Psi \to y_\Psi)$; we will define an open subset of $\mathbb{P}^1(\mathbb{Q}_p)$ by $U(v) := \{$points corresponding to ends of $\mathcal{T}$ intersecting $(x_\Psi \to y_\Psi)$ precisely at $v\}$. It is possible to label the vertices in $(x_\Psi \to y_\Psi)$ such that

$$U(v_j) = \left\{ t \in \mathbb{P}^1(\mathbb{Q}_p) - \{x_\Psi, y_\Psi\} \text{ such that } \operatorname{ord}_p\big(M_\Psi(t)\big) = j \right\}.$$

Letting $e_j = v_{j-1} \to v_j$, then $U(v_j) = U(e_j) - U(e_{j+1})$.

A fundamental region for the action of γ_Ψ on $\mathbb{P}^1(\mathbb{Q}_p) - \{x_\Psi, y_\Psi\}$ is given by

$$\begin{aligned}\mathcal{F}_\Psi &= \{t \in \mathbb{P}^1(\mathbb{Q}_p) : 0 \le \operatorname{ord}_p M_\Psi(t) < s\} \\ &= U(v_0) \cup U(v_1) \cup \cdots \cup U(v_{s-1}).\end{aligned}$$

Again, we can't define a period I_Ψ without a multiplicative integral, but we can define objects that behave like $\log_p I_\Psi$ and $\operatorname{ord}_p I_\Psi$. These are respectively:

Definition 6.1

$$LI_\Psi := \operatorname{lc}_f(\gamma_\Psi)\{x_\Psi \to y_\Psi\}(P_\Psi) = \int_z^{\gamma_\Psi z} \int_{x_\Psi}^{y_\Psi} (P_\Psi)\omega$$

for any $z \in \mathcal{H}_p$.

Definition 6.2

$$W_\Psi := \operatorname{oc}_f(\gamma_\Psi)\{x_\Psi \to y_\Psi\}(P_\Psi) = \sum_{e \in v \to \gamma_\Psi v} \kappa_f^{w_\infty}\{x_\Psi \to y_\Psi\}(e, P_\Psi)$$

where v is any vertex of $\mathcal{T}$.

The definitions as special values of cohomology classes are valid because for any coboundary of the form $b(\gamma) = m - \gamma \cdot m$,

$$b(\gamma_\Psi)\{x_\Psi \to y_\Psi\}(P_\Psi) = m\{x_\Psi \to y_\Psi\}(P_\Psi) - m\{\gamma_\Psi^{-1} x_\Psi \to \gamma_\Psi^{-1} y_\Psi\}(P_\Psi|\gamma_\Psi) = 0$$

6.2 Evaluation of W_Ψ

Darmon uses specific embeddings of conductor c, given by Ψ_ν, for some ν prime to c, such that

$$\Psi_\nu(a,a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \Psi_\nu(c,0) = \begin{pmatrix} c & \nu \\ 0 & 0 \end{pmatrix}$$

These are oriented optimal embeddings. Moreover, the embedding Ψ_ν is uniquely determined by the class of ν in $(\mathbb{Z}/c\mathbb{Z})^*/\langle p^2 + c\mathbb{Z}\rangle$, and these represent all the Γ-conjugacy classes of oriented optimal embeddings.

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The integer s is taken to be

$$s = 2 \cdot \big(\text{order of } p^2 \text{ in } (\mathbb{Z}/c\mathbb{Z})^*\big)$$

and s' will be the order of p in $(\mathbb{Z}/c\mathbb{Z})^*$. Thus either s' is even and $s = s'$ or s' is odd and $s = 2s'$.

Now fix one such $\Psi = \Psi_\nu$. Then the following are known:

$$x_\Psi = \infty, \quad y_\Psi = -\nu/c,$$
$$\gamma_\Psi = \Psi_\nu(p^{s/2}, p^{-s/2})$$
$$= \begin{pmatrix} p^{s/2} & (p^{s/2} - p^{-s/2})\nu/c, \\ 0 & p^{-s/2} \end{pmatrix}$$
$$P_\Psi(z) = \big((p^{s/2} - p^{-s/2})/c\big)^{\frac{k-2}{2}} (cz+\nu)^{\frac{k-2}{2}}.$$

For simplicity we can remove the factor $((p^{s/2} - p^{-s/2})/c)^{\frac{k-2}{2}}$ which is independent of ν, so

$$P_\Psi(z) = (cz+\nu)^{\frac{k-2}{2}},$$

and we can choose

$$M_\Psi(t) = t + \nu/c.$$

Proposition 6.1

$$W_{\Psi_\nu} = \beta \sum_{a \in J_\nu} w^{j(a)} \lambda^{w_\infty}(f, z^{\frac{k-2}{2}}; a, c),$$

where J_ν is the set $J_\nu = \{b \in (\mathbb{Z}/cZ)^* : \exists j = j(b)$ such that $b/\nu \equiv p^j \bmod c\}$, and

$$\beta = \begin{cases} 1 & \text{if } s = s' \text{(i.e. } s' \text{ is even)} \\ 2 & \text{if } s = 2s' \text{(i.e. } s' \text{ is odd), and } w = +1 \\ 0 & \text{if } s = 2s' \text{and } w = -1. \end{cases}$$

Proof Then the edge e_j is given by $\gamma^{-1}e_*$ where $\gamma = \left(\begin{smallmatrix} 1 & -\nu' \\ 0 & p^j \end{smallmatrix}\right)$, where ν' is an integer with $\nu' \equiv -\nu/c \bmod p^s$. We have $\gamma x_\Psi = \infty$ and $\gamma y_\Psi = \frac{(-\nu - c\nu')/p^j}{c}$

$$W_\Psi = \sum_{i=0}^{s-1} \kappa_f^{w_\infty}\{x_\Psi \to y_\Psi\}(e_i, P_\Psi).$$

We can evaluate each term of this sum:

$$\kappa_f^{w_\infty}\{x_\Psi \to y_\Psi\}(e_j, P_\Psi) = w^{|\gamma|}\big(\phi_f^{w_\infty}(P_\Psi|\gamma^{-1}, \gamma y_\Psi) - \phi_f^{w_\infty}(P_\Psi|\gamma^{-1}, \gamma x_\Psi)\big)$$
$$= w^j \phi_f^{w_\infty}\left(P_\Psi|\gamma^{-1}, \frac{(-\nu - c\nu')/p^j}{c}\right).$$

A calculation gives

$$P_\Psi|\gamma^{-1}(z) = \left(cz + (c\nu' + \nu)/p^j\right)^{\frac{k-2}{2}}.$$

Hence

$$\kappa_f^{w_\infty}\{x_\Psi \to y_\Psi\}(e_j, P_\Psi) = w^j \lambda^{w_\infty}\left(f, z^{\frac{k-2}{2}}; (c\nu' + \nu)/p^j, c\right).$$

The ν/p^j run over the set J_ν: once if $s = s'$ and twice if $s = 2s'$. In the latter case the w^j's have the opposite sign in the second occurrence if and only if $w = -1$. So we have

$$W_{\Psi_\nu} = \beta \sum_{a \in J_\nu} w^{j(a)} \lambda^{w_\infty}(f, z^{\frac{k-2}{2}}; a, c). \qquad ■$$

Assume we are not in the case $\beta = 0$.

Corollary 6.1 *If χ is a primitive Dirichlet character of conductor c, and with $\chi(p) = w$ then*

$$\sum_{\nu \in (\mathbb{Z}/cZ)^*} \chi(\nu) W_{\Psi_\nu} = s \sum_{a \in (\mathbb{Z}/cZ)^*} \chi(a) \lambda^{w_\infty}(f, z^{\frac{k-2}{2}}; a, c)$$
$$= \begin{cases} s\Lambda(f, \chi, \frac{k-2}{2}) & \text{if } \chi(-1) = w_\infty \\ 0 & \text{if } \chi(-1) = -w_\infty. \end{cases}$$

Proof As $\beta \neq 0$ we have $\beta = s/s'$. The sets J_ν for $\nu \in (\mathbb{Z}/c\mathbb{Z})^*$ cover $(\mathbb{Z}/c\mathbb{Z})^*$, with each element repeated s' times. The result follows from equation (6). ■

Note that if we had defined W_Ψ using κ_f instead of $\kappa_f^{w_\infty}$ then this sum would have evaluated to a multiple of the complex L-function, rather than its algebraic part.

6.3 Comparison of Distributions

The Mazur-Tate-Teitelbaum distribution and the Darmon distributions are both defined in terms of modular symbols, so should be related in some way.

Again we fix $\Psi = \Psi_\nu$. On the Darmon side we will integrate over

$$\mathcal{F}_\Psi = \bigcup_{j=0}^{s-1} U(v_j) = \bigcup_{j=0}^{s-1} \bigcup_{a \in (\mathbb{Z}/p^n\mathbb{Z})^*} U_{j,a},$$

where

$$U_{j,a} = \{t \in U(v_j) : p^{-j}(t + \nu/c) \equiv a \bmod p^n\}.$$

This subdivision will correspond to the division of

$$J_{\infty,\nu} := (\pi_0^\infty)^{-1} J_\nu \cap \mathbb{Z}_{p,c}^*$$
$$= \{b \in \mathbb{Z}_{p,c}^* : b/\nu \equiv p^j \bmod c \text{ for some } j = j(b)\}$$

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as

$$J_{\infty,\nu} = \bigcup_{j=0}^{s-1} J_{\infty,\nu,j} = \bigcup_{j=0}^{s-1} \bigcup_{a\in(\mathbb{Z}/p^n\mathbb{Z})^*} D(A_{a,j}, n)$$

where $A_{a,j} = (\nu + c\nu')/p^j + ac$ and

$$J_{\infty,\nu,j} := \{b \in \mathbb{Z}_{p,c}^* : b/\nu \equiv p^j \bmod c\}.$$

Then $U_{j,a} = \gamma^{-1}U(e_*)$ with

$$\gamma = \begin{pmatrix} 1 & -\nu' - p^j a \\ 0 & p^{n+j} \end{pmatrix}, \quad \gamma^{-1} = p^{-(n+j)} \begin{pmatrix} p^{n+j} & \nu' + p^j a \\ 0 & 1 \end{pmatrix},$$

and $\nu' \in \mathbb{Z}/p^{n+s}\mathbb{Z}$ defined by $\nu' \equiv -\nu/c \bmod p^{n+s}$.

Lemma 6.1 *If F is a locally analytic function on $\mathbb{Z}_p^*$, then*

$$w^j p^{j(\frac{k-2}{2})} \int_{U(\nu_j)} F\Big(\frac{cz+\nu}{p^j}\Big)\, d\mu_f^{w_\infty}\{x_\Psi \to y_\Psi\}(z) = \int_{J_{\infty,\nu,j}} F(x_p)\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(x),$$

and so

$$\int_{\mathcal{F}_\Psi} p^{j(z)(\frac{k-2}{2})} F\left(\frac{cz+\nu}{p^{j(z)}}\right) d\mu_f^{w_\infty}\{x_\Psi \to y_\Psi\}(z) = \beta \int_{J_{\infty,\nu}} w^{j(x)} F(x_p)\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(t).$$

Note that $j(t)$ and $j(x)$ are locally constant functions on $J_{\infty,\nu}$ and $\mathcal{F}_\Psi$.

Proof For the first:

Let P be a polynomial of degree $\le k-2$. Then calculations with modular symbols using (5) give

$$\int_{D(A,n)} P(t)\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(t) = w^n p^{-n(\frac{k-2}{2})} \phi_f^{w_\infty}\big(P(p^n cz + A), -A/p^n c\big)$$
$$= w^j p^{j(\frac{k-2}{2})} \int_{U_{j,a}} P\Big(\frac{cz+\nu}{p^j}\Big)\, d\mu_f^{w_\infty}\{x_\Psi \to y_\Psi\}(z),$$

where $A = A_{a,j}$ so that $\gamma y_\Psi = -A/cp^n$ and $\gamma x_\Psi = \infty$.

For a general F, both integrals are defined as a limit as n increases of integrals of truncations to polynomials of degree $k-2$. It is enough to show that

$$\mathrm{Trunc}^h_{D(A,n),A}\big(F(y_p)\big)\Big|_{y_p=(cz+\nu)/p^j} = \mathrm{Trunc}^h_{U_{j,a},\nu'+p^j a}\left(F\Big(\frac{cz+\nu}{p^{j(z)}}\Big)\right).$$

This holds by Lemma 3.1 with $\gamma^{-1} = \left(\begin{smallmatrix} c & \nu \\ 0 & p^j \end{smallmatrix}\right)$.

For the second:

As j varies the $J_{\infty,\nu,j}$ cover the set $J_{\infty,\nu}$: once if $s = s'$ and twice if $s = 2s'$, in the latter case with opposite sign for w^j if and only if $w = -1$. ■

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6.4 Evaluation of LI_Ψ

By definition,

$$LI_\Psi = \int_{\mathbb{P}}^{1} (\mathbb{Q}_p) \log_p\left(\frac{t-\gamma_\Psi z}{t-z}\right) P_\Psi(t)\, d\mu_f^{w_\infty}\{x_\Psi \to y_\Psi\}(t).$$

Again we fix $\Psi = \Psi_\nu$. To simplify the notation, set $\mu = \mu_f^{w_\infty}\{x_\Psi \to y_\Psi\}$, and $\kappa = \kappa_f^{w_\infty}\{x_\Psi \to y_\Psi\}$.

By the definition of the integral for locally analytic functions F, to evaluate it we need to divide $\mathbb{P}^1(\mathbb{Q}_p)$ into smaller open sets, and approximate the integral on each set by the evaluation of κ on a truncation of F.

Our choice of divisions will be as follows:

$$\mathbb{P}^1(\mathbb{Q}_p) = U^-(n) \amalg \coprod_{-n}^{+n} \gamma_\Psi^j \mathcal{F}_\Psi \amalg U^+(n)$$

where $U^+(n) = U(e_{(n+1)s})$, $U^-(n) = U(\overline{e_{-ns}})$, definitions as in 6.1. The middle divisions will be refined later, and we will take a limit as n increases.

First we show that the end divisions can be ignored:

Lemma 6.2

$$\lim_{n\to\infty}\left(\kappa\left(e_{(n+1)s}, \mathrm{Trunc}^{k-2}_{U^+(n),y_\Psi}\left[\log_p\left(\frac{t-\gamma_\Psi z}{t-z}\right)P_\Psi(t)\right]\right)\right)$$
$$= \lim_{n\to\infty}\left(\kappa\left(\overline{e_{-ns}}, \mathrm{Trunc}^{k-2}_{U^-(n),y_\Psi}\left[\log_p\left(\frac{t-\gamma_\Psi z}{t-z}\right)P_\Psi(t)\right]\right)\right) = 0$$

Proof Note that in the second term because $\infty \in U^-(n)$, we can choose to truncate at $a_0 = y_\Psi \notin U^-(n)$.

$$\kappa\left(e_{-ns}, \mathrm{Trunc}^{k-2}_{U^-(n),y_\Psi}\left[\log_p\left(\frac{t-\gamma_\Psi z}{t-z}\right)P_\Psi(t)\right]\right)$$
$$= \kappa\left(e_0, \mathrm{Trunc}^{k-2}_{U(\overline{e_*}),y_\Psi}\left[\log_p\left(\frac{\gamma_\Psi^{-n}t-\gamma_\Psi z}{\gamma_\Psi^{-n}t-z}\right)P_\Psi(t)\right]\right) \quad \text{by } \Gamma\text{-invariance of } \kappa$$
$$= \kappa\left(e_0, \mathrm{Trunc}^{k-2}_{U(\overline{e_*}),y_\Psi}\left[\log_p\left(\frac{t-\gamma_\Psi^{n+1}z}{t-\gamma_\Psi^{n}z}\right)P_\Psi(t)\right]\right),$$

as each of the expressions inside the "$\log_p$"s are Möbius transforms of t taking $\gamma_\Psi^{n+1}z \mapsto 0$, $\gamma_\Psi^n z \mapsto \infty$ and $\infty \mapsto 1$, hence they are equal.

Now,

$$\log_p\left(\frac{t-\gamma_\Psi^{n+1}z}{t-\gamma_\Psi^{n}z}\right) = \log_p\left(1+\frac{y_\Psi-\gamma_\Psi^{n+1}z}{t-y_\Psi}\right) - \log_p\left(1+\frac{y_\Psi-\gamma_\Psi^{n}z}{t-y_\Psi}\right)$$
$$= -\sum_{i\geq 1}\frac{1}{i}\left(\frac{\gamma_\Psi^{n+1}z-y_\Psi}{t-y_\Psi}\right)^i + \sum_{i\geq 1}\frac{1}{i}\left(\frac{\gamma_\Psi^{n}z-y_\Psi}{t-y_\Psi}\right)^i$$

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and $P_\Psi(t) = (ct+\nu)^{\frac{k-2}{2}} = c^{\frac{k-2}{2}}(t-y_\Psi)^{\frac{k-2}{2}}$, so

$$\begin{aligned}
&\mathrm{Trunc}^{k-2}_{U(e_*),y_\Psi}\Big[\log_p\Big(\frac{t-\gamma_\Psi^{n+1}z}{t-\gamma_\Psi^{n}z}\Big)P_\Psi(t)\Big] \\
&\qquad = \sum_{i=1}^{\frac{k-2}{2}}\big(-(\gamma_\Psi^{n+1}z-y_\Psi)^i+(\gamma_\Psi^{n}z-y_\Psi)^i\big)\cdot\frac{c^{\frac{k-2}{2}}}{i}(t-y_\Psi)^{(\frac{k-2}{2}-i)}.
\end{aligned}$$

But $\gamma_\Psi^{n+1}z \to y_\Psi$ as $n\to\infty$, and $\gamma_\Psi^{n}z \to y_\Psi$ as $n\to\infty$, so these coefficients go to zero as n increases. κ is linear in P, so it follows that the limit of the second term is zero.

Similarly, for the first term,

$$\begin{aligned}
&\kappa\Big(e_{(n+1)s}, \mathrm{Trunc}^{k-2}_{U^+(n),y_\Psi}\Big[\log_p\Big(\frac{t-\gamma_\Psi z}{t-z}\Big)P_\Psi(t)\Big]\Big) \\
&\quad = \kappa\Big(e_0, \mathrm{Trunc}^{k-2}_{U(e_*),y_\Psi}\Big[\log_p\Big(\frac{\gamma_\Psi^{(n+1)}t-\gamma_\Psi z}{\gamma_\Psi^{(n+1)}t-z}\Big)P_\Psi(t)\Big]\Big) \quad \text{by } \Gamma\text{-invariance of } \kappa \\
&\quad = \kappa\Big(e_0, \mathrm{Trunc}^{k-2}_{U(e_*),y_\Psi}\Big[\log_p\Big(\frac{t-\gamma_\Psi^{-n}z}{t-\gamma_\Psi^{-(n+1)}z}\Big)P_\Psi(t)\Big]\Big),
\end{aligned}$$

as each of the expressions inside the "$\log_p$"s are Möbius transforms of t taking $\gamma_\Psi^{-n}z \mapsto 0$, $\gamma_\Psi^{-(n+1)}z \mapsto \infty$ and $\infty \mapsto 1$, hence they are equal.

Now,

$$\begin{aligned}
\log_p\Big(\frac{t-\gamma_\Psi^{-n}z}{t-\gamma_\Psi^{-(n+1)}z}\Big) &= \log_p\Big(1+\frac{t-y_\Psi}{y_\Psi-\gamma_\Psi^{-n}z}\Big) \\
&\qquad - \log_p\Big(1+\frac{t-y_\Psi}{y_\Psi-\gamma_\Psi^{-(n+1)}z}\Big) + \log_p\Big(\frac{y_\Psi-\gamma_\Psi^{-n}z}{y_\Psi-\gamma_\Psi^{-(n+1)}z}\Big) \\
&= -\sum_{i\ge 1}\frac{1}{i}\Big(\frac{t-y_\Psi}{\gamma_\Psi^{-n}z-y_\Psi}\Big)^i + \sum_{i\ge 1}\frac{1}{i}\Big(\frac{t-y_\Psi}{\gamma_\Psi^{-(n+1)}z-y_\Psi}\Big)^i,
\end{aligned}$$

where the last term is zero because

$$\begin{aligned}
\log_p\Big(\frac{y_\Psi-\gamma_\Psi^{-n}z}{y_\Psi-\gamma_\Psi^{-(n+1)}z}\Big) f &= \log_p\Big(\frac{M_\Psi(\gamma_\Psi\cdot\gamma_\Psi^{-(n+1)}z)}{M_\Psi(\gamma_\Psi^{-(n+1)}z)}\Big) \\
&= \log_p(p^s) \text{as } M_\Psi(\gamma_\Psi z) = p^s M_\Psi(z) \quad \text{for any } z \\
&= 0.
\end{aligned}$$

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We have $P_\Psi(t) = (ct+\nu)^{\frac{k-2}{2}} = c^{\frac{k-2}{2}}(t-y_\Psi)^{\frac{k-2}{2}}$, so

$$\mathrm{Trunc}^{k-2}_{U(e_*),y_\Psi}\left[\log_p\left(\frac{t-\gamma_\Psi^{-n}z}{t-\gamma_\Psi^{-(n+1)}z}\right)P_\Psi(t)\right]$$
$$= \sum_{i=1}^{\frac{k-2}{2}}\left(-\left(\frac{1}{\gamma_\Psi^{-n}z-y_\Psi}\right)^i+\left(\frac{1}{\gamma_\Psi^{-(n+1)}z-y_\Psi}\right)^i\right)\frac{c^{\frac{k-2}{2}}}{i}(t-y_\Psi)^{i+\frac{k-2}{2}}.$$

Again, these coefficients tend to zero as n increases, so the first limit is also zero. ■

To evaluate the remaining parts we need the lemma:

Lemma 6.3

$$\int_{\mathcal{F}_\Psi} P_\Psi(t)\,d\mu(t) = 0.$$

Proof This follows immediately because

$$\int_{\mathcal{F}_\Psi} P_\Psi(t)\,d\mu(t) = \kappa(e_s,P_\Psi)-\kappa(e_0,P_\Psi) = 0.$$ ■

Hence the integral LI_Ψ is approximated by

$$LI_{\Psi,n} = \sum_{-n}^{n}\int_{\gamma_\Psi^j\mathcal{F}_\Psi}\log_p\left(\frac{t-\gamma_\Psi z}{t-z}\right)P_\Psi(t)\,d\mu(t)$$
$$= \int_{\mathcal{F}_\Psi}\log_p\left(\frac{t-\gamma_\Psi^{1+n}z}{t-\gamma_\Psi^{-n}z}\right)P_\Psi(t)\,d\mu(t) \quad \text{by a similar method to Darmon's case.}$$

Lemma 6.4 *As $n\to\infty$, the limit of the above is*

$$LI_\Psi = \int_{\mathcal{F}_\Psi}\log_p\big(M_\Psi(t)\big)P_\Psi(t)\,d\mu(t).$$

Proof The integrand in $LI_{\Psi,n}$ involves a Möbius transform taking $\gamma_\Psi^{1+n}z$ to 0 and $\gamma_\Psi^{-n}z$ to ∞. But $\gamma_\Psi^{1+n}z \to y_\Psi$ and $\gamma_\Psi^{-n}z \to x_\Psi$. The idea is that in the limit we can replace the Möbius transform by one taking y_Ψ to 0 and x_Ψ to ∞, *i.e.*, the transform M_Ψ. More precisely, write

$$M_n(t) = \frac{-y_\Psi\gamma_\Psi^{-n}z}{\gamma_\Psi^{n+1}z}\cdot\frac{t-\gamma_\Psi^{1+n}z}{t-\gamma_\Psi^{-n}z};$$

then

$$LI_{\Psi,n} = \int_{\mathcal{F}_\Psi}\log_p\big(M_n(t)\big)P_\Psi(t)\,d\mu(t)$$

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by Lemma 6.3. The multiplier inside the $\log_p$ is chosen such that all the Möbius maps take $0 \mapsto -y_\Psi$.

Then $M_n(t) \to M_\Psi(t)$ as $n \to \infty$, for any $t \in \mathfrak{F}_\Psi$, so by continuity of the integral,

$$\int_{\mathfrak{F}_\Psi} \log_p\big(M_n(t)\big)\, P_\Psi(t)\, d\mu(t) \to \int_{\mathfrak{F}_\Psi} \log_p\big(M_\Psi(t)\big)\, P_\Psi(t)\, d\mu(t). \qquad \blacksquare$$

Proposition 6.2

$$LI_{\Psi_\nu} = \beta \int_{J_{\infty,\nu}} w^{j(t)} t_p^{\frac{k-2}{2}} \log_p(t_p)\, d\mu^{w_\infty}_{f,\mathrm{MTT}}(t),$$

where $\mu^{w_\infty}_{f,\mathrm{MTT}}$ is the Mazur-Tate-Teitelbaum measure associated to f_0, β is as in Proposition 6.1, and

$$\begin{aligned} J_{\infty,\nu} &= (\pi_0^\infty)^{-1} J_\nu \cap \mathbb{Z}^*_{p,c} \\ &= \{b \in \mathbb{Z}^*_{p,c} : b/\nu \equiv p^j \bmod c \text{ for some } j = j(b)\}. \end{aligned}$$

Proof We have

$$\begin{aligned} LI_{\Psi_\nu} &= \int_{\mathfrak{F}_\Psi} \log_p(z+\nu/c))(cz+\nu)^{\frac{k-2}{2}}\, d\mu(z) \\ &= \int_{\mathfrak{F}_\Psi} \log_p(cz+\nu)(cz+\nu)^{\frac{k-2}{2}}\, d\mu(z) \quad \text{by Lemma 6.3} \\ &= \int_{\mathfrak{F}_\Psi} p^{j(z)(\frac{k-2}{2})} \log_p\Big(\frac{cz+\nu}{p^{j(z)}}\Big)\Big(\frac{cz+\nu}{p^{j(z)}}\Big)^{\frac{k-2}{2}} d\mu(z) \quad \text{as } \log_p(p^j) = 0 \\ &= \beta \int_{J_{\infty,\nu}} w^{j(t)} t_p^{\frac{k-2}{2}} \log_p(t_p)\, d\mu^{w_\infty}_{f,\mathrm{MTT}}(t) \quad \text{by Lemma 6.1.} \end{aligned} \qquad \blacksquare$$

Now assume again that we are not in the case $\beta = 0$.

Corollary 6.2 *If χ is a primitive Dirichlet character of conductor c, and $\chi(p) = w$ then*

$$\begin{aligned} \sum_{\nu \in (\mathbb{Z}/cZ)^*} \chi(\nu) LI_{\Psi_\nu} &= s \int_{\mathbb{Z}^*_{p,c}} \chi(t) t_p^{\frac{k-2}{2}} \log_p(t_p)\, d\mu^{w_\infty}_{f,\mathrm{MTT}}(t) \\ &= \begin{cases} s\frac{d}{dt}\mathbf{L}_p\big(f_0, wp^{\frac{k-2}{2}}, \chi(x)x_p^{\frac{k-2}{2}}, t\big)\big|_{t=0} & \text{if } \chi(-1) = w_\infty, \\ 0 & \text{if } \chi(-1) = -w_\infty. \end{cases} \end{aligned}$$

Proof First, $\chi(\nu)w^{j(t)} = \chi(t)$. Secondly, each coset $a + pc\mathbb{Z}_{p,c}$ which is contained in $\mathbb{Z}^*_{p,c}$ is contained in precisely s of the sets $J_{\infty,\nu}$, for $\nu \equiv a, a/p, \dots, a/p^{s'-1} \bmod c$, and $\beta s' = s$.

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Note that if $\chi(-1) = w_\infty$,

$$L_p^{w_\infty}\big(f_0, wp^{\frac{k-2}{2}}, x_p^{\frac{k-2}{2}}\chi(x), t\big) = \mathbf{L}_p\big(f_0, wp^{\frac{k-2}{2}}, x_p^{\frac{k-2}{2}}\chi(x), t\big),$$

as then $(-1)_p^{\frac{k-2}{2}}\chi(-1) = (-1)^{\frac{k-2}{2}} w_\infty$.

If $\chi(-1) = -w_\infty$ then

$$L_p^{w_\infty}\big(f_0, wp^{\frac{k-2}{2}}, x_p^{\frac{k-2}{2}}\chi(x), t\big) = 0,$$

so assume we are in the former case.

By definition,

$$\begin{aligned}
&\frac{d}{dt}L_p^{w_\infty}\big(f_0, wp^{\frac{k-2}{2}}, \chi(x)x_p^{\frac{k-2}{2}}, t\big)\big|_{t=0}\\
&\qquad= \Big(\frac{d}{dt}\int_{\mathbb{Z}_{p,c}^*}\chi(x)x_p^{\frac{k-2}{2}}\langle x\rangle^t\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(x)\Big)\Big|_{t=0}\\
&\qquad= \Big(\frac{d}{dt}\int_{\mathbb{Z}_{p,c}^*}\chi(x)x_p^{\frac{k-2}{2}}\Big(\sum_{r=0}^{\infty}\frac{t^r}{r!}(\log_p\langle x\rangle)^r\Big)\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(x)\Big)\Big|_{t=0}\\
&\qquad= \int_{\mathbb{Z}_{p,c}^*}\chi(x)x_p^{\frac{k-2}{2}}\log_p\langle x\rangle\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(x)\\
&\qquad= \int_{\mathbb{Z}_{p,c}^*}\chi(x)x_p^{\frac{k-2}{2}}\log_p(x_p)\, d\mu_{f,\mathrm{MTT}}^{w_\infty}(x). \qquad \blacksquare
\end{aligned}$$

If we had constructed the distribution using κ_f instead of $\kappa_f^{w_\infty}$, so that it took values in $V_f = \mathbb{C}_p \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}L_f$, then the algebraic part $\mathbf{L}_p$ would be replaced by L_p in this Corollary. This is not as useful, because the eigenspaces of the cohomology classes constructed in this way may not be 1-dimensional (see the next section).

7 Cohomology and Hecke Operators

7.1 Spaces Involved and Exact Sequences

Let $V_{k-2} = \mathcal{P}_{k-2}(\mathbb{C}_p)^* \cong \mathrm{Hom}(\mathcal{P}_{k-2}, \mathbb{C}_p)$. Let $\mathcal{F} = \mathcal{F}_{k-2} = \mathrm{Hom}(\mathbb{D}, V_{k-2})$ and $\mathcal{M} = \mathcal{M}_{k-2} = \mathrm{Hom}(\mathbb{D}_0, V_{k-2})$. Then there is an exact sequence:

$$0 \to V_{k-2} \to \mathcal{F} \to \mathcal{M} \to 0.$$

This leads to the long exact sequence

$$\begin{aligned}
(11)\qquad & 0 \to V_{k-2}^{\Gamma_0(M)} \to \mathcal{F}^{\Gamma_0(M)} \to \mathcal{M}^{\Gamma_0(M)} \to\\
&\to H^1\big(\Gamma_0(M), V_{k-2}\big) \to H^1\big(\Gamma_0(M), \mathcal{F}\big) \to H^1\big(\Gamma_0(M), \mathcal{M}\big) \to\\
&\to H^2\big(\Gamma_0(M), V_{k-2}\big) \to \cdots .
\end{aligned}$$

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This sequence is compatible with the action of Hecke operators $T(l)$ for l prime to N, and W_∞. Its behaviour for $k = 2$ is considered in Darmon's paper, so assume $k > 2$ is even. Several terms in (11) are known:

$V_{k-2}^{\Gamma_0(M)} = 0$, as V_{k-2} is an irreducible $\Gamma_0(M)$-module ([Hi] p. 165 Lemma 2). (This is not true when $k = 2$ as then $V_{k-2} = \mathbb{C}_p$.)

$H^2(\Gamma_0(M), V_{k-2}) = 0$ by [Hi, p. 162 Proposition 1].

We have

$$\mathcal{F} = \oplus_x \operatorname{Ind}_{\Gamma_x}^{\Gamma_0(M)} V_{k-2}$$

where x runs over the distinct cusps, so

$$H^0\big(\Gamma_0(M), \mathcal{F}\big) = \bigoplus_x V_{k-2}^{\Gamma_x} \quad \text{and}$$

$$H^1\big(\Gamma_0(M), \mathcal{F}\big) = \bigoplus H^1(\Gamma_x, V_{k-2}).$$

Each Γ_x is free on a generator π_x, so $V_{k-2}^{\Gamma_x} = V_{k-2}^{\pi_x}$ and

$$H^1(\Gamma_x, V_{k-2}) = V_{k-2}/(\pi_x - 1)V_{k-2}.$$

Note that by [Hi, p. 166 (2a)], each of these has dimension 1. This shows that the dimensions of $H^0(\Gamma_0(M), \mathcal{F})$ and $H^1(\Gamma_0(M), \mathcal{F})$ are both equal to S, the number of cusps.

By definition of the parabolic cohomology groups H_P^1 as in [Hi], the sequence (11) now divides into the following parts:

$$0 \to \mathcal{F}^{\Gamma_0(M)} \to \mathcal{M}^{\Gamma_0(M)} \to H_P^1\big(\Gamma_0(M), V_{k-2}\big) \to 0, \tag{12}$$

$$0 \to H_P^1\big(\Gamma_0(M), V_{k-2}\big) \to H^1\big(\Gamma_0(M), V_{k-2}\big) \to H^1\big(\Gamma_0(M), \mathcal{F}\big) \to H^1\big(\Gamma_0(M), \mathcal{M}\big) \to 0. \tag{13}$$

Over the complex numbers there are Eichler-Shimura isomorphisms ([Hi, Section 6.2])

$$H^1\big(\Gamma_0(M), V_{k-2}(\mathbb{C})\big) \cong S_k\big(\Gamma_0(M)\big) \oplus S_k^c\big(\Gamma_0(M)\big) \oplus E_k\big(\Gamma_0(M)\big)$$
$$H_P^1\big(\Gamma_0(M), V_{k-2}(\mathbb{C})\big) \cong S_k\big(\Gamma_0(M)\big) \oplus S_k^c\big(\Gamma_0(M)\big).$$

where $E_k(\Gamma_0(M))$ is the space of Eisenstein series.

We have

$$\dim_{\mathbb{C}_p} H^1\big(\Gamma_0(M), V_{k-2}\big) = \dim_{\mathbb{C}} H^1\big(\Gamma_0(M), V_{k-2}(\mathbb{C})\big)$$

and similarly for H_P^1.

But the dimension of $E_k(\Gamma_0(M))$ is also S, by [Miy, p. 179–180, 1°], so looking at the alternating sum of dimensions in (13) we see $H^1(\Gamma_0(M), \mathcal{M}) = 0$. (In the case $k = 2$ the dimension of $E_k(\Gamma_0(M))$ is $S - 1$).

Now we look at sequence (12). We know the dimensions of the Hecke eigenspaces in $H_P^1(\Gamma_0(M), V_{k-2}(\mathbb{C}))$ by the Eichler-Shimura isomorphism, so also of the Hecke eigenspaces in $H_P^1(\Gamma_0(M), V_{k-2})$.

7.2 Description of Cusps and Hecke Action on $\mathcal{F}^{\Gamma_0(M)}$

According to [Mil] the cusps of $\Gamma_0(M)$ can be described as follows: there is a cusp $\binom{\tilde{a}}{d}$ for each $d|M$ and each $\tilde{a} \in (\mathbb{Z}/t\mathbb{Z})^*$ where $t = (d, M/d)$. Write a for a representative in $\mathbb{Z}$ of the class $\tilde{a}$, which is prime to M. Then a/d is a representative of the cusp $\binom{\tilde{a}}{d}$.

To tell whether two rational numbers p/q and r/s are equivalent under $\Gamma_0(M)$ ([Cr]) we write $p/q = P\infty$ and $r/s = R\infty$ with $P = \left(\begin{smallmatrix} p & u \\ q & v \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $R = \left(\begin{smallmatrix} r & w \\ s & z \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then they are equivalent if and only if there exists $h \in \mathbb{Z}$ such that $R^{-1}\gamma P = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ for some $\gamma \in \Gamma_0(M)$, *i.e.*,

$$\gamma = R \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} P^{-1} \in \Gamma_0(M). \tag{14}$$

So they are equivalent if and only if there is a solution h to the congruence $qsh \equiv sv - qz \mod M$. Assuming r and p are prime to M this is equivalent to

$$rspqh \equiv rs - pq + qs(ru - sw) \mod M$$

Lemma 7.1 *The Hecke operator $T(l)$ acts on the cusps by*

$$T(l)\begin{pmatrix} \tilde{a} \\ d \end{pmatrix} = l\begin{pmatrix} l\tilde{a} \\ d \end{pmatrix} + \begin{pmatrix} l^{-1}\tilde{a} \\ d \end{pmatrix}.$$

Proof Use representatives for the cosets

$$\delta_j = \begin{pmatrix} 1 & j \\ 0 & l \end{pmatrix} \quad 0 \le j \le l-1 \text{ and } \delta_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}.$$

First, $\delta_l(a/d) = (la/d)$ is clearly a representative of the cusp $\binom{l\tilde{a}}{d}$.

Now assume l does not divide $a + jd$. Then to show $\delta_j(a/d) = (a + jd)/ld \sim la/d$ we need to solve the above congruence with $p = a + jd$, $q = ld$, $r = la$, and $s = d$. Then

$$\begin{aligned} rspqh &\equiv rs - pq + qs(ru - sw) \bmod M \\ &\iff lad(a + jd)ldh \equiv lad - (a + jd)ld + ld^2(lau - dw) \bmod M \\ &\iff l^2a(d/t)(a + jd)h \equiv -j(d/t)l + l(d/t)(lau - dw) \bmod M/dt, \end{aligned}$$

which is soluble as l, a, (d/t) and $a + jd$ are all prime to M/dt.

Finally, for some j, l does divide $a+jd$. Then $(a+jd)/l \equiv al^{-1} \bmod t$, so $\delta_j(a/d) = ((a + jd)/l)/d$ represents the cusp $\binom{l^{-1}\tilde{a}}{d}$. ■

We have a decomposition

$$\mathcal{F}^{\Gamma_0(M)} = \bigoplus_x V_{k-2}^{\Gamma_x}$$

$$\phi \mapsto \bigoplus_x \phi(x),$$

where x runs over a set of representatives of the cusps, and each component is 1-dimensional.

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Lemma 7.2 *There is a natural basis element $v(x)$ of $V_{k-2}^{\Gamma_x}$ given by $v(x)(P) := P(x)$ (or $v(x) = 1 \in \mathbb{C}_p$ if $k = 2$).*

Thus for $\phi \in \mathcal{F}^{\Gamma_0(M)}$ we can write $\phi(x) = \lambda_\phi(x)v(x)$, so

$$\phi = \sum \lambda_\phi(a/d)v(a/d),$$

where d runs over divisors of M and a is a representative as above of $\tilde{a}$ which runs through $(\mathbb{Z}/t\mathbb{Z})^*$.

Lemma 7.3 *The $\lambda_\phi(a/d)$ are independent of the choice of representative $a \in \mathbb{Z}$ of the class $\tilde{a}$.*

Proof This follows from the fact that for $\gamma \in \Gamma_0(M)$,

$$\lambda_\phi(\gamma x) = j(\gamma, x)^{k-2}\lambda_\phi(x), \tag{15}$$

with the usual notation

$$j(\gamma, x) = \frac{(Cx + D)}{(\det \gamma)^{1/2}}$$

if $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})^+$.

This follows from the definition of the λ_ϕ and the $\Gamma_0(M)$-invariance of ϕ.

Now suppose a and a' are two representatives of the class $\tilde{a}$, prime to M. Multiplying out the equation (14) for γ, with $p/q = a/d$ and $r/s = a'/d$, we see that if $\gamma a/d = a'/d$, then $j(\gamma, a/d) = 1$. ■

Lemma 7.4 *The action of the Hecke operator $T(l)$ on $\mathcal{F}^{\Gamma_0(M)}$ is such that*

$$\lambda_{T(l)\phi}(a/d) = l\lambda_\phi(la/d) + \lambda_\phi(l'a/d)$$

where a, d are as above and l' is an integer prime to M with $ll' \equiv 1 \bmod t$.

Proof Fix $\phi \in \mathcal{F}^{\Gamma_0(M)}$ and let $\lambda(x) := \lambda_\phi(x)$ for all $x \in \mathbb{P}^1(\mathbb{Q})$. Then

$$\begin{aligned} T(l)\phi(x)(P) &= l^{\frac{k-2}{2}} \sum_j (\phi|\delta_j)(x)(P) \\ &= l^{\frac{k-2}{2}} \Big(\sum_j \lambda(\delta_j x) j(\delta_j, x)^{-(k-2)}\Big) P(x). \end{aligned}$$

So when $\delta_j(a/d) \sim a'/d$ we need to know how $\lambda(\delta_j(a/d))$ is related to $\lambda(a'/d)$. As above we have $\gamma \in \Gamma_0(M)$ such that $\gamma\delta_j(a/d) = a'/d$. Then by (15),

$$\lambda\big(\delta_j(a/d)\big) = \lambda(a'/d)\, j(\gamma^{-1}, a'/d)^{k-2}.$$

When $j = l$ and $a' = la$ then $\gamma = 1$ and so $\lambda(\delta_l(a/d)) = \lambda(la/d)$.

When l does not divide $a + jd$, and $a' = la$,

$$\gamma^{-1} = \begin{pmatrix} a+jd & u \\ dl & v \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & -w \\ -d & a' \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ dlz - (-hdl + v)\,d & -wdl + a'(-hdl + v) \end{pmatrix}$$

so $j(\gamma^{-1}, a'/d) = a'lz - wdl = l$ and $\lambda(\delta_j(a/d)) = l^{k-2}\lambda(la/d)$

When l divides $a + jd$, and $a' = l'a$ then $\gamma = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ for some h and $\lambda(\delta_j(a/d)) = \lambda(l'a/d)$.

So we have overall

$$\begin{aligned}\lambda_{T(l)\phi}(a/d) &= l^{\frac{k-2}{2}}\Big(\sum_j \lambda\big(\delta_j(a/d)\big)\, j\big(\delta_j, (a/d)\big)^{-(k-2)}\Big)\\ &= l^{\frac{k-2}{2}}\big(l^{\frac{k-2}{2}}\lambda(la/d) + (l-1)l^{-\frac{k-2}{2}}\cdot l^{k-2}\lambda(la/d) + l^{-\frac{k-2}{2}}\lambda(l'a/d)\big)\\ &= \big(l^{k-2}\lambda(la/d) + \lambda(l'a/d)\big).\end{aligned}$$

■

Then writing $v\binom{\tilde{a}}{d}$ for $v(a/d)$ for a suitable choice of representative a, we have

$$\mathcal{F}^{\Gamma_0(M)} = \bigoplus_{\text{cusps}} v\binom{\tilde{a}}{d}\mathbb{C}_p$$

and

$$T(l)v\binom{\tilde{a}}{d} = l^{k-1}v\binom{l^{-1}\tilde{a}}{d} + v\binom{l\tilde{a}}{d}.$$

This action is diagonalizable with eigenvalues

$$l^{k-1}\chi(l) + \chi^{-1}(l),$$

where χ is any Dirichlet character of conductor $t = (d, M/d)$ for any $d|M$.

From [Miy, p. 179–180] we see that these are precisely the eigenvalues of the Hecke operator acting on Eisenstein series.

Remark The results and methods of this section still hold for $k = 2$. There is an error here in [Dar], where equation (150) in fact only holds if N is squarefree, so that all $T(l)$ act trivially (*i.e.*, as multiplication by $(l+1)$) on the cusps. In general, one can only say that the eigenvalues of $T(l)$ on $\mathcal{F}^{\Gamma_0(N)}$ and on the Eisenstein series are of the form $l\chi(l) + \chi^{-1}(l)$, for χ as above.

7.3 Dimension of Eigenspaces

Let f be a newform for $\Gamma_0(N)$ with $T(l)f = a_l f$. For a module A with Hecke operators acting on it, then A^{f,w_∞} will denote the eigenspace in A where $T(l) = a_l$ for l prime to N, and $W_\infty = w_\infty$.

The results of the previous section together with sequence (12) shows that the non-zero eigenspaces in $\mathcal{M}^{\Gamma_0(M)}$ correspond either to eigenvalues of Eisenstein series, or to eigenvalues of newforms for $\Gamma_0(M)$, so $(\mathcal{M}^{\Gamma_0(M)})^{f,w_\infty} = 0$.

A similar argument using the equivalent sequence for $\Gamma_0(N)$, or reference to [GS], shows that $(\mathcal{M}^{\Gamma_0(N)})^{f,w_\infty}$ is 1-dimensional. Now we know enough to prove

Proposition 7.1

$$\dim_{\mathbb{C}_p}(H^1(\Gamma,\mathcal{M})^{f,w_\infty}) = 1.$$

Proof We use the exact sequence from [Ser]

$$\mathcal{M}^{\Gamma_0(M)} \oplus \mathcal{M}^{\Gamma_0(M)'} \to \mathcal{M}^{\Gamma_0(N)} \to H^1(\Gamma,\mathcal{M}) \to H^1\big(\Gamma_0(M),\mathcal{M}\big) \oplus H^1\big(\Gamma_0(M)',\mathcal{M}\big)$$

where $\Gamma_0(M)' = \mathrm{Stab}_\Gamma(\alpha v_*) = \alpha\Gamma_0(M)\alpha^{-1}$. Thus

$$\begin{aligned}\big(\mathcal{M}^{\Gamma_0(M)}\big)^{f,w_\infty} \oplus (\mathcal{M}^{\Gamma_0(M)'})^{f,w_\infty} \to \big(\mathcal{M}^{\Gamma_0(N)}\big)^{f,w_\infty} \to \big(H^1(\Gamma,\mathcal{M})\big)^{f,w_\infty}\\ \to H^1\big(\Gamma_0(M),\mathcal{M}\big)^{f,w_\infty} \oplus H^1\big(\Gamma_0(M)',\mathcal{M}\big)^{f,w_\infty}\end{aligned}$$

We know the first and last terms here are zero (using similar methods for the conjugate subgroup $\Gamma_0(M)'$), and that $\dim(\mathcal{M}^{\Gamma_0(N)})^{f,w_\infty} = 1$. ■

7.4 The Exceptional Zero Conjecture

The p-adic L-function attached to the newform f for $\Gamma_0(N)$ satisfies

$$\mathbf{L}_p(f,\omega^j\chi, j) = e_p(\chi, j)K(\chi, j)L(f_{\bar{\chi}}, j+1)/\Omega_f^{w_\infty},$$

where ω is the Teichmüller character, χ is a primitive Dirichlet character of conductor c,

$$e_p(\chi, j) = \frac{1}{a_p^{\nu(c)}}\left(1 - \frac{\chi(p)p^j}{a_p}\right)$$

is the p-adic multiplier, an algebraic number, and

$$K(\chi, j) = \frac{c^{j+1}j!}{(-2\pi i)^j\tau(\bar{\chi})}$$

is an element of $\mathbb{C}$.

When $e_p(\chi, j) = 0$, we say there is an exceptional zero. So there is an exceptional zero when $j = \frac{k-2}{2}$ and $\chi(p) = w$.

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Proposition 7.2 **(The Exceptional Zero is of Local Type)** *There is a constant $\mathcal{L}_p^{w_\infty}(f) \in \mathbb{C}_p$ such that for χ any primitive Dirichlet character of conductor c prime to N satisfying $\chi(p) = w$ and $\chi(-1) = w_\infty$,*

$$\mathbf{L}_p'(f, \omega^{\frac{k-2}{2}}\chi, t)|_{t=\frac{k-2}{2}} = \mathcal{L}_p^{w_\infty}(f)\Lambda\Big(f, \chi, \frac{k-2}{2}\Big).$$

Proof We know that the cohomology classes lc_f and oc_f are both contained in the same 1-dimensional eigenspace. Hence, provided oc_f is nonzero, we can choose a constant $\mathcal{L}_p^{w_\infty}(f) \in \mathbb{C}_p$ such that $\mathrm{lc}_f = \mathcal{L}_p^{w_\infty}(f)\,\mathrm{oc}_f$. Hence for any $\Psi = \Psi_\nu$, $LI_\Psi = \mathcal{L}_p^{w_\infty}(f)W_\Psi$. Thus, summing and applying Corollaries 6.1 and 6.2, we get the result.

By Corollary 6.1, oc_f will be non-zero provided there exists a conductor c and some Dirichlet character χ as above satisfying

$$L(f, \chi, k/2) \neq 0.$$

This holds by a general result of Rohrlich, the theorem in the introduction of [Ro]. ■

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