Bull. Aust. Math. Soc. **83** (2011), 22–29
doi:10.1017/S000497271000184X

NOTES ON GRAPH-CONVERGENCE FOR MAXIMAL MONOTONE OPERATORS

FILOMENA CIANCIARUSO, GIUSEPPE MARINO, LUIGI MUGLIA and HONG-KUN XU ✉

(Received 12 November 2009)

Abstract

We construct a sequence $\{A_n\}$ of maximal monotone operators with a common domain and converging, uniformly on bounded subsets, to another maximal monotone operator A; however, the sequence $\{t_n^{-1}A_n\}$ fails to graph-converge for some null sequence $\{t_n\}$.

2010 *Mathematics subject classification*: primary 47H05; secondary 47H09, 47H10.

Keywords and phrases: maximal monotone operator, graph-convergence, Mann iteration.

1. Introduction

It is well known that graph-convergence plays an important role in solving many nonlinear problems, in particular, those governed by maximal monotone operators [1, 2]. In this regard, Lions [3] proves a very interesting and useful result which implies that $t^{-1}A$ graph-converges as $t \to 0$ to $N_{A^{-1}(0)}$, where A is a maximal monotone operator in a Hilbert space H such that $A^{-1}(0) := \{x \in D(A) : 0 \in Ax\} \neq \emptyset$, and N_K denotes the normal cone associated with a closed convex subset $K \subset H$. This result has many applications in variational inequalities and fixed points (see, for example [4, 6]).

On the other hand, since nonlinear problems are usually ill-posed, perturbation techniques are needed. A natural question thus arises: if $\{A_n\}$ is a sequence of maximal monotone operators which converge (in a certain sense, for instance, uniform convergence on bounded sets) to another maximal monotone operator A with $A^{-1}(0) \neq \emptyset$, and if $\{t_n\}$ is a null sequence of positive real numbers, does the sequence $\{t_nA_n\}$ graph-converge to $N_{A^{-1}(0)}$? In other words, is Lions' result stable in terms of perturbation?

The purpose of this note is to give a negative answer to this question. More precisely, we will construct a sequence $\{A_n\}$ of maximal monotone operators with a

The second author was supported in part by Ministero dell'Universitá e della Ricerca of Italy. The fourth author was supported in part by NSC 97-2628-M-110-003-MY3.

common domain and which converges to the zero operator uniformly on bounded sets; however, the sequence $\{t_n^{-1}A_n\}$ fails to graph-converge for some null sequence $\{t_n\}$.

2. Preliminaries

Let H be a real Hilbert space and let A be an operator (possibly multi-valued) with domain $D(A)$ and range $R(A)$ in H. The graph of A is

$$\mathrm{Gr}(A) = \{(x, y) \in H \times H : x \in D(A),\ y \in Ax\}.$$

We say that A is monotone if $\mathrm{Gr}(A)$ is a monotone set; that is,

$$\langle x_1 - x_2, y_1 - y_2\rangle \geq 0 \quad \forall (x_i, y_i) \in \mathrm{Gr}(A),\ i = 1, 2.$$

A monotone operator A is maximal monotone if its graph $\mathrm{Gr}(A)$ is not properly contained in the graph of any other monotone operators. Equivalently, a monotone operator A is maximal monotone if and only if the following implication holds:

$$(x, y) \in H \times H,\ \langle x - \xi, y - \eta\rangle \geq 0 \quad \forall (\xi, \eta) \in \mathrm{Gr}(A) \implies (x, y) \in \mathrm{Gr}(A).$$

The resolvent of a monotone operator A is defined as

$$J_\lambda^A = (I + \lambda A)^{-1}$$

where $\lambda > 0$. Maximal monotonicity can be characterized by the resolvent.

PROPOSITION 2.1 [2]. *Let A be an operator in H. The following are equivalent:*

(i) *A is a maximal monotone operator;*
(ii) *A is monotone and $R(I + \lambda A) = H$ for all $\lambda > 0$;*
(iii) *for every $\lambda > 0$, $J_\lambda^A : H \to H$ is nonexpansive.*

Monotone operators find many applications in various disciplines. The following result, due to Lions, is a useful tool in many areas of mathematical analysis, such as variational calculus and iterative methods for nonexpansive mappings.

PROPOSITION 2.2 [3]. *Consider the net $(J_{t^{-1}}^A(x + t^{-1}u))_{0<t<1}$. Then:*

(I) *the following properties are equivalent:*
  (a) *$u \in R(A)$;*
  (b) *$(J_{t^{-1}}^A(x + t^{-1}u))_{0<t<1}$ is bounded;*
  (c) *there exists a strictly positive subsequence $(t_n)_{n\in\mathbb{N}}$ convergent to 0 such that $(J_{t_n^{-1}}^A(x + t_n^{-1}u))_{n\in\mathbb{N}}$ is bounded;*
  (d) *$\lim_{t\to 0^+} J_{t^{-1}}^A(x + t^{-1}u)$ exists;*

(II) *if any one of these conditions is satisfied, then*

$$\lim_{t\to 0^+} J_{t^{-1}}^A(x + t^{-1}u) = P_{A^{-1}(u)}(x). \tag{2.1}$$

https://doi.org/10.1017/S000497271000184X Published online by Cambridge University Press

Nonlinear problems are often ill-posed; perturbations are thus needed. This means that one should consider a sequence of perturbed problems whose solutions would converge in some sense to a solution of the original problem. Graph-convergence is usually used.

DEFINITION 2.3 [1]. Let A_n, A be maximal monotone operators in H. The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be graph-convergent to A, denoted $A_n \to^G A$, if, for every $(x, y) \in \mathrm{Gr}(A)$, there exists a sequence $(x_n, y_n) \in \mathrm{Gr}(A_n)$ such that $x_n \to x$ and $y_n \to y$, as $n \to \infty$. Equivalently, $A_n \to^G A$ if and only if

$$\limsup_{n\to\infty} \mathrm{Gr}(A_n) \subset \mathrm{Gr}(A) \subset \liminf_{n\to\infty} \mathrm{Gr}(A_n).$$

The next proposition will be a relevant tool for our purposes.

PROPOSITION 2.4 [1]. *Let $(A_n)_{n\in\mathbb{N}}$, A be maximal monotone operators in H with $A_n \to^G A$, as $n \to \infty$. Then, for any sequence $(w_n, z_n) \in \mathrm{Gr}(A_n)$ such that $w_n \to w$ and $z_n \rightharpoonup z$, we have $(w, z) \in \mathrm{Gr}(A)$.*

The following proposition shows that graph-convergence is equivalent to resolvent convergence for maximal monotone operators.

PROPOSITION 2.5 [1]. *Let A_n and A be maximal monotone operators in H. The following are equivalent:*

(i) $A_n \to^G A$;
(ii) $J_\lambda^{A_n} \to J_\lambda^A$ *for every* $\lambda > 0$;
(iii) $J_{\lambda_0}^{A_n} \to J_{\lambda_0}^A$ *for some* $\lambda_0 > 0$.

Recall now that the metric projection $P_C : H \to C$ from H onto a closed convex subset $C \subset H$ is the mapping which assigns to each $x \in H$ the only point $P_C x$ in C with the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\|.$$

The following is a characterization of P_C.

LEMMA 2.6. *Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \le 0 \quad \forall y \in C. \tag{2.2}$$

REMARK 2.7. For every $y \in H$, $A^{-1}(y)$ is a closed and convex subset of H.

REMARK 2.8. It follows from (2.2) that

$$P_{A^{-1}(u)} = J_1^{N_{A^{-1}(u)}}$$

where $N_{A^{-1}(u)} : H \to \mathcal{P}(H)$ is the normal cone to $A^{-1}(u)$, that is,

$$N_{A^{-1}(u)} : x \mapsto \begin{cases} \{v \in H : \langle y - x, v\rangle \le 0\ \forall y \in A^{-1}(u)\}, & x \in A^{-1}(u), \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.3}$$

https://doi.org/10.1017/S000497271000184X Published online by Cambridge University Press

Moreover observing that $J^A_{t^{-1}} = J^{t^{-1}A}_1$, (2.1) becomes

$$\lim_{t\to 0^+} J_1^{t^{-1}A}(x + t^{-1}u) = J_1^{N_{A^{-1}(u)}}(x). \tag{2.4}$$

The aim of this paper is to demonstrate that Lions' conclusion in Proposition 2.2(II) is optimal in the sense that if $\{A_n\}$ is a sequence of maximal monotone operators A_n with a common domain and uniformly convergent on the bounded subsets of the common domain to another maximal monotone operator A, then it is not necessary true that $\lim_{n\to\infty} J^{A_n}_{t_n^{-1}}(x + t_n^{-1}u) = P_{A^{-1}(u)}(x)$ for all null sequences $\{t_n\}$ of positive numbers.

It is worth of noting that in the special case of $u = 0$, (2.4) is reduced to the following result.

PROPOSITION 2.9 [4, 6]. *Let A be a maximal monotone operator on H such that $A^{-1}(0) \neq \emptyset$. Then $t^{-1}A \to^G N_{A^{-1}(0)}$ as $t \to 0$.*

Finally, we need the following useful lemma.

LEMMA 2.10 [7]. *Suppose that a positive sequence $\{a_n\}$ satisfies the condition*

$$a_{n+1} \le (1 - \sigma_n)a_n + \sigma_n\delta_n, \quad n \ge 0,$$

where $\{\sigma_n\}$ is a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \sigma_n = \infty$ and $\{\delta_n\}$ is a sequence such that $\limsup_{n\to\infty} \delta_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.

3. A counterexample

PROPOSITION 3.1. *Let H be a Hilbert space and D a nonempty closed convex subset of H containing more than one point. Then there exist maximal monotone operators $A_n, A : D \to H$ for $n \ge 1$ such that $A_n \to A$ uniformly on bounded subsets of D. However, $t_n^{-1}A_n \not\to^G N_{A^{-1}(0)}$, where $\{t_n\}$ is some null sequence of positive numbers.*

PROOF. Take $d_0, d_1 \in D$ such that $d_0 \neq d_1$. Let A be the zero operator (that is, $Ax \equiv 0$ for all $x \in D$). For each $n \ge 1$, set

$$s_n = \frac{1}{(n+1)^{\alpha}}, \quad t_n = \frac{1}{(n+1)^{\beta}}, \quad 0 < \alpha < \beta \le 1.$$

Define A_n by

$$A_n x := \frac{s_n}{1 - s_n}(x - d_1), \quad x \in D.$$

Then it is easy to see that each A_n is maximal monotone operator defined on D; moreover, A_n tends as $n \to \infty$ to A uniformly on bounded subsets of D. We will prove that $\{t_n^{-1}A_n\}$ does not graph-converge to $N_{A^{-1}(0)}$. To this end we use Mann's

https://doi.org/10.1017/S000497271000184X Published online by Cambridge University Press

iteration method. Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = s_n d_1 + (1 - s_n)[t_n d_0 + (1 - t_n) x_n]. \end{cases} \tag{3.1}$$

We next discuss some properties of $\{x_n\}$.

Fact 1. $(x_n)_{n\in\mathbb{N}}$ is bounded. Indeed, by (3.1),

$$x_{n+1} = (1 - t_n)(1 - s_n) x_n + t_n (1 - s_n) d_0 + s_n d_1$$

is a convex combination of $\{x_n, d_0, d_1\}$. Hence,

$$\begin{aligned} \|x_{n+1}\| &\le (1 - t_n)(1 - s_n)\|x_n\| + t_n(1 - s_n)\|d_0\| + s_n\|d_1\| \\ &\le \max\{\|x_n\|, \|d_0\|, \|d_1\|\}. \end{aligned}$$

Now, by induction, it follows that

$$\|x_n - x_0\| \le \max\{\|x_0\|, \|d_0\|, \|d_1\|\}$$

for all $n \ge 0$, and (x_n) is bounded.

Fact 2. The following relation holds:

$$\|x_{n+1} - x_n\| = o\left(\frac{1}{n^\gamma}\right) \quad \text{as } n \to \infty, \text{ where } 0 < \gamma < 1 - \alpha. \tag{3.2}$$

Indeed, some manipulations give

$$\begin{aligned} x_{n+1} - x_n = {} & (1 - s_n)(1 - t_n)(x_n - x_{n-1}) + (s_n - s_{n-1})(d_1 - x_{n-1}) \\ & + [(1 - s_n)(t_n - t_{n-1}) - (s_n - s_{n-1})t_{n-1}](d_0 - x_{n-1}). \end{aligned}$$

Since $\{x_n\}$ is bounded, it turns out that, for a constant

$$M > 2\max\Big\{1, \|d_0\|, \|d_1\|, \sup_{n\ge 0}\|x_n\|\Big\},$$

we have

$$\|x_{n+1} - x_n\| \le (1 - s_n)(1 - t_n)\|x_n - x_{n-1}\| + M(|t_n - t_{n-1}| + |s_n - s_{n-1}|).$$

By multiplying both sides by n^γ, we obtain

$$\begin{aligned} n^\gamma\|x_{n+1} - x_n\| &\le (1 - s_n)(1 - t_n)n^\gamma\|x_n - x_{n-1}\| \\ &\qquad + Mn^\gamma(|t_n - t_{n-1}| + |s_n - s_{n-1}|) \\ &= (1 - s_n)(1 - t_n)(n - 1)^\gamma\|x_n - x_{n-1}\| \\ &\qquad + (1 - s_n)(1 - t_n)[n^\gamma - (n - 1)^\gamma]\|x_n - x_{n-1}\| \\ &\qquad + Mn^\gamma(|t_n - t_{n-1}| + |s_n - s_{n-1}|). \end{aligned}$$

Setting $a_n = (n-1)^\gamma \|x_n - x_{n-1}\|$, $\sigma_n = s_n + t_n - s_n t_n$, and

$$\delta_n = \frac{M}{\sigma_n}\{[n^\gamma - (n-1)^\gamma] + n^\gamma(|t_n - t_{n-1}| + |s_n - s_{n-1}|)\},$$

we obtain

$$a_{n+1} \le (1-\sigma_n)a_n + \sigma_n\delta_n. \tag{3.3}$$

Since, as $n \to \infty$,

$$n^\gamma - (n-1)^\gamma = O\Big(\frac{1}{n^{1-\gamma}}\Big), \quad |s_n - s_{n-1}| = O\Big(\frac{1}{n^{1+\alpha}}\Big),$$
$$|t_n - t_{n-1}| = O\Big(\frac{1}{n^{1+\beta}}\Big), \quad \sigma_n = O\Big(\frac{1}{n^{\alpha}}\Big),$$

we see that (as $0 < \gamma < 1-\alpha$)

$$\delta_n \approx \frac{1}{n^{1-\gamma}} + \frac{1}{n^{1-\gamma-\alpha}} + \frac{1}{n^{1+\beta-\alpha-\gamma}} \to 0.$$

Since we also have $\sum_{n=1}^\infty \sigma_n = \infty$, we can apply Lemma 2.10 to (3.3) to conclude that $\lim_{n\to\infty} n^\gamma \|x_{n+1} - x_n\| = 0$.

Suppose now that $\{t_n^{-1}A_n\}$ graph-converged to $N_{A^{-1}(0)}$; we would then get a contradiction as shown in Facts 3 and 4 below.

Fact 3. The sequence (x_n) is weakly convergent to d_0. Indeed, let $p \in \omega_w(x_n)$. It suffices to show that

$$\langle p - d_0, x - p\rangle \ge 0, \quad \forall x \in D,$$

or equivalently (see [4, 6])

$$0 \in (I-V)p + N_D p, \tag{3.4}$$

where V is the constant mapping $Vx \equiv d_0$. As a matter of fact, the definition of x_{n+1} implies that

$$\frac{1}{t_n(1-s_n)}(x_n - x_{n+1}) = \frac{s_n(x_n - d_1)}{t_n(1-s_n)} + x_n - d_0 = \frac{1}{t_n}A_n x_n + x_n - d_0. \tag{3.5}$$

Notice that (3.5) can be rewritten as (note $\beta \le \gamma$)

$$\Big((I-V) + \frac{1}{t_n}A_n\Big)x_n = \frac{n^\beta}{1-s_n}(x_n - x_{n+1}) \to 0 \quad \text{due to (3.2).} \tag{3.6}$$

Since we also have $(I-V) + t_n^{-1}A_n \to^G (I-V) + N_D$ as $n \to \infty$, it turns out from (3.6) that $0 \in (I-V)p + N_D p$. This is (3.4).

Fact 4. $x_n \to d_0$ as $n \to \infty$.

Indeed,

$$\begin{aligned}\|x_{n+1} - d_0\|^2 &= \|s_n(d_1 - d_0) + (1 - s_n)(1 - t_n)(x_n - d_0)\|^2 \\ &\le (1 - s_n)^2(1 - t_n)^2\|x_n - d_0\|^2 + 2s_n\langle d_1 - d_0, x_{n+1} - d_0\rangle \\ &\le (1 - s_n)\|x_n - d_0\|^2 + s_n\delta_n,\end{aligned}$$

where

$$\delta_n = 2\langle d_1 - d_0, x_{n+1} - d_0\rangle \to 0 \quad \text{as } x_n \to d_0$$

weakly. By Lemma 2.10, we obtain that $\|x_n - d_0\| \to 0$.

Fact 5. Consider now the sequence z_n defined by

$$\begin{cases} z_0 = x_0, \\ z_{n+1} = s_n d_1 + (1 - s_n) z_n. \end{cases} \tag{3.7}$$

Then $z_n \to d_1$. Indeed, this is a very particular case of the algorithm introduced in [5]. So we get $z_n \to d_1$.

Fact 6. $\|x_n - z_n\| \to 0$. Indeed,

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= \|(1 - s_n)[(1 - t_n)(x_n - z_n) + t_n(d_0 - z_n)]\| \\ &\le (1 - s_n)\|x_n - z_n\| + \gamma t_n,\end{aligned}$$

where $\gamma > 0$ is a constant such that $\gamma > \sup\{\|d_0 - z_n\| : n \ge 0\}$. Noticing that

$$\frac{t_n}{s_n} = \frac{1}{(n+1)^{\beta-\alpha}} \to 0,$$

we can apply Lemma 2.10 to conclude that $\|x_n - z_n\| \to 0$.

Therefore, the sequences (x_n) and (z_n) converge to the same limit which contradicts the fact that $x_n \to d_0$ (Fact 4) and $z_n \to d_1$ (Fact 5). □

References

[1] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series (Pitman (Advanced Publishing Program), Boston, MA, 1984).

[2] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, 5. Notas de Matemática (50) (North Holland, Amsterdam, 1973).

[3] P. L. Lions, 'Two remarks on the convergence of convex functions and monotone operators', *Nonlinear Anal.* **2**(5) (1978), 553–562.

[4] P. E. Maingé and A. Moudafi, 'Strong convergence of an iterative method for hierarchical fixed point problems', *Pac. J. Optim.* **3**(3) (2007), 529–538.

[5] G. Marino and H. K. Xu, 'A general iterative method for nonexpansive mappings in Hilbert spaces', *J. Math. Anal. Appl.* **318**(1) (2006), 43–52.

[6] A. Moudafi and P. E. Maingé, 'Towards viscosity approximations of hierarchical fixed-point problems', *Fixed Point Theory Appl.* **2006** (2006), 10; Article ID 95453.

[7] H. K. Xu, 'An iterative approach to quadratic optimization', *J. Optim. Theory Appl.* **116**(3) (2003), 659–678.

https://doi.org/10.1017/S000497271000184X Published online by Cambridge University Press

FILOMENA CIANCIARUSO, Dipartimento di Matematica,
Universitá della Calabria, 87036 Arcavacata di Rende (CS), Italy
e-mail: cianciaruso@unical.it

GIUSEPPE MARINO, Dipartimento di Matematica, Universitá della Calabria,
87036 Arcavacata di Rende (CS), Italy
e-mail: gmarino@unical.it

LUIGI MUGLIA, Dipartimento di Matematica, Universitá della Calabria,
87036 Arcavacata di Rende (CS), Italy
e-mail: muglia@mat.unical.it

HONG-KUN XU, Department of Applied Mathematics,
National Sun Yat-Sen University, Kaohsiung 80424, Taiwan
and
Department of Mathematics, College of Science, King Saud University,
PO Box 2455, Riyadh 11451, Saudi Arabia
e-mail: xuhk@math.nsysu.edu.tw