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THE INJECTIVE HULL AND THE $\mathscr{CL}$-COMPACTIFICATION OF A CONTINUOUS POSET

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Introduction. In **[57]** (2.12), D. S. Scott showed that the continuous lattices, invented by him in his study of a mathematical theory of computation **[56]**, are precisely (when they are made into topological spaces via the Scott topology) the injective T_0-spaces, i.e., the injective objects in the category $\mathbf{T}_0$ of T_0-spaces and continuous maps with regard to the class of all embeddings. Moreover, the sort of morphisms between continuous lattices Scott considered are precisely the continuous maps with regard to the respective Scott topologies. These are fairly non-Hausdorff topologies. (Indeed, the Scott topology induces the partial order of the lattice L via $x \leqq y$ if and only if $x \in \mathrm{cl}\{y\}$, the "specialization order" of the topology; hence L is Hausdorff in the Scott topology if and only if L has at most one element.) In topological algebra, compact Lawson semilattices (= compact Hausdorff topological $\wedge$-semilattices such that the $\wedge$-preserving continuous maps into the unit interval, with its ordinary topology and the min-semilattice structure, separate the points) with a unit element 1 have attracted considerable interest. In **[40]**, K. H. Hofmann and A. R. Stralka essentially proved that they are precisely the continuous lattices; their (compact Hausdorff) topology is uniquely determined by the lattice-structure: it is called the *$\mathscr{CL}$-topology* or the *Lawson topology* of the continuous lattice; cf. **[47]**, **[20]** VI-3.4. (J. D. Lawson **[45]** showed that a semilattice admits at most one compact Hausdorff topology making it into a topological semilattice.)

Interest in continuous posets is more recent than that in continuous lattices themselves. It was primarily initiated by theoretical computer scientists (**[59]**, **[49]**). But soon continuous posets equipped with the Scott topology were recognized as a significant class of topological spaces: They are the projective sober spaces (**[27]**, 2.19), i.e., the retracts of the free objects of a functor, the "specialization order" functor, which naturally arises in the study of sober spaces (and, more generally, T_0-spaces). They are the prime spectra of the completely distributive complete lattices (**[46]**, **[30]**, 2.5), a fact which establishes a bijective correspondence between these two classes of structures. In **[28]**, a characterization of continuous posets in terms of adjunctions (between partially ordered sets) has been given.

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In [**5**] Section 2, B. Banaschewski showed that in $\mathbf{T}_0$ every space X has a greatest essential extension

$$X \to \lambda X$$

(extension = topological embedding) and he noted that the spaces X satisfying

(*) Whenever $x \in V \in \mathfrak{O}(X)$, the lattice of open sets of X (ordered by inclusion), then there is an open neighborhood W of x in X such that

$$V \cap \cap\{\mathrm{cl}\{z\} \mid z \in W\} \text{ not } = \emptyset$$

have an *injective hull* in $\mathbf{T}_0$, i.e., λX is an injective T_0-space ([**5**] Corollary 2, p. 240), i.e., λX is (by D. Scott's result [**57**], 2.12) a continuous lattice in its Scott topology. In [**27**], 3.14, it is established that the sober spaces satisfying (*) are precisely the continuous posets P endowed with their Scott topology σ_P. Thus, for a continuous poset P, the injective hull $(P, \sigma_P) \to \lambda(P, \sigma_P)$ has the form

$$e:(P, \sigma_P) \to (I(P), \sigma_{I(P)})$$

for some continuous lattice $I(P)$, and the order-extension

$$e:P \to I(P)$$

is uniquely determined up to an isomorphism. This will be called the *injective hull* of the continuous poset P. (The stronger assertion, in [**5**] Corollary 2, p. 240, that every space with an injective hull in $\mathbf{T}_0$ satisfies (*) is false, as K. H. Hofmann and M. W. Mislove [**28**] have observed. In an appendix to this paper we give a corrected treatment explaining also its impact on the results in [**27**] and [**30**] which are partly in need of reformulation.) This terminology should not be confused with the result of B. Banaschewski and G. Bruns ([**7**] Section 4) that the MacNeille completion is the injective hull in the category **Poset** of partially ordered sets and isotone maps with regard to the class of all order-embeddings.

In Section 1, we provide an intrinsic characterization of the injective hull of a continuous poset P in order-theoretic and algebraic terms, the proof of which heavily relies on the universal properties of this concept. For a continuous poset P and a complete lattice L, an order-embedding $e:P \to L$ is an (the) injective hull of P if and only if (i) $e[P]$ is join-dense in L, (ii) $e:P \to L$ preserves suprema of non-empty up-directed subsets, (iii) $e:P \to L$ preserves the way below relation, and (iv) $e[P]$ generates L, i.e., there is no proper subset of L containing $e[P]$ which is stable in L under arbitrary infima and under suprema of non-empty up-directed subsets. None of these conditions can be omitted.

In the following sections, we make a study of (the analogue of) the $\mathscr{CL}$-topology ζ on a continuous poset. R. L. Wilson ([**61**, **62**]) has dealt with continuous posets which are compact (Hausdorff) in their $\mathscr{CL}$-

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topology ζ. Generally, however, ζ need not be compact, but it is always completely regular Hausdorff ([**37**], p. 243). Indeed, the $\mathscr{CL}$-topology on a continuous poset P is the trace of the (compact Hausdorff) $\mathscr{CL}$-topology of the injective hull $I(P)$ of P.

Every embedding of a space X into a compact Hausdorff space Y induces a natural Hausdorff compactification of X, viz. the closure of X in Y. We endow the closure C of a continuous poset P in its injective hull L with regard to the $\mathscr{CL}$-topology ζ_L with the partial order inherited from L (but not with any topology). The order-extension

$$P \to C$$

will be called the *$\mathscr{CL}$-compactification* of P. Commenting on an earlier draft of this paper, K. H. Hofmann and M. W. Mislove [**38**] gave an example to show that C need not be a continuous poset. Here we show by examples that $\sigma_L|C$ need not be the intrinsic Scott topology of C with (1) C non-continuous and (2) a continuous poset C with ascending chain condition (= a.c.c.).

A continuous 1, $\wedge$-semilattice is compact (Hausdorff) in its $\mathscr{CL}$-topology if and only if it is a complete lattice (hence a continuous lattice). It results that the $\mathscr{CL}$-compactification of a continuous 1, $\wedge$-semilattice coincides with the injective hull. This leads to another intrinsic characterization of this construction (which recently has received an interesting application in [**33**], Theorem 2.5).

In Section 4, we show that the injective hull of an algebraic poset P is an algebraic lattice. We also provide a representation of the $\mathscr{CL}$-compactification of an algebraic poset.

In Section 5, the $\mathscr{CL}$-compactification $P \to C$ of a continuous poset P is shown to be equivalent to the underlying order-embedding of a "Hausdorff compactification" of the locally quasicompact space (P, σ_P) (not an ordinary compactification) obtained by J. M. G. Fell ([**17**] Section 2, [**18**]). Thus it results from [**31**], 3.7.1 that the second factor in the splitting

$$P \to C \to L$$

of the injective hull $e:P \to L$ is the *MacNeille completion* ([**51**]) of C, i.e., the smallest completion of C (cf. [**4**]). This shows that $\sigma_L|C$ and $\zeta_L|C$ are intrinsic topologies of C, an observation which provides justification for considering the $\mathscr{CL}$-compactification of a continuous poset (merely) as an order-extension. Incidentally, it also results that, for continuous posets P whose $\mathscr{CL}$-topology is compact, the injective hull coincides with the MacNeille completion.

Thus it comes out somewhat as a surprise to what kind of structure the $\mathscr{CL}$-compactification leads: posets Q with continuous MacNeille completion M which are closed (as a subset of M, embedded via $x \to \downarrow x$, the principal ideal generated by x) with regard to the $\mathscr{CL}$-topology of M, hence receiving a compact Hausdorff topology, viz. $\zeta_M|Q$. This may be viewed as another reasonable generalization of the concept of a continuous lattice. Every continuous poset admits a canonical mapping into such a poset which is, both topologically and order-theoretically, a (join-) dense embedding, the $\mathscr{CL}$-compactification. To what extent is this extension determined by these properties? We have a partial result (not to be established here): For a continuous poset P, an order-embedding $e:P \rightarrow Q$ is (equivalent to) the $\mathscr{CL}$-compactification of P if (1) Q is a continuous poset with compact $\mathscr{CL}$-topology ζ_Q, (2) $e:(P, \zeta_P) \rightarrow (Q, \zeta_Q)$ is a topological embedding, (3) $e[P]$ is (topologically) dense in (Q, ζ_Q), and (4) $e[P]$ is join-dense in Q. Condition (1) is, as noted above, not a necessary requirement.

The results of this article have been communicated to the ("write-in") *Seminar on Continuity in Semilattices* (SCS) in several memos: 1. *The $\mathscr{CL}$-compactification of a continuous poset* (Sept. 10, 1981), 2. *Continuous posets: Injective hull and MacNeille completion* (Dec., 1981), 3. *Two remarkable continuous posets and an appendix to "The $\mathscr{CL}$-compactification and the injective hull of a continuous poset"* (July 28, 1982). I am greatly indebted to K. H. Hofmann for his comments on earlier drafts of this paper, in particular for providing 1.4 (which has led to an extension of my previous characterization of the injective hull of a continuous 1, $\wedge$-semilattice, stated in Theorem 3.6, to arbitrary continuous posets) and for eliminating a serious error (and, thereby, discovering the error in **[5]** Corollary 2, p. 240).

0. Prerequisites. Basic concepts and some lemmas. We restrict ourselves here to the basic definitions. For further information the reader may consult **[20]**. We add some lemmas (which are known, but not in **[20]**).

i) For an arbitrary partially ordered set ($\equiv$ *poset*, for short) $(P, \leqq)$ we have

$$x \ll y \quad (\text{“}x \text{ is } \textit{way below } y\text{”})$$

if and only if whenever $y \leqq \sup D$ (the supremum of D) for some non-empty, up-directed subset D (i.e., $a, b \in D$ implies $a, b \leqq c$ for some $c \in D$) of P, then $x \leqq d$ for some $d \in D$ (cf. **[57]**, p. 110).

ii) A poset $(P, \leqq)$ is said to be a *continuous poset* (**[59]**, **[49]**) if and only if

1) P is "*up-complete*", i.e., for every non-empty up-directed subset D of $(P, \leqq)$, $\sup D$ exists;

2) for every $x \in P$, $\{y \in P | y \ll x\}$ is non-empty and up-directed, and

$$\sup\{y \in P | y \ll x\} = x.$$

A *continuous lattice* ([**56**], [**57**]) is a continuous poset which is a complete lattice (or, equivalently, a 0, $\vee$-semilattice). In a continuous poset P, $\ll$ has the interpolation property: If $x \ll y$ in P, then $x \ll z \ll y$ for some $z \in P$ ([**49**], 2.5).

iii) For an arbitrary poset $(P, \leqq)$, a subset M is said to be open in the *Scott topology* (to be designated by σ_P) of P ([**57**], p. 101), if and only if

1) M is an *upper set*, i.e., $x \leqq y$, $x \in M$, $y \in P$ imply $y \in M$;

2) whenever sup $D \in M$ for a non-empty, up-directed subset D of P, then $D \cap M$ not $= \emptyset$.

For a continuous poset P, the sets of the form

$$\Uparrow x: = \{y \in P | x \ll y\}$$

with x ranging through P, form an open basis of the Scott topology σ_P. It results that, in a continuous poset, P, $x \ll y$ if and only if

$$y \in U \subseteq \uparrow x: = \{z \in P | x \leqq z\}$$

for some Scott-open subset U of P ([**49**], 3.2).

For up-complete posets P and Q, a map

$$f:(P, \sigma_P) \rightarrow (Q, \sigma_Q)$$

is continuous if and only if f preserves suprema of non-empty up-directed subsets, i.e., $f(\sup D) = \sup(f[D])$ for every non-empty up-directed subset D of P (cf. [**63**], 3.5).

iv) Every topology τ on a set M induces a pre-order ($=$ quasi-order), i.e., a transitive and reflexive relation, on this set

$$x \leqq y \text{ if and only if } x \in \text{cl}\{y\} \quad (x, y \in M),$$

the *specialization pre-order* ([**2**], IV, 4.2.2); this pre-order is antisymmetric (i.e., a partial order) if and only if (M, τ) is T_0. The *compatible* topologies on a pre-ordered set are those which induce the given pre-order. The Scott topology on a poset is always compatible.

The finest compatible topology of a poset P, the *A(lexandrov)-discrete topology* α_P on P, has as its open sets all upper sets of P [**1**].

There is also a weakest compatible topology on a poset P, the *weak topology* of P, which has the sets of the form

$$\downarrow x: = \{y \in P | y \leqq x\} \quad (x \in P)$$

as a subbasis of its closed sets ($=$ the *upper topology* $\nu(P)$ of P, [**20**], II-1.16). The weak topology on $(P, \leqq)^{\text{op}}$ ($=$ the "opposite" $(P, \leqq^*)$ of $(P, \leqq)$; where $x \leqq^* y$ if and only if $y \leqq x$) will be designated by ω_P ($=$ the *lower topology* of P, [**20**], III-1.1).

The common refinement $\sigma_P \vee \omega_P$ of the Scott topology σ_P of $(P, \leqq)$ and the weak topology ω_P of $(P, \leqq)^{\text{op}}$ is called the *$\mathcal{CL}$-topology* or the *Lawson topology* ([**20**], III-1.5); it will be designated here by ζ_P (instead of λ_P) in order to avoid any confusion with the space λX to be introduced below.

In view of the role played by ζ_L in continuous lattice theory, it may be noted that for a non-continuous compact Hausdorff topological 1, $\wedge$-semilattice (L, τ_L), τ_L is strictly finer than ζ_L (G. Gierz): By [**20**], 0-4.4, L is meet-continuous, hence ζ_L is T_2 if and only if L is continuous ([**20**], III-2.9). By ([**20**], VI-3.11) and (VI-1.6(ii)) ([**20**], VI-1.14) we have

$$\zeta_L = \sigma_L \vee \omega_L \subset \tau_L.$$

v) A topological space X is called *sober* ([**2**], IV, 4.2.1; cf. [**10**], II, p. 17 and also [**53**], [**22**, **23**, **24**]) if and only if every non-empty, irreducible, closed subspace A of X has a unique "generic" point x, i.e., a point x with $\mathrm{cl}\{x\} = A$. (A subspace A is irreducible if and only if it is not the union of two proper closed subsets.)

vi) A subset F of a 1, $\wedge$-semilattice L is a *filter of* L (or *in* L) if and only if F is an upper set, $1 \in F$, and $x \wedge y \in F$ whenever x and $y \in F$. Note that the improper filter $F = L$ is not excluded. The set ΦL of all filters of L, ordered by inclusion, is an algebraic lattice.

For a topological space X, let ΦX denote the lattice of all *open filters* of X, i.e., filters of $\mathfrak{O}(X)$, the lattice of open subsets of X (ordered by inclusion). The sets

$$\Phi_U := \{F \in \Phi X \mid U \in F\} \quad (U \in \mathfrak{O}(X))$$

form an open basis of the Scott topology of ΦX. The mapping $X \to \Phi X$ taking $x \in X$ into

$$\mathfrak{O}(x) = \{U \in \mathfrak{O}(X) \mid x \in U\},$$

the *open neighborhood filter* of x in X, induces on X the initial topology; it is an embedding if and only if X is a T_0-space, i.e., $x = y$ whenever $\mathfrak{O}(x) = \mathfrak{O}(y)$ for all $x, y \in X$ (cf. [**5**], Section 1).

vii) An open filter of a space X is said to be a *join filter* if and only if it is a join ($\equiv$ supremum), in ΦX, of a family of open neighborhood filters of X. For a T_0-space X, there is an embedding

$$\lambda_X : X \to \lambda X$$

into λX, the space of all join filters of X with the topology inherited from ΦX, taking $x \in X$ into $\mathfrak{O}(x)$. In [**5**], Section 2, B. Banaschewski shows that this is the greatest essential extension of X in the category $\mathbf{T}_0$. A continuous map $f:X \to Y$ is an *essential extension* if and only if

a) $f:X \to Y$ is an embedding ($\equiv$ extension),

b) whenever gf is an embedding for any continuous map $g:Y \to Z$, then g is an embedding.

A greatest essential extension is one from which any other essential extension may be obtained by co-restriction.

viii) Assigning to a join filter $\mathscr{F}$ of a T_0-space X the set

$$\text{conv}\,\mathscr{F} := \{x \in X | \mathfrak{O}(x) \subseteq \mathscr{F}\},$$

called the *convergence set* of $\mathscr{F}$ (even if $\mathscr{F}$ is improper), yields an equivalent representation

$$\gamma_X : X \rightarrow \gamma X$$

of the greatest essential extension of the T_0-space X. The points of γX are the convergence sets of X. (Note that, for every filter $\mathscr{G}$ on X,

$$\text{conv}\,\mathscr{G} = \text{conv}\,\mathscr{F}$$

with

$$\mathscr{F} := \sup\{\mathfrak{O}(x) \,|\, x \in X \text{ and } \mathfrak{O}(x) \subseteq \mathscr{G}\} \in \lambda X.)$$

The topology of γX may be obtained by transferring the topology of λX along the bijection

$$\text{conv}_X : \lambda X \rightarrow \gamma X.$$

Composing conv_X with $\lambda_X : X \rightarrow \lambda X$, we obtain the embedding

$$\gamma_X : X \rightarrow \gamma X, \quad x \mapsto \text{cl}\{x\}.$$

Cf. [**26**], Section 3.

Note that every convergence set is closed. Thus there is a canonical order-embedding of γX into the lattice $\mathfrak{A}(X)$ of all closed subsets of X.

The specialization order of ΦX, λX and γX is given by the inclusion relation.

ix) For posets P and Q, a map $f:P \rightarrow Q$ is *isotone* if and only if $f(x) \leqq f(y)$ for $x, y \in P$, whenever $x \leqq y$; f is an *order-embedding* (= *order-extension*) if and only if, for $x, y \in P$, $x \leqq y$ is equivalent to $f(x) \leqq f(y)$ (then f is necessarily one-to-one).

The following lemmas will be used in this paper on several occasions. (Some of them are consequences of the representation theory for partially ordered sets developed e.g. by J. R. Büchi [**11**], B. Banaschewski [**4**] and G. Bruns [**10**].)

0.1. LEMMA. *An embedding $X \rightarrow Y$ of topological spaces is an order-embedding with regard to the specialization pre-order of X and Y, respectively.*

A subset K of a poset Q is *join-dense* in Q if and only if every $x \in Q$ is a supremum of a subset of K. An order embedding $e:P \rightarrow Q$ is join-dense if and only if $e[P]$ is join-dense in Q.

The following result is (the dual of) [**26**], 2.8.

0.2. LEMMA. *Suppose K is a join-dense subset of a poset Q, then the weak topology ω_K of K^{op} is the trace $\omega_Q|K$ of the weak topology of Q^{op}.*

0.3. LEMMA. *Suppose $e:M \rightarrow N$ is a join-dense order-embedding, then e preserves all infima, to the extent they exist.*

Proof. Let $K \subseteq M$ and $\inf_M K = x \in M$. If $y \in N$ and y not $\leqq x$, then, for some $m \in M$, $m \leqq y$, but m not $\leqq x$. Thus m fails to be a lower bound of K in M. As a consequence, y cannot be a lower bound of K. By contraposition, this proves $x = \inf_N K$.

0.4. LEMMA. *The injective hull $P \rightarrow L$ of a continuous poset P preserves suprema of non-empty up-directed subsets.*

Proof. Every continuous map $(P, \sigma_P) \rightarrow (L, \sigma_L)$ between sober spaces preserves suprema of this type (cf. e.g. **[63]**).

The following lemma has an analogue for meet-continuous lattices in **[20]**, III-2.1(i), but needs a different proof for continuous posets (cf. **[20]**, p. 144/145).

0.5. LEMMA. *For a continuous poset P, $U \in \zeta_P$ implies that*

$$\uparrow U: = \{x \in P | y \leqq x \text{ for some } y \in U\} \in \sigma_P.$$

Proof. Suppose $y \leqq x$ for some $y \in U$, $x \in P$. Then there is a Scott-open set $\Uparrow v$ and $u_1, \ldots, u_n \in P \quad (n \geqq 0)$ such that

$$y \in \Uparrow v - (\uparrow u_1 \cup \ldots \cup \uparrow u_n) \subseteq U,$$

since the sets of this type form an open basis of (P, ζ_P). By the interpolation property there is some $z \in P$ with $v \ll z \ll y$, hence $z \in U$. It results that $\Uparrow z$ is a σ_P-open neighborhood of x contained in $\uparrow U$. As a consequence, $\uparrow U$ is σ_P-open.

0.6. LEMMA. *Suppose $e:P \rightarrow K$ and $j:P \rightarrow Q$ are order-embeddings such that $j[P]$ is join-dense in Q. Then there is at most one order-embedding $f:K \rightarrow Q$ with $fe = j$.*

Proof. Let $f:K \rightarrow Q$ be an order embedding with $fe = j$. Since f is isotone, $e(x) \leqq k$ implies $fe(x) = j(x) \leqq f(k)$ for every $x \in P$ and every $k \in K$, hence $f(k)$ is an upper bound of

$$M_k: = \{j(x) | x \in P, e(x) \leqq k\}.$$

On the other hand, $j(x) \leqq f(k)$ implies $e(x) \leqq k$, since f is an order-embedding and $j(x) = fe(x)$. Thus

$$M_k = \{j(x) | x \in P, j(x) \leqq f(k)\},$$

hence

$$f(k) = \sup M_k$$

is uniquely determined, since $j[P]$ is join-dense in Q, and the definition of M_k does not depend on f.

0.7. LEMMA. *Suppose L is a continuous lattice and $C \subseteq L$ is stable in L under the formation of arbitrary infima and under suprema of non-empty up-directed subsets. Then the inclusion $(C, \sigma_C) \rightarrow (L, \sigma_L)$ is a topological embedding.*

Proof. Since C is a complete lattice and since the inclusion $d:C \rightarrow L$ preserves arbitrary infima, d has a left adjoint $g:L \rightarrow C$ ("lower adjoint", [**20**], 0-3.4) which, of course, preserves arbitrary suprema ([**20**], 0-3.3). Since d is one-to-one, we have $g \circ d = \mathrm{id}_C$ ([**20**], 0-3.7). Both d and g are continuous with regard to the Scott topologies σ_C and σ_L of C and L, respectively. Thus we can infer from $g \circ d = \mathrm{id}_C$ that

$$d:(C, \sigma_C) \rightarrow (L, \sigma_L)$$

is a topological embedding.

1. Intrinsic characterization of the injective hull of a continuous poset.

1.0. Suppose $e:P \rightarrow L$ is (a representation of) the injective hull of the continuous poset P, then e is an order-embedding into the continuous lattice L satisfying

i) $e[P]$ is *join-dense* in L, i.e., every member of L is a join (= supremum) of a family of members of $e[P]$ (= the image of P under e).

This is readily clear from the very construction of λX as a subspace of the filter space ΦX (0.vii; [**5**], Section 2). Furthermore we have

ii) e preserves suprema of non-empty up-directed subsets (cf. 0.4).

1.1. LEMMA. *For a continuous poset P, the injective hull $e:P \rightarrow L$ preserves and reflects the way below relation, i.e., $x \ll_P y$ if and only if $e(x) \ll_L e(y)$ for every $x, y \in P$.*

Proof. The way below relation in P and L is denoted by $\ll_P$ and $\ll_L$, respectively.

We represent the injective hull of (P, σ_P) by Banaschewski's construction $\lambda(P, \sigma_P)$ ([**5**], Section 2, Proposition 2, p. 237) briefly described in 0.vii of this paper.

As noted in 0.iii, in a continuous poset P,

$$x \ll_P y$$

if and only if the following condition (**) is satisfied:

There is some σ_P-open set U with $y \in U$ and

$$x \leqq z$$

for every $z \in U$, i.e., every neighborhood of x contains U. As a consequence, in the injective hull $\lambda(P, \sigma_P)$ we have $\mathfrak{O}(y) \in \Phi_U$ and

$$\mathfrak{O}(x) \subseteq \mathscr{F}$$

for every join filter $\mathscr{F}$ contained in Φ_U.

Thus the description of $\ll$ (in 0.iii) applies again, now to the injective T_0-space $\lambda(P, \sigma_P)$, and we have

$$\mathfrak{O}(x) \ll \mathfrak{O}(y)$$

in $\lambda(P, \sigma_P)$.

The other implication is easily established along the same lines or may be, alternatively, deduced from 1.2.

1.2. LEMMA. *Suppose P and K are (up-complete) posets and $f:P \to K$ is an order-embedding preserving suprema of non-empty up-directed subsets, then f reflects the way below relation, i.e., $f(x) \ll_K f(y)$ implies $x \ll_P y$ for every $x, y \in P$.*

Proof. Suppose $y \leqq \sup_P D$ for a non-empty up-directed subset D of P, then $f(y) \leqq f(\sup_P D)$ by the isotonicity of f. Since

$$f(\sup_P D) = \sup_K f[D],$$

we have $f(x) \leqq f(d)$ for some $d \in D$, hence $x \leqq d$.

1.3. In the following, a subset M of a complete lattice L is said to *generate* (a complete lattice) $S \subseteq L$ if and only if S is the smallest subset of L which contains M and is stable in L under arbitrary infima and under suprema of non-empty up-directed subsets. If L is a continuous lattice, then (as is readily clear from the "equational" characterization of continuous lattices (cf. [**12**], [**20**], I-2.3)) so is S in the partial order inherited from L, i.e., S is the "subobject" of L "generated" by $M \subseteq L$ in the equational category **ContLat** of continuous lattices and those maps which preserve suprema of non-empty up-directed subsets and arbitrary infima. By the way note that neither condition (D) of [**12**], p. 53 (= (DD) in [**20**], I-2.3) nor (DD*) of [**20**], I-2.3 is an equational description of **ContLat**, because it involves an operation ("up-directed supremum") which is not everywhere defined. However, by [**12**] and [**48**], **ContLat** has an equational characterization, since it is monadic over the category **Ens** of sets and maps.

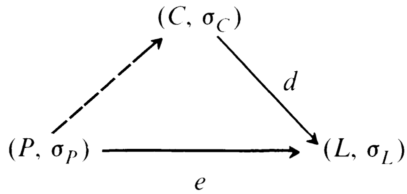
The following result has been observed by K. H. Hofmann in a seminar report [**34**] commenting on the first draft of the present paper.

1.4. LEMMA. *Suppose $e:P \to L$ is an injective hull of the continuous poset P. Then $e[P]$ generates L.*

Proof. Suppose $C \subseteq L$ contains $e[P]$ and is stable in L under the formation of suprema of non-empty up-directed subsets and under arbitrary infima. Then C is a continuous lattice in the partial order induced from L. By 0.7, the inclusion

$$d:(C, \sigma_C) \to (L, \sigma_L)$$

is a topological embedding. Thus we have a splitting



Since e is an essential extension and since d is also an embedding, we can infer from [5], Lemma 2 (p. 235) that d is an essential extension. The injective T_0-space (C, σ_C), of course, has no non-trivial essential extensions, hence $C = L$, as claimed.

1.5. LEMMA. *Suppose P is a continuous poset, and K is a complete lattice. Suppose there is an order-embedding $e:P \to K$ satisfying*

(i) *$e[P]$ is join-dense in K;*

(ii) *$e:P \to K$ preserves suprema of non-empty up-directed subsets;*

(iii) *$e:P \to K$ preserves the way below relation.*

Then K is a continuous lattice.

Proof. Since $e[P]$ is join-dense in K, for every $x \in K$ there is a family of elements $x_i \in P \quad (i \in I)$ with

$$x = \sup\{e(x_i) \mid i \in I\}$$

in K.

Since P is a continuous poset, we have, for every $i \in I$,

$$x_i = \sup\{y \in P \mid y \ll x_i \text{ in } P\}$$

in P. By hypothesis, $e:P \to K$ preserves the way below relation. Consequently, for

$$X: = \{y \in P \mid y \ll x_i \text{ in } P \text{ for some } i \in I\},$$

$e[X]$ consists of elements way below x in K. Now, we can infer from hypothesis (ii) that

$$\begin{aligned} x &= \sup\{e(x_i) \mid i \in I\} \\ &= \sup\{\sup\{e(y) \mid y \ll x_i \text{ in } P\} \mid i \in I\} \\ &= \sup(e[X]), \end{aligned}$$

since, for every $i \in I$, $\{y \in P \mid y \ll x_i \text{ in } P\}$ is a non-empty and up-directed subset of P.

1.6. LEMMA. *Suppose P and K are continuous posets, and $f:P \to K$ is an order-embedding such that*

a) *$f:P \to K$ preserves suprema of non-empty up-directed subsets, and*

b) *$f:P \to K$ preserves the way below relation.*

Then the Scott topology σ_P of P is the trace (via $f:P \to K$) of the Scott topology σ_K of K.

Proof. We observe first that $f:P \to K$ is Scott-continuous (cf. e.g. **[63]**). As noted in 0.iii, the sets

$$\Uparrow_P x: = \{y \in P | x \ll {}_P y\}$$

with x ranging through P form an open basis of the Scott topology σ_P. Thus it suffices to show that such a set is an inverse image of some Scott-open set of K. In order to see that

$$\Uparrow_P x = f^{-1}[\Uparrow_K f(x)],$$

let, firstly, $y \in \Uparrow_P x$, i.e., $x \ll {}_P y$, hence $f(x) \ll {}_K f(y)$ by hypothesis (b), i.e.,

$$y \in f^{-1}[\Uparrow_K f(x)].$$

Conversely, let

$$z \in f^{-1}[\Uparrow_K f(x)],$$

i.e., $f(x) \ll {}_K f(z)$. As a consequence, $x \ll {}_P z$ by 1.2.

Note that 1.6(b) is not a necessary requirement: For a continuous lattice L and every $x \in L$, $\uparrow x$ is stable in L under arbitrary infima and under suprema of non-empty up-directed subsets, hence $\uparrow x$ is a continuous lattice, which, by 0.7, inherits the Scott topology from σ_L. As a zero element of $\uparrow x$, x is compact in $\uparrow x$, but not necessarily in L.

1.7. THEOREM. *Suppose P is a continuous poset and K is a complete lattice. An order-embedding $e:P \to K$ is (equivalent to) the injective hull of P if and only if the following conditions are satisfied:*

(i) *$e[P]$ is join-dense in K;*

(ii) *$e:P \to K$ preserves suprema of non-empty up-directed subsets;*

(iii) *$e:P \to K$ preserves the way below relation;*

(iv) *$e[P]$ generates K.*

Proof. It remains to verify the sufficiency.

Note first that K is a continuous lattice, by 1.5. By 1.6, $e:(P, \sigma_P) \to (K, \sigma_K)$ is a topological embedding.

Let $j:P \to L$ denote the injective hull of P. Then, by the injectiveness of (K, σ_K) in $\mathbf{T}_0$, there is a continuous map

$$f:(L, \sigma_L) \to (K, \sigma_K)$$

rendering

$$\begin{array}{ccc} (L, \sigma_L) & & \\ {\scriptstyle j}\uparrow & \overset{f}{\searrow} & \\ & & (K, \sigma_K) \\ (P, \sigma_P) & \underset{e}{\nearrow} & \end{array}$$

commutative.

Since $j:(P, \sigma_P) \rightarrow (L, \sigma_L)$ is an essential extension and

$$e:(P, \sigma_P) \rightarrow (K, \sigma_K)$$

is an embedding, we can infer that

$$f:(L, \sigma_L) \rightarrow (K, \sigma_K)$$

is a (topological) embedding. Thus $f:L \rightarrow K$ is an order-embedding (by 0.1) which preserves suprema of non-empty up-directed subsets (by 0.iii).

Since $e[P] = fj[P]$ is join-dense in K by (i), $f[L]$ is also join-dense in K. Thus $f:L \rightarrow K$ preserves arbitrary infima, by 0.3. Consequently, f identifies L with a subposet $f[L]$ of K which contains $e[P]$ and is stable in K under arbitrary infima and under suprema of non-empty up-directed subsets. Thus, by (iv), $f[L] = K$, hence $f:L \rightarrow K$ is an isomorphism, as claimed.

1.8. COROLLARY. *Suppose P is a continuous poset, K is a complete lattice, and $e:P \rightarrow K$ is an order-embedding satisfying*

i) *$e[P]$ is join-dense in K,*

ii) *$e:P \rightarrow K$ preserves suprema of non-empty up-directed subsets,*

iii) *$e:P \rightarrow K$ preserves the way below relation.*

Then K is a continuous lattice, and the continuous lattice K' generated by $e[P]$ in K determines, by co-restriction of $e:P \rightarrow K$, an injective hull $e':P \rightarrow K'$.

Proof. By 1.5, K is a continuous lattice, hence so is K'. Now it suffices to show that $e':P \rightarrow K$ preserves the way below relation.

If $x \ll_P y$ in P, then $e(x) \ll_K e(y)$ in K by (iii). The inclusion $K' \rightarrow K$ is a join-dense order-embedding preserving suprema of non-empty up-directed subsets. Thus

$$e'(x) \ll_{K'} e'(y)$$

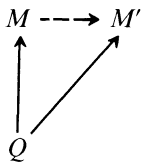
in K', by 1.2.

1.9. *Remark.* It is readily clear from the proofs that (ii) and (iii) in 1.7 and 1.8 can be replaced by the assumption that

a) K is a continuous lattice, and

b) $e:(P, \sigma_P) \rightarrow (K, \sigma_K)$ is a (topological) embedding with regard to the respective Scott topologies.

The *MacNeille completion* $Q \to M$ of a poset Q [**51**] is an order-embedding where M is a complete lattice and (the image of) Q is join-dense (and meet-dense) in M, and, whenever $Q \to M'$ is a join-dense completion of Q then there exists a (unique) order-embedding $M \to M'$ rendering



commutative. These properties characterize the MacNeille completion (cf. [**4**]).

The following result has been first established in [**32**].

1.10. COROLLARY. *For the MacNeille completion $e:P \to M$ of a continuous poset P, the following are equivalent*:

(a) *The MacNeille completion $e:P \to M$ coincides with the injective hull of P*;

(b) *$e:P \to M$ preserves the way below relation*;

(c) *M is a continuous lattice and*

$$e:(P, \sigma_P) \to (M, \sigma_M)$$

is a topological embedding.

Proof. It suffices to consider "(b) implies (a)" and "(c) implies (a)".

(b) $\Rightarrow$ (a): Condition (i) of 1.7 is clear. As for condition (ii) note that a meet-dense order-embedding preserves all suprema to the extent they exist (by 0.3) and that P is up-complete, by hypothesis. Since $e[P]$ is meet-dense in M, every subobject of M containing $e[P]$ coincides with M. Thus condition (iv) is established.

The proof of (c) $\Rightarrow$ (a) is similar (use 1.9).

1.11. *Remark.* The following examples show that one cannot dispense with any of the conditions (i), (ii), (iii) and (iv) in 1.7, even if K is required to be a continuous lattice.

(i) For a poset $P = \{a, b, c\}$ of three pairwise incomparable elements, let the set Q of all subsets of P be ordered by reverse inclusion. The completion

$$P \to Q, \quad x \to \{x\} \quad (x \in P)$$

preserves suprema of non-empty up-directed subsets as well as the way below relation. Since P is meet-dense in Q, every subobject of the continuous lattice Q containing P coincides with Q. However, the MacNeille completion M of P, which in this case coincides with the injective hull of P, is strictly smaller than Q, viz.

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$$M = \{\emptyset, \{a\}, \{b\}, \{c\}, P\}.$$

(Recall that every finite poset is a continuous poset in which the way below relation coincides with the order relation, since every finite non-empty up-directed subset contains a greatest element.)

(ii) For the set $\mathbf{N}_\infty$ of natural numbers, with a greatest element ∞ adjoined, in its usual order, let $\mathscr{L}_0(\mathbf{N}_\infty)$ denote the set of all non-empty lower sets of $\mathbf{N}_\infty$ (a subset M of a poset P is a *lower set* if and only if $a \in M, b \in P, b \leqq a$ imply $b \in M$): Both $\mathbf{N}_\infty$ and $\mathscr{L}_0(\mathbf{N}_\infty)$ are continuous lattices. The order-extension

$$e:\mathbf{N}_\infty \to \mathscr{L}_0(\mathbf{N}_\infty), \quad x \mapsto \downarrow x: = \{y \in \mathbf{N}_\infty | y \leqq x\}$$

is join-dense, preserves the way below relation, but fails to preserve the supremum of $(n)_{n \in \mathbf{N}}$. Since $\mathbf{N}$ is the only member of $\mathscr{L}_0(\mathbf{N}_\infty)$ which has not the form $\downarrow x$, there is no proper subobject of $\mathscr{L}_0(\mathbf{N}_\infty)$ containing the image of $\mathbf{N}_\infty$.

(iii) The MacNeille completion (object) M of a continuous poset P need not be a continuous lattice ([**15**], example 3, p. 53-54). Also, there are continuous posets P whose MacNeille completion $P \to M$ fails to be (equivalent to) the injective hull of P, although M is a continuous lattice; cf. [**32**] (see also 2.9 below).

(iv) Clearly $P^{\text{op}} \to Q^{\text{op}}$, with P, Q as in (i) and the orders reversed, is a join-dense completion of the continuous poset $P = P^{\text{op}}$ which preserves suprema of non-empty up-directed subsets and the way below relation, but, by (i), it fails to be the injective hull of P.

2. The $\mathscr{CL}$-compactification of a continuous poset. As in Section 1, $P \to L$ will denote an arbitrary representation of the injective hull of the continuous poset P. In order to simplify notation, in the proofs "embeddings" sometimes will be tacitly assumed to be inclusions of subsets (subposets, subspaces, etc.).

2.1. PROPOSITION. *The $\mathscr{CL}$-topology on a continuous poset P is the trace of the $\mathscr{CL}$-topology of the injective hull L of P.*

Proof. By [**27**], 3.14 (see introduction), we have an embedding

$$(P, \sigma_P) \to (L, \sigma_L)$$

for the Scott topologies σ_P and σ_L respectively, hence σ_P is the trace of σ_L on P. Thus it suffices to consider the trace of the weak topology ω_L of L^{op}. Since, by the very definition of λX, $\lambda_X:X \to \lambda X$ is a join-dense order-embedding (with regard to the respective specialization orders) for every T_0-space X, we have (by 0.2)

$$\omega_P = \omega_L|P.$$

Consequently,

$$\zeta_L|P = (\sigma_L \vee \omega_L)|P = (\sigma_L|P) \vee (\omega_L|P) = \sigma_P \vee \omega_P = \zeta_P.$$

Since the $\mathcal{CL}$-topology on a continuous lattice is compact Hausdorff ([**20**], III-1.10), we can infer the following result which has been obtained by K. H. Hofmann and M. W. Mislove ([**37**], 5.6) via a different, but similar argument.

2.2. COROLLARY. *A continuous poset P in its $\mathcal{CL}$-topology is a completely regular Hausdorff space.*

2.3. Every embedding $e:X \to Y$ of a space X into a compact Hausdorff space Y leads in a natural way to a Hausdorff compactification of X, viz. the closure of X in Y. Here we study, for a continuous poset P, the closure

$$C: = \text{cl } P$$

with regard to the $\mathcal{CL}$-topology of the injective hull L of P.

By an abuse of language, the poset C, equipped with the partial order induced from L, and the order-embedding $P \to C$ will be both referred to as the *$\mathcal{CL}$-compactification* of the continuous poset P.

For a continuous poset P with compact $\mathcal{CL}$-topology ζ_P, the $\mathcal{CL}$-compactification of P is the identity on P.

Recall that the *ℓ-topology* [**58**] associated with a space X is the topology on (the underlying set of) X, an open subbasis of which consists of all open sets and all closed sets of X.

2.4. LEMMA. *The $\mathcal{CL}$-topology on an arbitrary poset Q is weaker (= coarser) than the ℓ-topology associated with the Scott topology σ_Q of Q.*

Proof. Since, for every $x \in Q$,

$$Q - \uparrow x = \cup\{\downarrow y | y \in Q, x \text{ not } \leqq y\},$$

every subbasic $\mathcal{CL}$-open set $Q - \uparrow x$ is a union of point-closures, with regard to the Scott topology σ_Q, $\downarrow y$; hence it is a union of subbasic ℓ-open sets, hence ℓ-open.

By [**53**], 3.2 or [**22**], (1.1, 1.5), a subspace of a sober space X is sober if and only if it is *ℓ-closed*, i.e., closed in the ℓ-topology of X.

2.5. PROPOSITION. *The space $(C, \sigma_L|C)$ is sober and has an injective hull in $\mathbf{T}_0$.*

Proof. a) By definition, C is $\mathcal{CL}$-closed in L. Hence, by 2.4, it is also ℓ-closed in (L, σ_L). Since (L, σ_L) is sober, so is $(C, \sigma_L|C)$, by the remark preceding the proposition.

b) Since the composite of the embeddings

$$(P, \sigma_P) \to (C, \sigma_L|C) \to (L, \sigma_L)$$

is an essential extension, so is the second factor ([**5**], Section 1, Lemma 2, p. 235). Since (L, σ_L) is an injective T_0-space, the embedding

$$(C, \sigma_L|C) \to (L, \sigma_L)$$

is an injective hull.

2.6. PROPOSITION. *The $\mathscr{CL}$-compactification $P \to C$ of a continuous poset P coincides with the injective hull $P \to L$ of P if and only if C is a complete lattice.*

Proof. Since $(C, \sigma_L|C)$ is sober, the order-embedding $C \to L$ preserves suprema of non-empty up-directed subsets (cf. [**63**]). Since $P \to L$ is join-dense, so is $C \to L$. Thus $C \to L$ preserves arbitrary infima (by 0.3). If C is a complete lattice, the proof of 1.4 goes through, hence $C = L$.

2.7. *Remark.* If the trace $\sigma_L|C$ of the Scott topology σ_L of the injective hull L of a continuous poset P on its $\mathscr{CL}$-compactification C coincides with the intrinsic Scott topology σ_C of C, then

$$\zeta_L|C = \zeta_C.$$

Indeed,

$$\zeta_L|C = (\sigma_L \vee \omega_L)\,|C = \sigma_L|C \vee \omega_L|C = \sigma_C \vee \omega_C = \zeta_C,$$

since $\omega_L|C = \omega_C$ by 0.2. In particular, then, the inclusion

$$(P, \zeta_P) \to (C, \zeta_C)$$

is an embedding.

We now turn to some significant examples deferring the abstract justification for our definition of the $\mathscr{CL}$-compactification until Section 5.

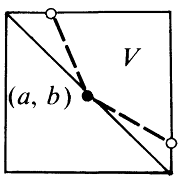
2.8. *Example.* In [**38**], K. H. Hofmann and M. W. Mislove show (in order to correct an error in an earlier draft of the present paper) that the poset P which is obtained from the square I^2 of $I = [0, 1]$ by deleting the square of $(0, 1/2]$ is a continuous poset P whose $\mathscr{CL}$-compactification C fails to be a continuous poset, but $\sigma_L|C = \sigma_C$. In [**35**], a modification of this example yields a poset P with a sober compatible topology having an injective hull in $\mathbf{T}_0$ such that (P, σ_P) fails to have an injective hull. The following further modification yields a continuous poset P whose $\mathscr{CL}$-compactification C is not a continuous poset such that (C, σ_C) fails to have an injective hull (whence $\sigma_L|C \neq \sigma_C$, by 2.5).

$$P = \{ (x, y) \in I^2 | x + y > 1\} \cup (\{0\} \times I) \cup (I \times \{0\})$$

where I denotes the unit interval [0, 1] and P receives the natural order from $I^2 = I \times I$. From 1.10 it is readily clear that the inclusion $P \to I^2$ is (both the MacNeille completion and) the injective hull of P. Since the $\mathscr{CL}$-topology of I^2 is the ordinary Euclidean topology, we have

$$C = P \cup \{ (x, y) \in I^2 | x + y = 1\}$$

for the $\mathscr{CL}$-compactification C of P. Modifying an argument of [**35**] we see that V in



is a σ_C-open neighborhood of $(a, b) \in C$ with $a + b = 1$ $(0 < a < 1)$ which is not the trace of a σ_L-open neighborhood of (a, b) (where $L: = I^2$). Indeed, it results from 6.1 (iii) below that (C, σ_C) fails to have an injective hull in $\mathbf{T}_0$.

2.9. *Example.* Let

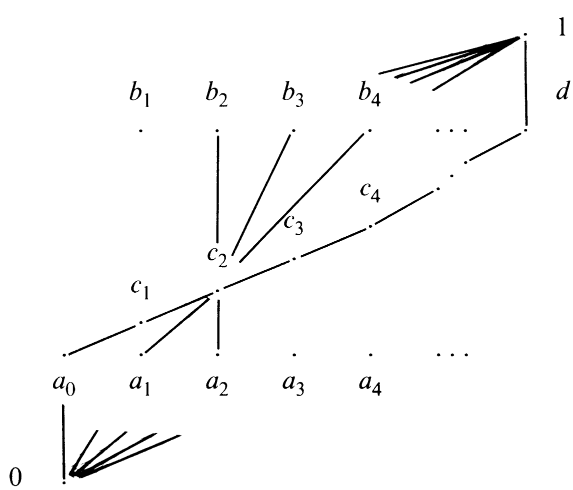
$$L = \{a_n | n \in \mathbf{N}\} \cup \{b_n | n \in \mathbf{N}\} \cup \{c_n | n \in \mathbf{N}\} \cup \{a_0, d, 0, 1\}$$

(with $\mathbf{N} = \{1, 2, 3, \dots \}$) be partially ordered by

$$a_k < c_l < b_m \text{ if and only if } k \leqq l \leqq m \text{ and } l, m \geqq 1$$

$$c_m \leqq c_n < d \text{ if and only if } 1 \leqq m \leqq n$$

such that 0, 1 are the smallest and the greatest element of L, respectively.



The only non-empty up-directed subsets of L which do not contain their supremum are cofinal subsets of

$$\downarrow c_1 \cup \downarrow c_2 \cup \downarrow c_3 \cup \dots$$

which have d as their supremum. Thus all elements of L are compact except for d. It results that L is a continuous lattice, indeed an algebraic lattice.

$$\{c_n\} = \uparrow c_n - (\uparrow b_n \cup \uparrow c_{n+1}) \quad (n \in \mathbf{N}).$$

Thus c_n is isolated in the $\mathscr{CL}$-topology of L.

Every neighborhood of d in (L, ζ_L) contains a basic $\mathscr{CL}$-open neighborhood of the form

$$\uparrow c_n - (\uparrow b_{k_1} \cup \dots \cup \uparrow b_{k_j})$$

for natural numbers $n, k_1, \dots, k_j$ (finitely many).

Every neighborhood of 0 in (L, ζ_L) contains a basic $\mathscr{CL}$-open neighborhood of the form

$$L - (\uparrow a_{k_1} \cup \dots \cup \uparrow a_{k_n})$$

for finitely many natural numbers $k_1, \dots, k_n$.

We consider the sub-poset P of L

$$P = \{a_0\} \cup \{a_n | n \in \mathbf{N}\} \cup \{b_n | n \in \mathbf{N}\}.$$

Since P satisfies the a.c.c. (= ascending chain condition), it is a continuous poset. The order-embedding

$$e:P \to L$$

is an injective hull of P by 1.7, since

(i) P is join-dense in L, since

$$c_n = a_n \vee a_{n-1}, \quad d = \sup\{c_n | n \in \mathbf{N}\},$$

$$1 = \sup\{b_n | n \in \mathbf{N}\}, \quad 0 = \sup \emptyset,$$

(ii) $e:P \to L$ preserves suprema of non-empty up-directed subsets,

(iii) $e:P \to L$ preserves the way below relation ($x \ll y$ in P if and only if $x \leqq y$),

(iv) P generates L, since

$$c_n = b_n \wedge b_{n+1}, \quad 0 = a_0 \wedge a_1, \quad 1 = \inf \emptyset$$

and d is the supremum of the chain $(c_n)_{n \in \mathbf{N}}$.

The $\mathscr{CL}$-compactification C of P is

$$C = P \cup \{0, d\}.$$

Note that C has the a.c.c., hence (C, σ_C) has an injective hull in $\mathbf{T}_0$, but

$\sigma_L|C \neq \sigma_C$. (Every σ_L-neighborhood of $d \in L$ contains some $\uparrow c_n$, hence it contains

$$\{b_n, b_{n+1}, \dots\}$$

for some n. Since d is compact in C, $\{d\}$ is σ_C-open, but it is not the trace of any σ_L-open set.)

Thus a continuous poset C with ascending chain condition may carry a compatible sober topology $\sigma_L|C$ different from the Scott topology σ_C in which it has an injective hull in $\mathbf{T}_0$ (by 2.5). (By 6.5, this phenomenon is excluded for continuous lattices.) This contradicts the present wording of [**27**], 3.14.

We note in passing that (here) the embedding $C \to L$ is the MacNeille completion, a fact which will be established in full generality in Section 5. Note that this is not the injective hull of the continuous poset C, since $\sigma_L|C \neq \sigma_C$. The poset $P \cup \{d\}$ and related posets have been introduced into the study of the MacNeille completion by M. Erné [**15**]. It may be worth pointing out that the MacNeille completion $M = L - \{d\}$ of P fails to be a continuous lattice (since every b_n fails to be compact, but has a direct predecessor, viz. c_n).

2.10. *Remark*. The preceding example 2.9 also shows that one implication of [**26**], 4.3 is false (this is irrelevant for [**26**], 4.2): A T_D-space (i.e., points are locally closed or, equivalently, the ℓ-topology is discrete, [**3**], [**60**], [**10**], II, p. 7; cf. also [**53**], [**23**, **24**]) may have an injective hull in $\mathbf{T}_0$ without carrying the Alexandrov-discrete topology:

$$\{d\} \cup \{c_n, c_{n+1}, \dots\} = \uparrow c_n \cap \downarrow d.$$

Thus d is isolated in the ℓ-topology of $(C, \sigma_L|C)$. Clearly, every compact element of L is ℓ-isolated in (L, σ_L), and conversely. Thus the ℓ-topology of $(C, \sigma_L|C)$ is discrete, but $\sigma_L|C$ fails to be Alexandrov-discrete.

2.11. *Example*. A finite poset P is compact Hausdorff in its $\mathcal{CL}$-topology, hence P coincides with its $\mathcal{CL}$-compactification C. If P fails to be a lattice or $P = \emptyset$, then C is different from the injective hull L of P.

2.12. *Example*. Let P be an *antichain* (= a poset in which $x \leqq y$ implies $x = y$ for every $x, y \in P$). Clearly, P is a continuous poset, since every non-empty up-directed subset of P is a singleton. The Scott topology of P is discrete, hence so is the $\mathcal{CL}$-topology. If card $P \geqq 2$, then the injective hull L of P is obtained by adding a greatest element 1 and a smallest element 0 (cf. [**26**], Section 5): $\{1\}$ is a Scott-open set, whereas $P \cup \{0, 1\}$ is the only neighborhood of 0 in the Scott topology. Thus, if P is infinite, the $\mathcal{CL}$-compactification C of P is $P \cup \{0\}$ with $0 \leqq x$ for every $x \in P \cup \{0\}$.

Thus, for every infinite antichain P, we have $P \neq C \neq L$.

3. The $\mathscr{CL}$-compactification of a continuous 1, $\wedge$-semilattice. A *continuous* 1, $\wedge$*-semilattice* is a continuous poset which is a 1, $\wedge$-semilattice (in the given partial order).

By [**20**], III-2.13, a continuous lattice L is a topological 1, $\wedge$-semilattice in its $\mathscr{CL}$-topology, i.e., the binary infimum $\wedge : L \times L \to L$ is a continuous map (with regard to the product topology).

3.1. PROPOSITION. *A continuous* 1, $\wedge$*-semilattice S is a completely regular Hausdorff topological* $\wedge$*-semilattice in its* $\mathscr{CL}$*-topology.*

Proof. The injective hull $S \to L$ is join-dense (1.0(i)), hence it preserves all infima to the extent they exist (by 0.3). Thus $S \to L$ preserves 1 and $\wedge$, i.e., S is a 1, $\wedge$-subsemilattice of L. Since (L, ζ_L) is compact Hausdorff ([**20**], III-1.10), the assertion is now clear from 2.1.

A compact Hausdorff (semi-)topological 1, $\wedge$-semilattice is a complete lattice ([**20**], VI-1.13(v)). Thus we have

3.2. PROPOSITION. *A continuous*, 1, $\wedge$*-semilattice is a continuous lattice if and only if it is compact in its* $\mathscr{CL}$*-topology.*

3.3. THEOREM. *The* $\mathscr{CL}$*-compactification* $S \to C$ *of a continuous* 1, $\wedge$*-semilattice S coincides with the injective hull of S. In particular, C is a continuous lattice.*

Proof. It is well known that the closure of a $\wedge$-subsemilattice of a topological $\wedge$-semilattice is a $\wedge$-(sub-)semilattice. Thus C is a compact Hausdorff topological 1, $\wedge$-semilattice in the topology induced from the $\mathscr{CL}$-topology ζ_L of the injective hull L of S, hence C is a complete lattice. Now 2.6 applies.

3.4. COROLLARY. *A continuous*, 1, $\wedge$*-semilattice S is dense in its injective hull L with regard to the* $\mathscr{CL}$*-topology of L.*

3.5. *Remark.* Examples of continuous 1, $\wedge$-semilattices which are not continuous lattices naturally arise in the study of the "dual" of a continuous poset (in the sense of [**46**], [**30**]: the dual of a continuous 1, $\wedge$-semilattice is a continuous 1, $\wedge$-semilattice).

The dual of the unit interval may be represented as $(0, 1] \cup \{2\}$ (in its natural order) and fails to be a complete lattice, hence, by 3.2, it is non-compact in its $\mathscr{CL}$-topology (= Euclidean topology). (Cf. [**46**], 9.6, [**30**], 3.13 (b) for more detailed information.)

By 3.3 and 1.7 we have the following criterion which will be used in [**33**] (Theorem 2.5).

3.6. THEOREM. *Suppose S is a continuous* 1, $\wedge$*-semilattice and L is a complete lattice. An order-embedding* $e:S \to L$ *is* (*equivalent to*) *the injective hull of S if and only if*

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(i) *$e[S]$ is join-dense in L;*

(ii) *$e:S \to L$ preserves suprema of non-empty up-directed subsets;*

(iii) *$e:S \to L$ preserves the way below relation;*

(iv) *$e[S]$ is dense in (L, ζ_L).*

Proof. A subset of a continuous lattice which is stable under arbitrary infima and under suprema of non-empty up-directed subsets is σ_L-closed ([**20**], III-1.11). Thus (iv) implies 1.7 (iv).

3.7. *Remark.* In view of 1.9, conditions (ii) and (iii) of 3.6 can be replaced by the requirement that L is a continuous lattice and $e:(S, \sigma_S) \to (L, \sigma_L)$ is an embedding for the Scott topologies.

3.8. THEOREM. *Suppose S is a continuous 1, $\wedge$-semilattice and L is a continuous lattice. An order-embedding $e:S \to L$ is (equivalent to) the injective hull of S if and only if*

(1) *$e[S]$ is join-dense in L;*

(2) *$e:(S, \zeta_S) \to (L, \zeta_L)$ is a topological embedding with regard to the respective $\mathscr{CL}$-topologies;*

(3) *$e[S]$ is (topologically) dense in (L, ζ_L).*

The proof of 3.8 is immediate from 3.6 and from the following Lemmas 3.9 and 3.10 which will be established under the following (more general) hypothesis: Suppose P and K are continuous posets and $e:P \to K$ is an order-embedding such that

(1) $e[P]$ is join-dense in K;

(2) $e:(P, \zeta_P) \to (K, \zeta_K)$ is a topological embedding;

(3) $e[P]$ is (topologically) dense in (L, ζ_L).

3.9. LEMMA. *Under the above hypotheses, we have*

(a) *$e:(P, \sigma_P) \to (K, \sigma_K)$ is continuous;*

(b) *$e:(P, \leqq) \to (K, \leqq)$ preserves suprema of non-empty up-directed subsets.*

Proof. (a) The Scott-open sets of an arbitrary poset are precisely the $\mathscr{CL}$-open upper sets ([**20**], III-1.21). Thus, by (2), the inverse image of a Scott-open subset of K is ζ_P-open. It is also an upper set, since the isotone maps are precisely the continuous maps for the respective Alexandrov-discrete topologies ([**1**]).

(b) This is immediate from (a), since a map between up-complete posets is Scott-continuous if and only if it preserves suprema of non-empty up-directed subsets ([**63**], 3.5).

3.10. LEMMA. *Under the above hypotheses, $e:P \to K$ preserves the way below relation, i.e., $x \ll_P y$ in P implies $e(x) \ll_K e(y)$ in K.*

Proof. We use $\leqq$, $\ll$ (without subscript), in order to designate the order and the way below relation in K, respectively; e.g.

$$\uparrow x = \{k \in K | x \leqq k\}.$$

We consider $e:P \rightarrow K$ as the (canonical) inclusion of a subset and thus omit the symbol e.

Suppose $x_0 \ll_P y_0$ in P. Since P is a continuous poset, this means (cf. 0.iii) that there is a σ_P-open set U such that

$$y_0 \in U \subseteq P \cap \uparrow x_0.$$

Since U is σ_P-open, it is also ζ_P-open, hence $U = P \cap V$ for some ζ_K-open subset V by hypothesis (2). As a consequence, there are $a_0, a_1, \ldots, a_n \in K$ ($n \geqq 0$) such that

$$y_0 \in \Uparrow a_0 - (\uparrow a_1 \cup \ldots \cup \uparrow a_n) \subseteq V.$$

By way of contradiction, suppose now that

$$x_0 \text{ not } \leqq z$$

for every $z \in K$ with $z \ll y_0$.

By the interpolation property, there is some $z' \in K$ with

$$z \ll z' \ll y_0.$$

Clearly, z' not $\in \uparrow a_i$ for every $i \in \{1, \ldots, n\}$ (otherwise $y_0 \in \uparrow a_i$, since $z' \ll y_0$). Also, x_0 not $\leqq z'$ by assumption (since $z' \ll y_0$). Thus

$$\Uparrow z - (\uparrow x_0 \cup \uparrow a_1 \cup \ldots \cup \uparrow a_n)$$

is a non-empty, ζ_K-open set for every $z \in K$ with $z \ll y_0$, hence

$$P \cap \Uparrow z - (\uparrow x_0 \cup \uparrow a_1 \cup \ldots \cup \uparrow a_n)$$

is also non-empty, since, by hypothesis (3), P is a (topologically) dense subset of (K, ζ_K).

Since $a_0 \ll y_0$, this contradicts the fact that

$$P \cap \Uparrow a_0 - (\uparrow a_1 \cup \ldots \cup \uparrow a_n) \subseteq P \cap V = U \subseteq P \cap \uparrow x_0.$$

Thus $x_0 \leqq z$ for some $z \in K$ with $z \ll y_0$, hence $x_0 \ll y_0$, as claimed.

From the preceding Lemmas 3.9 and 3.10 together with 5.8 one may obtain a proof of the partial characterization of the $\mathscr{CL}$-compactification of a continuous poset mentioned in the introduction.

3.11. *Remark*. Every topological 1, $\wedge$-semilattice has a compact Hausdorff 1, $\wedge$-semilattice reflection, sometimes called the Bohr compactification. By the universal property it results that the Bohr compactification of $(S, 1, \wedge, \zeta_S)$ for a continuous 1, $\wedge$-semilattice S is a (dense) topological embedding. I wonder whether it is different from the injective hull of S.

4. The injective hull and the $\mathscr{C}\mathscr{L}$-compactification of an algebraic poset.

4.0. (a) An element a of an arbitrary poset P is *compact* if and only if $a \ll a$.

A poset $(P, \leqq)$ is said to be *algebraic* if and only if

i) P is up-complete, i.e., for every non-empty up-directed subset D, the supremum, sup D, exists,

ii) for every $x \in P$, the set

$$K_x := \{y \in P \mid y \text{ compact}, y \leqq x\}$$

is non-empty and up-directed, and

$$x = \sup K_x.$$

A poset P is an algebraic poset if and only if it is a continuous poset in which, for every $x, y \in P$, $x \ll y$ (if and) only if $x \leqq c \leqq y$ for some compact element c of P.

Concerning the definition of an algebraic poset, a caveat may be in order (which, mutatis mutandis, also applies to continuous posets): It may happen that all of the axioms for an algebraic poset are satisfied except that the sets K_x fail to be up-directed (**[50]**, 4.2 or **[49]**, 4.5). Even when "enough" compact elements are readily available, it sometimes remains a delicate problem to verify the up-directedness of the sets K_x.

The concept of an algebraic poset arose in theoretical computer science (**[50]**, **[54]**, cf. also **[14]**). It is a natural extension of the familiar notion of a (complete) "algebraic lattice" (cf. **[9]**, **[20]**, I-4).

(b) A subset J of a poset Q is an *ideal* if and only if J is a non-empty up-directed lower set of Q, cf. e.g. **[25]**, p. 126. (J is a lower set of Q if and only if it is an upper set of Q^{op}.) For an arbitrary poset Q, the set $\mathfrak{I}(Q)$ of all ideals of Q, partially ordered by the inclusion relation, is an algebraic poset; $\mathfrak{I}(Q)$ endowed with the Scott topology is the (*universal*) *sobrification space* ${}^s(Q, \alpha_Q)$ of (Q, α_Q), where α_Q denotes the Alexandrov-discrete topology on Q, cf. **[25]**, 2.1. Indeed, by **[50]**, 3.2, 3.3 or **[25]**, 1.10, 1.9, every algebraic poset P is isomorphic to $\mathfrak{I}(Q)$ for a unique (up to an isomorphism) poset Q, the subposet of all compact elements of P.

(c) Recall that a subset M of a poset Q is a *Frink ideal* if and only if for every finite subset F of M,

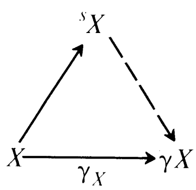
$$\{y \in Q \mid y \leqq x \text{ for every upper bound } x \text{ of } F \text{ in } Q\}$$

is contained in M. (A Frink ideal is not generally an ideal in the sense of **[25]**, Section 1. However, for 0, $\vee$-semilattices, the two concepts coincide with the ordinary concept of an ideal.)

It is known that the Frink ideals of a poset Q form, ordered by inclusion, an algebraic lattice $\mathfrak{F}(Q)$ (cf. **[4]**, p. 126): Clearly, $\mathfrak{F}(Q)$ is stable under arbitrary (set-) intersections. The (set-theoretic) union of every non-empty up-directed family of Frink ideals is a Frink ideal, hence it is the supremum of this family. It immediately results that the principal ideals $\downarrow x$ of P are compact in $\mathfrak{F}(Q)$ and every compact element of $\mathfrak{F}(Q)$ is a finite supremum, in $\mathfrak{F}(Q)$, of principal ideals (since it is clearly a supremum of some family of principal ideals).

By [**26**], 4.2, the maximal essential extension $\gamma(Q, \alpha_Q)$ of a poset Q in its Alexandrov-discrete topology α_Q is the Frink ideal completion $\mathfrak{F}(Q)$ of Q ([**19**]; cf. also [**4**]) and it carries the Scott topology.

4.1. Recall from [**26**], Section 3 that the greatest essential extension $\gamma_X: X \to \gamma X$ of a T_0-space X factors through the (universal) sobrification $X \to {}^sX$ of X (cf. [**2**], IV-4.2.1)



By [**5**], Section 1, Lemma 2 (p. 235), ${}^sX \to \gamma X$ is an essential extension, and every essential extension of ${}^sX$ induces, by composition, an essential extension of X. Thus ${}^sX \to \gamma X$ is (equivalent to) the greatest essential extension of ${}^sX$. In particular,

$$\gamma({}^sX) \cong \gamma X.$$

(Also, X has an injective hull in $\mathbf{T}_0$ if and only if so has ${}^sX$.)

4.2. THEOREM. *The injective hull of an algebraic poset P is an algebraic lattice, the lattice of Frink ideals of the sub-poset of compact elements of P.*

Proof. In 4.1, let $X = (Q, \alpha_Q)$ where Q denotes the subposet of compact elements of P and α_Q is the Alexandrov-discrete topology of Q; then ${}^sX \cong (P, \sigma_P)$ by [**25**], (2.1) and γX is the Frink ideal completion $\mathfrak{F}(P)$ of P, an algebraic lattice, endowed with its Scott topology ([**26**], 4.2).

It is noted in [**38**] that the $\mathcal{CL}$-compactification of an algebraic poset need not be a continuous poset: Replace the unit interval in the construction in 2.8 by the Cantor discontinuum.

4.3. PROPOSITION. *Suppose C is the $\mathcal{CL}$-compactification of an algebraic poset P. Then every element of C is a supremum of compact elements.*

Proof. By 4.2, every element of the injective hull L of P is a supremum of compact elements of L which are contained in P. By 1.2 these are compact in C, since the embedding $C \to L$ preserves suprema of non-empty up-directed subsets (by sobriety of $(C, \sigma_L|C)$, 2.5).

In the following, we provide a representation of the $\mathcal{CL}$-compactification C of the algebraic poset P in terms of the subposet of compact elements of P.

4.4. LEMMA. *For the $\mathcal{CL}$-topology of an algebraic poset P, the sets*

$$\uparrow c \text{ and } L - \uparrow c$$

with c ranging through the compact elements of P, form an open subbasis. Thus an algebraic poset is 0-dimensional in its $\mathcal{CL}$-topology (i.e., has a basis of open-closed sets).

Proof. (cf. [**20**], III-1.12). a) For a continuous poset P, the sets

$$\Uparrow x = \{y \in P | x \ll y\}$$

form an open basis of the Scott topology σ_P (cf. 0.iii). If P is algebraic, then we may interpolate a compact element c such that $x \ll c \ll y$. Thus the sets of the form

$$\uparrow c = \{y \in P | c \leqq y\}$$

with c compact, form an open basis of the Scott topology of an algebraic poset P.

b) Since, in an algebraic poset P, $x = \sup K_x$ for $x \in P$, we have

$$\uparrow x = \cap \{\uparrow c | c \in K_x\}.$$

Thus the sets of the form

$$L - \uparrow c$$

with c compact, form an open subbasis of ω_P, the weak topology of P^{op}.

c) Combining a) and b), we see that the $\mathcal{CL}$-topology

$$\zeta_P = \sigma_P \vee \omega_P$$

is 0-dimensional.

As noted above, a subspace of a sober space is sober if and only if it is ℓ-closed. Thus, for every T_0-space X, ${}^sX$ is ℓ-closed in γX. On the other hand, the embedding $X \to {}^sX$ is ℓ-dense ([**53**], [**22**], 1.1). (Clearly, the ℓ-topology of a subspace is the relative ℓ-topology.) Consequently ${}^sX$ is the ℓ-closure of X in γX.

4.5. LEMMA. *Suppose X is a T_0-space such that ${}^sX$ is a continuous poset in its Scott topology. Then the $\mathcal{CL}$-closure of ${}^sX$ in γX, i.e., the $\mathcal{CL}$-compactification of the continuous poset underlying ${}^sX$, coincides with the $\mathcal{CL}$-closure of X in γX.*

Proof. Note first that γX is a continuous lattice in its Scott topology. By 2.4, the $\mathcal{CL}$-topology of the continuous lattice underlying γX is coarser than the ℓ-topology. Thus the ℓ-closure of a subset is contained in the

$\mathcal{CL}$-closure. Consequently, the $\mathcal{CL}$-closure of ${}^{s}X$ (= the ℓ-closure of X) in γX coincides with the $\mathcal{CL}$-closure of X in γX.

As in 4.2 we represent the injective hull of an algebraic poset $\mathfrak{I}(P)$ by means of the Frink ideal completion $\mathfrak{F}(P)$ of P, where P is an arbitrary poset. By the remarks in 4.0 (b), (c) we can infer from 4.5:

4.6. LEMMA. *For an arbitrary poset P, the $\mathcal{CL}$-compactification of $\mathfrak{I}(P)$ is the $\mathcal{CL}$-closure of $\{\downarrow x | x \in P\}$ in the Frink ideal completion $\mathfrak{F}(P)$ of P.*

4.7. PROPOSITION. *For an arbitrary poset P, a subset K of P belongs to the $\mathcal{CL}$-compactification of $\mathfrak{I}(P)$ if and only if the following condition (*) is fulfilled:*

Whenever $x_1, \dots, x_m \in K$ and $y_1, \dots, y_n$ not $\in K$, then there is some $z \in P$ such that

$$x_i \leqq z \text{ for every } i \in \{1, \dots, m\},$$

and

$$y_k \text{ not} \leqq z \text{ for every } k \in \{1, \dots, n\},$$

where m and n are natural numbers $\geqq 0$.

Proof. Note first that a subset K of P is a Frink ideal if and only if (*) is satisfied for $n = 1$ and all natural numbers m.

We use the open subbasis of the $\mathcal{CL}$-topology on the algebraic lattice $\mathfrak{F}(P)$ described in 4.4. For a compact element C of $\mathfrak{F}(P)$ we have (as noted in 4.0 (c))

$$C = \downarrow x_1 \vee \dots \vee \downarrow x_k$$

where $x_1, \dots, x_k \in P$, $k \in \mathbf{N} \cup \{0\}$, and $\vee$ denotes the join in $\mathfrak{F}(P)$. Now it is readily clear from 4.4 that the sets

$$\mathfrak{U}(x) := \{M \in \mathfrak{F}(P) \mid \downarrow x \subseteq M\} = \{M \in \mathfrak{F}(P) \mid x \in M\} \text{ and}$$
$$\mathfrak{V}(x) := \{M \in \mathfrak{F}(P) \mid \downarrow x \text{ not} \subseteq M\} = \{M \in \mathfrak{F}(P) \mid x \text{ not} \in M\}$$

with x ranging through P, form an open subbasis of the $\mathcal{CL}$-topology of $\mathfrak{F}(P)$, from which an open basis is obtained by taking intersections of any finite number of members.

By 4.6, the $\mathcal{CL}$-compactification of $\mathfrak{I}(P)$ is the $\mathcal{CL}$-closure of $\{\downarrow x | x \in P\}$ in $\mathfrak{F}(P)$. Clearly, a Frink ideal K of P belongs to the $\mathcal{CL}$-closure of $\{\downarrow x | x \in P\}$ if and only if every basic $\mathcal{CL}$-open neighborhood

$$\mathfrak{U}(x_1) \cap \dots \cap \mathfrak{U}(x_m) \cap \mathfrak{V}(y_1) \cap \dots \cap \mathfrak{V}(y_n)$$

of K contains some $\downarrow z$, $z \in P$, i.e., if and only if condition (*) is satisfied.

It is not known whether there are continuous non-algebraic posets in which the compact elements form a join-dense subset. In the following, we exclude an anomaly of this kind in a special context.

4.8. LEMMA. (cf. [**20**], II-1.23). *A continuous poset P is an algebraic poset if and only if the Scott topology σ_P of P is an algebraic lattice.*

Proof. If P is an algebraic poset, then $P \cong \mathfrak{X}(Q)$ for the subposet Q of compact elements of P and

$$(P, \sigma_P)\ 1 \stackrel{\wedge}{=} {}^s(Q, \alpha_Q),$$

by [**25**], 2.1. Since α_Q is stable under arbitrary intersections, it is a complete ring of sets, hence it is an algebraic lattice. This proves the assertion, since $\mathfrak{O}({}^sX) \cong \mathfrak{O}(X)$ for every space X.

Now suppose that σ_P is an algebraic lattice. Since P is a continuous poset, σ_P is completely distributive. A completely distributive algebraic lattice K is (isomorphic to) a complete ring of sets (K. H. Hofmann, see also [**16**] and references given there), i.e., σ_P is isomorphic to α_R, the Alexandrov-discrete topology of some poset R. From $\sigma_P \cong \alpha_R$ we can infer that (P, σ_P) is the sobrification space of (R, α_R), since (P, σ_P) is sober. Thus, by [**25**], 2.1, $P \cong \mathfrak{X}(R)$ is an algebraic poset.

A subset M of a topological space X is said to be "*saturated*" if and only if it is the intersection of its open neighborhoods, i.e., if and only if $y \in M$ and $y \leqq x \in X$ always imply $x \in M$ (cf. [**20**], V-5.2 and V-5.17) where $\leqq$ is the specialization (pre-) order of X. The *saturation* of a subset K of a space X is given by

$$\uparrow K = \{x \in X | a \leqq x \text{ for some } a \in K\},$$

the intersection of all open neighborhoods of K in X. (A space is T_1 if and only if every singleton is saturated; then every subset is saturated.) The saturated subsets of a space are precisely the upper sets with regard to the specialization pre-order.

A subset K of a space X is quasi-compact if and only if the saturation $\uparrow K$ of K in X is quasi-compact. By 0.5, in a continuous poset P, the σ_P-saturation of a ζ_P-open subset is σ_P-open.

4.9. LEMMA. *Suppose P is a continuous poset whose $\mathscr{CL}$-topology ζ_P is locally compact (Hausdorff). Then P is an algebraic poset if and only if (P, ζ_P) is a 0-dimensional space.*

Proof. (a) If P is algebraic, then (P, ζ_P) is 0-dimensional by 4.4.

(b) Suppose (P, ζ_P) is 0-dimensional.

Let V be open in (P, σ_P), and let $x \in V$. Then V is ζ_P-open, hence, by local compactness, in (P, ζ_P) there is a compact neighborhood K of x contained in V. Since (P, ζ_P) is 0-dimensional, there is an open-closed

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subset W_x of (P, ζ_P) with $W_x \subseteq K$ and $x \in W_x$, hence W_x is compact in (P, ζ_P). Since $\sigma_P \subseteq \zeta_P$, W_x is quasi-compact in (P, σ_P), hence, by the remarks preceding this proposition, the σ_P-saturation $\uparrow W_x$ of W_x is both open and quasi-compact in (P, σ_P).

As a consequence, $\uparrow W_x$ is an (algebraically) compact element in σ_P with $x \in \uparrow W_x \subseteq V$, hence

$$V = \cup\{\uparrow W_x | x \in X\}.$$

In all, this says that σ_P is an algebraic lattice, hence, by 4.8, $(P, \leqq)$ is an algebraic poset.

The proof of 4.9 uses an argument given in the proof of [**20**], III-2.16 "(4) implies (2)" (p. 157; remedying, incidentally, an inaccuracy of formulation).

4.10. PROPOSITION. *Suppose the $\mathcal{CL}$-compactification C of an algebraic poset P is a continuous poset. If the trace $\zeta_L|C$ of the $\mathcal{CL}$-topology ζ_L of the injective hull L of P on C coincides with the intrinsic $\mathcal{CL}$-topology ζ_C of C, then C is an algebraic poset.*

Proof. By 4.2, the injective hull of L of P is an algebraic lattice, hence (L, ζ_L) is 0-dimensional, by 4.4. Thus $(C, \zeta_L|C)$ is 0-dimensional and, by definition, compact Hausdorff. Since, by hypothesis, C is a continuous poset and $\zeta_L|C = \zeta_C$, the assertion is clear from 4.9.

5. The $\mathcal{CL}$-compactification of a continuous poset P corresponds to the Fell compactification of (P, σ_P).

5.0. The $\mathcal{CL}$-compactification of a continuous poset is, as we shall prove in the following, a special case of a construction due to J. M. G. Fell ([**17**], Section 2, [**18**]), to be referred to, by a certain abuse of language (since it is not an ordinary compactification), as the Fell compactification. (Fell provides, in a special case, an interpretation of his construction in functional-analytic terms, [**17**], 2.2.)

For a space X, Fell considers the lattice $\mathfrak{A}(X)$ of all closed subsets, ordered by the inclusion relation, and defines a certain topology on $\mathfrak{A}(X)$ for which the sets

$$U(C; V_1, \ldots, V_n) = \{A \in \mathfrak{A}(X) | A \cap C = \emptyset, A \cap V_i \neq \emptyset \text{ for } i = 1, \ldots, n\}$$

(with C quasi-compact and V_i open in X, $n \in \mathbf{N} \cup \{0\}$) form an open basis.

As noted in [**20**], p. 151/152, the *Fell topology* is, for a locally quasi-compact space X, the $\mathcal{CL}$-topology ζ of the lattice $\mathfrak{O}(X)$ of open subsets of X (ordered by inclusion) transferred to $\mathfrak{A}(X)$ along the bijection

$\mathfrak{O}(X) \to \mathfrak{A}(X)$, $V \mapsto X - V$, where a space X is said to be *locally quasi-compact* if and only if every point has a neighborhood basis consisting of quasi-compact (but not necessarily open) subsets.

5.1. *Definition*. For a locally quasi-compact space X, the *Fell compactification* is the map

$$x \to \mathfrak{H}(X), \quad x \mapsto \mathrm{cl}\{x\}$$

into the closure $\mathfrak{H}(X)$ of the set

$$\{\mathrm{cl}\{x\} \mid x \in X\}$$

with regard to the Fell topology of $\mathfrak{A}(X)$. The latter set $\mathfrak{H}(X)$ will be endowed with the partial order induced from $\mathfrak{A}(X)$ and with the trace of the Fell topology of $\mathfrak{A}(X)$.

Note that $\mathfrak{H}(X)$ is a compact p(artially) o(rdered) space in the sense of L. Nachbin ([**52**], [**20**], VI-1.1), since so is $(\mathfrak{O}(X), \zeta_{\mathfrak{O}(X)})$ by [**20**], VI-3.4 (i) and VI-1.14 (reversing the order is an admissible operation for p.o. spaces). If X is a T_0-space, then the map $X \to \mathfrak{H}(X)$, $X \mapsto \mathrm{cl}\{x\}$ is an order-embedding for the specialization order of X.

5.2. The Scott topology σ_P on a continuous poset P is a completely distributive complete lattice ([**46**], [**30**]), hence, a fortiori ([**20**], I-2.5), a continuous lattice. Since (P, σ_P) is sober, it results that (P, σ_P) is locally quasi-compact ([**20**], V-5.6). We apply Fell's construction to $X := (P, \sigma_P)$:

For a completely distributive lattice K, the $\mathscr{CL}$-topology ζ_K agrees with the $\mathscr{CL}$-topology $\zeta_{K^{\mathrm{op}}}$ of K^{op} ([**44**]; [**20**], VII-2.9 (2)). Thus the Fell compactification of X is the closure of

$$\{\mathrm{cl}\{x\} \mid x \in X\}$$

with regard to the $\mathscr{CL}$-topology of $\mathfrak{A}(X)$. Note that $\mathfrak{A}(X)$ is completely distributive, since the dual of a completely distributive lattice is also completely distributive ([**55**]); hence, a fortiori, $\mathfrak{A}(X)$ is a continuous lattice (cf. [**20**], I-3.15).

5.3. Recall that, for every T_0-space X, we have an embedding

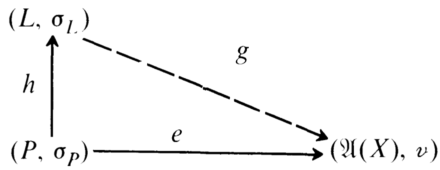
$$e: X \to (\mathfrak{A}(X), \upsilon), \quad x \mapsto \mathrm{cl}\{x\}$$

where υ denotes the weak topology on $\mathfrak{A}(X)$ (the "upper topology", [**20**], II-1.16). For a continuous poset P and $X = (P, \sigma_P)$, υ coincides with the Scott topology of $\mathfrak{A}(X)$ ([**20**], III-3.23(2) and IV-2.31), since $\mathfrak{A}(X)$ is completely distributive. Thus $(\mathfrak{A}(X), \upsilon)$ is an injective T_0-space.

Let h: $(P, \sigma_P) \to (L, \sigma_L)$ be any representation of the injective hull of (P, σ_P), then, since $(\mathfrak{A}(X), \upsilon)$ is injective, we obtain a continuous map

$$g:(L, \sigma_L) \to (\mathfrak{A}(X), \upsilon)$$

such that



commutes. Since e is an embedding and h is an essential extension, we infer that g is an embedding.

(a) Since g is Scott-continuous, g preserves suprema of non-empty, up-directed subsets.

(b) Since $e[P]$ is join-dense in $\mathfrak{A}(X)$, so is $g[L]$.

(c) Since g is a topological embedding, it is an order-embedding (with regard to the specialization order), hence, by 0.3, g preserves arbitrary infima.

This provides a representation

$$k: (P, \sigma_P) \to (K, \sigma_K), \quad x \mapsto \mathrm{cl}\{x\}$$

of the injective hull such that $K := g[L] \subseteq \mathfrak{A}(X)$ inherits its partial order from $\mathfrak{A}(X)$ and is stable in $\mathfrak{A}(X)$ under non-empty up-directed suprema and under arbitrary infima. By [**20**], III-1.11, the intrinsic $\mathscr{CL}$-topology ζ_K of K is the trace of the $\mathscr{CL}$-topology of $\mathfrak{A}(X)$, i.e., (K, ζ_K) is a closed subspace of $(\mathfrak{A}(X), \zeta_{\mathfrak{A}(X)})$. Consequently, the closure of a subset of K in (K, ζ_K) is the same as the closure in $(\mathfrak{A}(X), \zeta_{\mathfrak{A}(X)})$.

Thus we obtain (from 5.1 and 5.2)

5.4. THEOREM. *For a continuous poset P, the space $X = (P, \sigma_P)$ is locally quasi-compact. The Fell compactification*

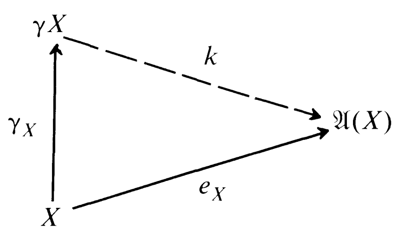
$$X \to \mathfrak{H}(X), \; x \mapsto \mathrm{cl}\{x\}$$

is (a representation of) the $\mathscr{CL}$-compactification of P and the topology of $\mathfrak{H}(X)$ is the trace of the $\mathscr{CL}$-topology of the injective hull K of P.

5.5. For the representation $\gamma_X:X \to \gamma X$ of the greatest essential extension of a T_0-space X, described in [**26**], Section 3, there is an obvious order-embedding

$$k: (\gamma X, \subseteq) \to \mathfrak{A}(X)$$

(the canonical inclusion) making

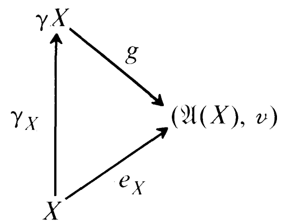


commutative, since every member of γX, i.e., every convergence set of X ([**26**], 3.9), is a closed subset of X. Since both γ_X and e_X are join-dense order-embeddings, by virtue of 0.6, k is uniquely determined as an order-embedding with $k\gamma_X = e_X$.

For a continuous poset P, we choose

$$(h: (P, \sigma_P) \to (L, \sigma_L)) = (\gamma_X : X \to \gamma X)$$

in 5.3. From the commutativity of



we deduce, by the uniqueness of k, that $g = k$, since a (topological) embedding is an order-embedding with regard to the induced specialization order.

In other words: the canonical order-embedding

$$\gamma X \to (\mathfrak{A}(X), \nu)$$

is (topologically) an embedding, if $X = (P, \sigma_P)$ for a continuous poset P. For general (non-Hausdorff) T_0-spaces X, this is still a delicate problem: All we know is that the topology of γX is coarser than the topology which γX inherits from $(\mathfrak{A}(X), \nu)$; cf. [**26**], 3.3, 3.14(2).

5.6. In [**31**], Section 3, there is associated to an arbitrary T_0-space X a T_0-space ψX and a (topological) embedding $\psi_X : X \to \psi X$ which is a corestriction of the greatest essential extension $\gamma_X : X \to \gamma X$, i.e., γX contains ψX as a subspace. It is shown there that

a) for a locally quasi-compact T_0-space X, $\psi_X : X \to \psi X$ is, on the level of the specialization order, the order-embedding induced by the Fell compactification $X \to \mathfrak{H}(X)$, $x \to \mathrm{cl}\{x\}$, where (as proposed in 5.1) $\mathfrak{H}(X)$ is considered as a compact partially ordered space ([**31**], 3.13);

b) the embedding $\psi X \to \gamma X$ is, on the level of the specialization order, the MacNeille completion ([**31**], 3.7.1).

By 5.4 and 5.5, we deduce from 5.6 (a):

5.7. LEMMA. *For a continuous poset P and $X := (P, \sigma_P)$, the embedding $\psi_X : X \to \psi X$ is (equivalent to) the $\mathcal{CL}$-compactification endowed with certain topologies*

$$(P, \sigma_P) \to (C, \sigma_L | C),$$

where L denotes the injective hull of P.

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Now it results from 5.6 (b)

5.8. THEOREM. *Let $P \to C$ denote the $\mathscr{CL}$-compactification of a continuous poset P, and let $P \to L$ denote the injective hull. Then the canonical embedding $C \to L$ is the MacNeille completion.*

5.9. From 5.8 it results that both $\sigma_L|C$ and $\zeta_L|C$ are intrinsic topologies of the $\mathscr{CL}$-compactification C of P. This justifies our definition of the $\mathscr{CL}$-compactification to be (merely) an order-extension.

5.10. COROLLARY. *Suppose a continuous poset P is compact (Hausdorff) in its $\mathscr{CL}$-topology, then the injective hull of P coincides with the MacNeille completion.*

Clearly, compactness of the $\mathscr{CL}$-topology of a continuous poset P is not a necessary requirement in order that the injective hull of P be its MacNeille completion (2.12). Indeed, this latter property is not an invariant of the $\mathscr{CL}$-topology, since both a (countably infinite) antichain and the poset P of 2.9 are discrete in their $\mathscr{CL}$-topology.

For further information on the extension

$$\psi_X: X \to \psi X$$

see [**31**], Section 3 and [**33**] where also the relationship of the work of K. H. Hofmann and J. D. Lawson [**36**] (Section 8) on pseudo-(meet-)prime elements (reported in [**20**], V-3) to this construction is explained.

6. Appendix: T_0-spaces which have an injective hull in the category $\mathbf{T}_0$. In [**5**], Section 2, B. Banaschewski showed that in the category $\mathbf{T}_0$ of T_0-spaces and continuous maps every space X has a greatest essential extension

$$\lambda_X: X \to \lambda X.$$

He also provides a criterion ([**5**], Section 3, Corollary 2, p. 240) which is sufficient in order to ensure that λX is an injective T_0-space, i.e., in order that X has an injective hull in $\mathbf{T}_0$. In [**38**], K. H. Hofmann and M. W. Mislove provided a counterexample to show that (other than claimed in [**5**]) this is not a necessary requirement (cf. also 2.8, 2.9 of the present paper). In the following, we provide necessary and sufficient conditions for a T_0-space X in order to have an injective hull

$$\lambda_X: X \to \lambda X.$$

We take [**5**], Section 2 for granted, but no information from [**5**], Section 3 will be used. For a somewhat different approach see [**35**].

Also, we comment on several results in [**27**], [**30**], and also on [**26**], 4.3 which are based on [**5**], Corollary 2, p. 240 and are therefore also in need of reformulation.

6.0. For a T_0-space X, λX is, by the very construction (cf. 0.vi, vii), stable in ΦX under the formation of arbitrary joins (= suprema). Thus there is a "kernel operator" $k:\Phi X \to \lambda X$ assigning to every open filter F of X the greatest join filter

$$\sup\{\mathfrak{O}(x) \,|\, x \in X, \mathfrak{O}(x) \subseteq F\}$$

contained in F. This map k is left inverse to the embedding $\lambda X \to \Phi X$. (Indeed, by **[5]** Proposition 3, p. 239, $k:\Phi X \to \lambda X$ is the only continuous left inverse of the embedding $\lambda X \to \Phi X$ if there exists any.)

Note that

$$\sup\{\mathfrak{O}(x) \,|\, x \in S\} = \{V \in \mathfrak{O}(X) \,|\, \text{there are } x_1, \dots, x_n \in S \ (n \geqq 0) \text{ and open neighborhoods } U_1, \dots, U_n \text{ of } x_1, \dots, x_n, \text{ respectively, with } U_1 \cap \dots \cap U_n \subseteq V\}$$

for every subset S of x.

6.1. THEOREM. *For a T_0-space X, the following are equivalent*:

(i) *The greatest essential extension λX of X is an injective T_0-space.*

(ii) *There is a (topological) embedding $e:X \to J$ into an injective T_0-space J which is join-dense with regard to the specialization partial order of J.*

(iii) *For every $x \in X$ and every open neighborhood V of x in X there exists an open neighborhood W of x in X, finitely many elements $y_1, y_2, \dots, y_n$ ($n \geqq 0$) of X and open neighborhoods $U_1, U_2, \dots, U_n$ of $y_1, \dots, y_n$, respectively, such that for every $i = 1, \dots, n$*

$$W \subseteq \uparrow y_i = \{z \in X \,|\, y_i \leqq z\}$$

for the specialization order $\leqq$ of X, and

$$U_1 \cap \dots \cap U_n \subseteq V.$$

Proof. (i) implies (ii): Evidently, $\lambda_X:X \to \lambda X$ is, by the very construction, join-dense with regard to the specialization order (which coincides with the inclusion relation of λX and ΦX, respectively).

(ii) implies (iii): By Scott's result, **[57]**, 2.12 (**[20]**, II-3.8), J is a continuous lattice L endowed with its Scott topology σ_L. The sets

$$\Uparrow q = \{p \in L | q \ll p\} \quad (q \in L)$$

form an open basis of σ_L (**[20]**, II-1.10 (i)). We may clearly restrict ourselves to the basic open subsets of X,

$$V = X \cap \Uparrow q$$

with q ranging through L.

Suppose $x \in V = X \cap \Uparrow q$ for some $q \in L$. By the interpolation property of $\ll$ in a continuous lattice (**[20]**, I-1.18), there is some $p \in L$ with $q \ll p \ll x$ in L, hence

$$x \in W := X \cap \Uparrow p \subseteq V.$$

Since, by hypothesis, $e:X \to J$ is join-dense, we have

$$p = \sup\{s \in X | s \leqq p\}.$$

On the other hand, since L is a continuous lattice, we have

$$y = \sup\{t \in L | t \ll y \text{ in } L\}$$

for every $y \in L$. Consequently,

$$p = \sup\{t \in L | t \ll y \leqq p \text{ for some } y \in X\}.$$

Since $q \ll p$, it results that there are finitely many $t_1, \ldots, t_n \in L$ ($n \geqq 0$) and $y_1, \ldots, y_n \in X$ with

$$q \leqq \sup\{t_1, \ldots, t_n\}$$

and

$$t_i \ll y_i \leqq p$$

for $i = 1, \ldots, n$. It results that every neighborhood, in X, of y_i contains $W = X \cap \Uparrow p$, and there are open (in X) neighborhoods $U_i = X \cap \Uparrow t_i$ of y_i ($i = 1, \ldots, n$) with

$$U_1 \cap \ldots \cap U_n \subseteq V = X \cap \Uparrow q.$$

(iii) implies (i): We shall prove that the kernel operator $k:\Phi X \to \lambda X$ is a continuous map, hence a retraction in $\mathbf{T}_0$. Since ΦX is an injective T_0-space, then so is its retract λX.

Suppose F is any open filter of X and, for some $V \in \mathfrak{O}(X)$,

$$k(F) \in \Phi_V.$$

Then there are $x_1, \ldots, x_m$ ($m \geqq 0$) and open neighborhoods $V_1, \ldots, V_m$ of $x_1, \ldots, x_m$, respectively, with

$$\mathfrak{O}(x_i) \subseteq F$$

for every $i = 1, \ldots, m$ and

$$V_1 \cap \ldots \cap V_m \subseteq V.$$

By (iii), for every $i = 1, \ldots, m$ there is an open neighborhood W_i of x_i and finitely many elements $y_i^1, \ldots, y_i^{n(i)}$ and open neighborhoods $U_i^1, \ldots, U_i^{n(i)}$ of $y_i^1, \ldots, y_i^{n(i)}$, respectively, with

$$W_i \subseteq \uparrow y_i^j$$

or, equivalently,

$$\mathfrak{O}(y_i^j) \subseteq W_i^{\varphi}$$

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(where $W^{\varphi} = \{M \in \mathfrak{O}(X) \mid W \subseteq M\}$ denotes the smallest member of Φ_W, the open filter generated by W) for every $j = 1, \ldots, n(i)$, and

$$U_i^1 \cap \ldots \cap U_i^{n(i)} \subseteq V_i.$$

It results that

$$\mathfrak{O}(y_i^j) \subseteq (W_1 \cap \ldots \cap W_m)^{\varphi}$$

for every $i = 1, \ldots, m$ and every $j = 1, \ldots, n(i)$, and

$$\cap \{U_i^j | i = 1, \ldots, m \text{ and } j = 1, \ldots, n(i)\}$$
$$\subseteq V_1 \cap \ldots \cap V_m \subseteq V.$$

Thus

$$k(W^{\varphi}) = \sup\{\mathfrak{O}(y) \mid y \in X, \mathfrak{O}(y) \subseteq W^{\varphi}\}$$

for $W := W_1 \cap \ldots \cap W_m$ contains V. Consequently, (because k is isotone and Φ_V is an upper set,) we have

$$k(G) \in \Phi_V$$

for every $G \in \Phi_W$.

Since $W_i \in \mathfrak{O}(x_i) \subseteq F$ for every $i = 1, \ldots, m$, we can infer $W \in F$, hence $F \in \Phi_W$.

In all, this says that $k:\Phi X \to \lambda X$ is continuous (at F).

This completes the proof.

6.2. *Remarks.* i) Note that in 6.1 (iii) necessarily

$$W \subseteq V.$$

ii) Suppose $e:X \to J$ is a join-dense topological embedding into an injective T_0-space $J = (L, \sigma_L)$. Let L' be the continuous lattice generated by $e[x]$ in J (in the sense of 1.3). Then the induced map

$$e':X \to J' := (L', \sigma_{L'})$$

is the injective hull of X. (The arguments given in Section 1 go through.) Note that 6.1 "(i) if and only if (ii)" can be established by some of the arguments in 1.7 and thus requires no information on the greatest essential extension except that it (exists and) is join-dense with regard to the specialization order.

6.3. *Definition.* Suppose X is a T_0-space with an injective hull $X \to \lambda X$. We say that

$$\deg(X) \leqq r,$$

i.e., X has *degree* at most r (a natural number $\geqq 0$) if and only if 6.1 (iii) can be fulfilled for every point x in X and every open neighborhood V of x in X by some $n \leqq r$.

6.4. *Remark*. A T_0-space X with an injective hull satisfies $\deg(X) \leqq 1$ if and only if for every $x \in X$ and every open neighborhood V of x there is some open neighborhood W of x and some $y \in V$ with

$$W \subseteq \uparrow y = \{z \in X \mid y \in \mathrm{cl}\{z\}\}.$$

B. Banaschewski ([**5**], Corollary 2, p. 240) observes that this class of T_0-spaces has an injective hull in $\mathbf{T}_0$, and he claims the other implication to be true, too. The error is hidden in the proof of [**5**], Corollary 1, p. 239 (line 3 from below)

$$\mathfrak{O}(x) = \bigvee k(\mathfrak{F}\{U\})$$

need not be a set-theoretic union if λX is injective (but this is true if every join filter of X is a neighborhood filter, as it is assumed there).

In [**27**], 3.14 it is established that the continuous posets in their Scott topology are precisely those sober spaces X with an injective hull in T_0 satisfying $\deg(X) \leqq 1$. All the statements in [**30**] on spaces X with an injective hull are correct provided the additional hypothesis $\deg(X) \leqq 1$ is imposed. On the other hand, several results are correct as they stand, (sometimes) requiring a different proof, as will be seen in the following.

6.5. PROPOSITION. *Suppose a T_0-space X is a conditional* 0, $\bigvee$-*semilattice with regard to its specialization order. If X has an injective hull, then* $\deg(X) \leqq 1$.

Proof. A poset is a conditional 0, $\bigvee$-semilattice if every finite subset which has an upper bound has a supremum. In 6.1 (iii) one may put

$$y = \sup\{y_1, \dots, y_n\},$$

where the "sup" is taken in the specialization order. Then

$$W \subseteq \uparrow y \quad \text{and} \quad y \in U_1 \cap \dots \cap U_n \subseteq V.$$

We thus obtain from (the proof of) [**30**], 2.8.

6.6. COROLLARY. *A T_0-space X is injective if and only if*
i) *X is sober,*
ii) *X has an injective hull in $\mathbf{T}_0$, and*
iii) *X is a* 0, $\bigvee$-*semilattice in its specialization order.*

6.7. COROLLARY. *If a T_1-space X has an injective hull in $\mathbf{T}_0$, then X is discrete.*

Proof. Suppose X has at least two points. For $x \in X$ choose some neighborhood $V \neq X$ of x. Then choose $W \in \mathfrak{O}(X)$ and $y_1, \dots, y_n \in X$ ($n \geqq 0$) and $U_1, \dots, U_n$ as in 6.1 (iii).

Since every point-closure in a T_1-space is a singleton,

$$x \in W \subseteq \uparrow y_i = \{z \in X \mid y_i \in \mathrm{cl}\{z\}\}$$

implies (if $n \neq 0$) that $y_1 = \ldots = y_n = x$, hence $W = \{x\}$ is open. If $n = 0$, then

$$X = U_1 \cap \ldots \cap U_n \subseteq V,$$

contradicting the hypothesis that $X \neq V$.

6.8. COROLLARY. *Suppose A is a closed subspace of a T_0-space X. If X has an injective hull in $\mathbf{T}_0$, then so has A.*

Proof. In order to verify 6.1 (iii) let $x \in V' \in \mathfrak{O}(A)$. Then $V' = V \cap A$ for some $V \in \mathfrak{O}(X)$, and we may choose $W \in \mathfrak{O}(x)$, some points y_1, $y_2, \ldots, y_n$ $(n \geqq 0)$ in X and open neighborhoods $U_1, \ldots, U_n$ (in X) of $y_1, \ldots, y_n$, respectively, satisfying 6.1 (iii). The requirement

$$x \in W \subseteq \{z \in X | y_i \in \mathrm{cl}\{z\}\}$$

(for $i = 1, \ldots, n$) guarantees

$$y_i \in \mathrm{cl}\{x\} \subseteq A$$

so that we may use $W' = W \cap A$ and $U_i' := U_i \cap A$ in order to fulfill 6.1 (iii) for A instead of X.

6.9. PROPOSITION. *Suppose $(X_i)_{i \in I}$ is a family of T_0-spaces which have an injective hull in $\mathbf{T}_0$. Then $\prod_{i \in I} X_i$ has an injective hull provided that*

$$K(I) = \{i \in I | X_i \textit{ does not have a smallest element in its specialization order}\}$$

is finite.

Proof. (1) First note that if X and Y have an injective hull, then so has $X \times Y$ (use 6.1 (iii)).

(2) Suppose now $K(I) = \emptyset$ and let o_i denote the smallest element of X_i in its specialization order. By 6.1 (ii), there are injective T_0-spaces J_i and join-dense (topological) embeddings $X_i \to J_i$. Clearly, $\prod_{i \in I} J_i$ is injective. Let

$$a_i \in J_i \quad (i \in I),$$

then, by hypothesis,

$$a_i = \sup A_i$$

for some subset A_i of X_i. We may assume that $o_i \in A_i$, hence $A_i \neq \emptyset$. Then

$$(a_i)_{i \in I} = (\sup A_i)_{i \in I} = \sup\left(\prod_{i \in I} A_i\right).$$

This proves that $\prod_{i \in I} X_i$ is join-dense in $\prod_{i \in I} J_i$, hence it has an injective hull in $\mathbf{T}_0$ by 6.1 (ii).

Combining (1) and (2), we establish the assertion.

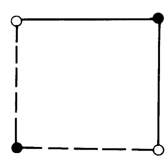
A product of discrete spaces may fail to be discrete, but it is always T_1. Thus (by 6.7) the class of all T_0-spaces with an injective hull in $\mathbf{T}_0$ fails to be productive. Incidentally, note that, by 6.1 (iii), the class of T_0-spaces with an injective hull in $\mathbf{T}_0$ is stable under coproducts (= sums).

The non-validity of one implication of [**5**], Corollary 2, p. 240 makes several results questionable which were based on this claim. (Note that all of these results are correct if one replaces the hypothesis "X has an injective hull in $\mathbf{T}_0$" by "X has the property stated in [**5**], Corollary 2, p. 240".) Contrary to [**26**], 4.3, a T_D-space with an injective hull in $\mathbf{T}_0$ need not be Alexandrov-discrete (cf. 2.10). K. H. Hofmann observes that the class of T_0-spaces with an injective hull is not open-hereditary (disproving [**5**], Corollary 4, p. 240). (With the notation of 6.10 below, $A - \{(0, 0)\}$ is an open subspace of B, but it fails to have an injective hull.) We modify his example in order to obtain

6.10. *Example.* By 6.1, the (ordinary) boundary B of the unit square $I^2 = I \times I$ endowed with the trace of the Scott topology of I^2 has an injective hull in $\mathbf{T}_0$, viz. I^2 in the Scott topology ([**35**]). The subspace

$$A = \{(0, 0)\} \cup ((0, 1] \times \{1\}) \cup (\{1\} \times (0, 1])$$

of B is stable in I^2 under arbitrary suprema, hence, by [**26**], 1.8, it is essentially complete, i.e., it has no nontrivial essential extension in $\mathbf{T}_0$. Since A fails to be a continuous lattice (in its specialization order), A does not have an injective hull in $\mathbf{T}_0$. However, the mapping $r:B \to A$ with $r(x) = x$ for $x \in A$ and $r(x) = (0, 0)$ for $x \in B - A$ is a continuous retraction



Thus the class of T_0-spaces with an injective hull in $\mathbf{T}_0$ (as well as the subclass consisting of those spaces which have a smallest element in their specialization order) fails to be stable under retracts. As a consequence, this class is neither a class of all injectives nor a class of all projectives with regard to any class of continuous maps. (The same applies to the sober members of this class; thus disproving the wording of [**27**], 3.17 (c).)

The requirement to have an injective hull in $\mathbf{T}_0$ does not impose any restriction on the specialization partial order of a T_0-space, since, for every poset P, (P, α_P) has an injective hull in $\mathbf{T}_0$ ([**26**], 4.2). For sober spaces, indeed for $\mathscr{d}$-spaces ([**63**]; "monotone convergence spaces", [**20**], II-3.9; i.e., T_0-spaces whose specialization order is up-complete and whose topology is coarser than the Scott topology of the specialization order), the situation is different. (Every sober space and every T_1-space is a $\mathscr{d}$-space, [**63**].) The following concept is implicit in [**49**].

6.11. *Definition.* A poset P is said to be *almost-continuous* if and only if it is up-complete and, for every $x \in P$,

$$x = \sup\{y \in P | y \ll x\}.$$

6.12. THEOREM. *Suppose a $\mathscr{d}$-space X has an injective hull in* $\mathbf{T}_0$, *then the specialization order of X is an almost-continuous poset.*

Proof. Every element F of λX is, by injectivity of λX, a supremum of elements way below F in λX. Since, by 1.0 (i), $\lambda_X:X \to \lambda X$, $x \mapsto \mathfrak{O}(x)$ is a join-dense order-embedding, it results that F is a supremum, in λX, of open neighborhood filters $\mathfrak{O}(x)$ which are way below F in λX. Since, by hypothesis, X is a $\mathscr{d}$-space, $\lambda_X:X \to \lambda X$ preserves suprema of non-empty up-directed subsets (cf. [**63**]), hence, by 1.2, λ_X reflects the way below relation. This proves the assertion.

The discussion in Section 2 shows that there are non-continuous, almost-continuous posets which arise from sober spaces with an injective hull in $\mathbf{T}_0$: In particular, the class of posets with continuous MacNeille completion which are compact in the topology inherited from the $\mathscr{CL}$-topology of the MacNeille completion. (The validity of Theorem 6.12, since it seems to be close to [**27**], 3.14, has possibly prevented an earlier discovery of the error in the formulation of the latter result.)

6.13. *Remark.* The notion of a degree for injective hulls leads to a natural (new?) dimension function $\imath$-dim for continuous lattices L themselves ("injectivity dimension"): $\imath$-dim L is at least n ($n \geqq 0$) if and only if (L, σ_L) is the injective hull of a sober space X of degree at least n.

The unit interval I has $\imath$-dimension 1. The example provided by K. H. Hofmann and M. W. Mislove [**38**] shows that $\imath$-dim $I^2 \geqq 2$ and, analogously, $\imath$-dim $I^n \geqq n$. Is it true that i-dim $I^n = n$? Are there continuous lattices L with $\imath$-dim $L = \infty$?

6.14. *Problem.* It is a natural question to what extent almost-continuous posets are related to the existence of an injective hull in $\mathbf{T}_0$. More specifically:

i) Is there an almost-continuous poset P

a) such that the Scott topology σ_P is non-sober, or

b) which is not induced by any sober topology?

(Examples of up-complete posets with non-sober Scott topology are not easy to find; cf. [**43**], [**42**].)

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ii) Is there an almost-continuous poset P

a) such that no join-dense continuous completion Q of P induces a sober topology $\sigma_Q|P$, or

b) such that no join-dense continuous completion $P \to Q$ preserves suprema of non-empty up-directed subsets (or, equivalently, $\sigma_Q|P$ is not coarser than σ_P)?

c) As in (a), but with the additional requirement that the MacNeille completion (or some join-dense completion which preserves suprema of non-empty up-directed subsets) of P is a continuous lattice.

6.15. *Problem.* One easily deduces from 6.1 (iii) that, for a given poset P, the supremum of every non-empty family of compatible topologies on P which have an injective hull in $\mathbf{T}_0$ also has an injective hull in $\mathbf{T}_0$. Is there always a coarsest compatible topology on a poset which has an injective hull in $\mathbf{T}_0$ (yielding the empty-indexed supremum)? The finest such topology is the Alexandrov-discrete topology ([**26**], correct implication of 4.3).

Is this also true for sober topological spaces or for $\mathscr{d}$-spaces?

6.16. *Remark.* Condition 6.1 (iii) may be compared with condition (4) of Proposition 6 of [**5**] (which characterizes, implicitly, those spaces X for which $\mathfrak{O}(X)$ is *hypercontinuous*, [**20**], pp. 166-167) which is studied further in [**6**], example 7, p. 158 (where these spaces are characterized as the "flat" spaces). This condition also appears, in a slightly different, but equivalent form in [**41**], p. 53 ("locally finite-bottomed spaces"). Recently, G. Gierz, J. D. Lawson and A. R. Stralka ([**21**], Section 6) have shown that the sober spaces X with $\mathfrak{O}(X)$ hypercontinuous are precisely the "quasi-continuous" posets endowed with their Scott topology.

Despite the similarity of these conditions, these classes of spaces are different and, moreover, incomparable:

i) Example 2.9 provides a sober space with an injective hull in $\mathbf{T}_0$ which does not carry the Scott topology, hence it is not flat (by [**21**], Section 6).

ii) Banaschewski's example ([**5**], p. 244) of a flat space which does not satisfy the condition of [**5**], Corollary 2, p. 240 apparently also violates 6.1 (iii) (check at $x = (0, 0)$), hence it does not have an injective hull in $\mathbf{T}_0$.

iii) The (ordinary) boundary of the unit square I^2 endowed with the trace of the Scott topology of I^2 (cf. 6.10) is sober, it has an injective hull in $\mathbf{T}_0$, and it is flat, but it fails to be a continuous poset.

Analogues of 6.3, 6.4, 6.7 (cf. [**5**] corollary on p. 244, [**41**], p. 41) and 6.9 can be easily established for flat spaces. The class of flat (= locally finite-bottomed) spaces is both open-hereditary and closed-hereditary. It is also stable under the formation of retracts. (The space A of 6.10 is both flat and essentially complete without being injective. Thus 6.5 and 6.6 do not carry over.)

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