This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P.Q.

A GENERALIZATION OF JACOBI'S THEOREM TO HYPERADJUGATES

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Jacobi's theorem expresses the adjugate of a determinant as a power of the determinant, or in a more general form a minor of the adjugate as the product of the complementary minor by a power of the determinant.

Can this result be generalized to the adjugate of the adjugate of the ... ?

Let Δ be a determinant of order n,

$\bar{\Delta}$ its adjugate,

$M_{r,\bar{\Delta}}$ a minor of $\bar{\Delta}$ of order r,

$M_{n-r,\Delta}$ the minor of Δ containing the $(n-r)$ rows and columns corresponding to those not chosen for $M_{r,\bar{\Delta}}$, and σ_r the sum of the indices of the rows and columns of $M_{r,\bar{\Delta}}$.

Jacobi's theorem states that $\bar{\Delta} = \Delta^{n-1}$ or, more generally, $M_{r,\bar{\Delta}} = (-1)^{\sigma_r} M_{n-r,\Delta} \Delta^{r-1}$.

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In the following generalization of this result,

$\Delta^{[1]}$ is the adjugate of Δ,

$\Delta^{[j+1]}$ the adjugate of $\Delta^{[j]}$ or the $(j+1)$st hyperadjugate of Δ,

$M_{r,\Delta}[k]$ a minor of $\Delta^{[k]}$ of order r,

$M_{r,\Delta}[\lambda]$, where $\lambda < k$, the minor of $\Delta^{[\lambda]}$ of order r, whose rows and columns correspond to those of $M_{r,\Delta}[k]$,

$M_{n-r,\Delta}[\mu]$, where $\mu < k$, the minor of $\Delta^{[\mu]}$ of order $n-r$, whose rows and columns correspond to those not chosen in $M_{r,\Delta}[k]$, and σ_{n-r} the sum of the rows and columns of $M_{r,\Delta}[k]$.

Thus, $\sigma_r + \sigma_{n-r} = 2n$

$$\Delta^{[\phi]} = (\Delta^{[\phi-1]})^{n-1}$$

$$= (\Delta^{[\phi-2]})^{(n-1)^2}$$

$$= \Delta^{(n-1)^{\phi}} .$$

Consider $M_{r,\Delta}[2] = (-1)^{\sigma_r} M_{n-r,\Delta}[1] (\Delta^{[1]})^{r-1}$

$$= (-1)^{\sigma_r}(-1)^{\sigma_{n-r}} M_{r,\Delta} (\Delta^{[1]})^{r-1} (\Delta)^{n-r-1} .$$

Note that σ_r and σ_{n-r} alternate in parity as do $(r-1)$ and $(n-r-1)$ and that $n-r$ and r alternate as suffixes in the development. Similarly, when k is even,

$$M_{r,\Delta}[k] = M_{r,\Delta} \{\Delta^{[k-1]}\}^{r-1} \{\Delta^{[k-2]}\}^{n-r-1} \ldots \{\Delta\}^{n-r-1}$$

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$$= M_{r,\Delta}\left(\{\Delta^{(n-1)^{(k-1)}}\}^{r-1}\{\Delta^{(n-1)^{(k-3)}}\}^{r-1}\dots\{\Delta^{n-1}\}^{r-1}\right)$$

$$\left(\{\Delta^{(n-1)^{k-2}}\}^{n-r-1}\{\Delta^{(n-1)^{(k-4)}}\}^{n-r-1}\dots\{\Delta\}^{n-r-1}\right)$$

$$= M_{r,\Delta}\Delta^{t}, \qquad \left(t = \frac{r(n-1)^{k} - 1}{n}\right).$$

When k is odd, $M_{r,\Delta}[k] = (-1)^{\sigma_r} M_{n-r,\Delta}\{\Delta^{[k-1]}\}^{r-1}\{\Delta^{[k-2]}\}^{n-r-1}$

$$\dots\{\Delta\}^{r-1} = \left(\{\Delta^{(n-1)^{(k-1)}}\}^{r-1}\{\Delta^{(n-1)^{(k-3)}}\}^{r-1}\dots\{\Delta\}^{r-1}\right)$$

$$\left(\{\Delta^{(n-1)^{(k-2)}}\}^{n-r-1}\dots\{\Delta^{(n-1)^{(k-4)}}\}^{n-r-1}\dots\right.$$

$$\left.\{\Delta^{n-1}\}^{n-r-1}\right)(-1)^{\sigma_r} M_{n-r,\Delta} = (-1)^{\sigma_r} M_{n-r,\Delta}\Delta^{t},$$

$$\left(t = \frac{r(n-1)^{k} + r - n}{n}\right).$$

REFERENCE

1. Ferrar, Algebra, Oxford, 1941, pp. 55-58.

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