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ON A CHARACTERIZATION OF PLANAR GRAPHS

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By introducing the concept of a polygon-extension of a planar graph, we provide a simple proof that a graph is planar if and only if every strict elegant ring in the graph is even.

1. Introduction

Throughout, we consider undirected graphs on a finite set of vertices. Following the notations used in [3], we denote the vertex set of a graph G by VG and its edge set by EG. If G is a directed graph, C is a directed circuit of G, and a, b are distinct vertices of VC, then we used the notation $C(a, b)$ to mean the direct subpath of C with origin a and terminus b. If $a = b$, then $C(a, b)$ means the subpath of C with vertex set $\{a\}$ and empty edge set. If P is a path in G with end vertices c, d, then we use IP to denote the set $VP - \{c, d\}$. Further, let X and Y be distinct paths or circuits of a graph G with $|VX \cap VY| \geq 2$. Then an $\overline{X}Y$-*path* is a maximal nondegenerate subpath P of Y for with $IP \cap VX = \emptyset$ and $EP \cap EX = \emptyset$. An XY-*path* is a maximal subpath P of Y for which $EP \subseteq EX \cap EY$ and $VP \subseteq VX \cap VY$.

Let S be a collection of circuits of G. If the edges of G can be directed so that every circuit of S is a directed circuit, then we say that S is *consistently orientable*. The cyclic sequence of circuits $S = (C_0, C_1, \ldots, C_{n-1})$ with $n \geq 3$ is a *ring* in the graph G, if

(R1) S is consistently orientable,

(R2) $EC_i \cap EC_j \neq \emptyset$ if and only if $i = j$, $i \equiv j + 1 \pmod n$

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or $i \equiv j - 1 \pmod{n}$,

(R3) no edge of G belongs to more than two circuits of G .

We note that (R2) implies (R3) except when $n = 3$.

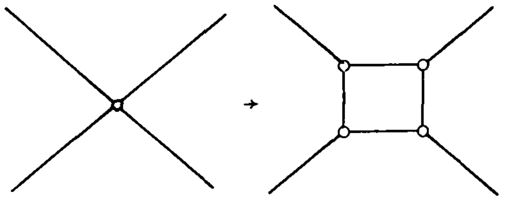
A ring $S = (C_0, C_1, \ldots, C_{n-1})$ with n circuits is called an *n-ring*. It is called an *odd ring* if n is odd, an *even ring* if n is even, a *strict ring* if $|VC_i \cap VC_j| \le 1$ whenever $EC_i \cap EC_j = \emptyset$, an *elegant ring* if, for each $i = 0, 1, \ldots, n-1$, there is a unique $\overline{C}_i C_{i+1}$-path (or equivalently, the common vertices on C_i and C_{i+1} are those on the path M_i where $EC_i \cap EC_{i+1} = EM_i$), a *perfect ring* if it is elegant and $VC_i \cap VC_j = \emptyset$ whenever $EC_i \cap EC_j = \emptyset$. We note here, and throughout this paper, that all subscripts are taken as being modulo n .

Let $P(G)$ be a planar embedding of a planar graph G . For each vertex x of G , we shall denote by $N(x)$ the set of all vertices y of G adjacent to x and $E(x)$ the set of all edges of G with one end at x . Let $e_1, e_2 \in E(x)$ (say $e_1 = \{x, y_1\}$, $e_2 = \{x, y_2\}$). We say that e_1, e_2 are *neighbouring edges* in $P(G)$ if and only if we can draw a curve C in the plane joining y_1 and y_2 such that C does not intersect any e in $E(x)$ except possibly at y_1 and y_2 and the open region in the plane bounded by C , e_1, e_2 is disjoint from $N(x)$. We now construct a graph $P^*(G)$ whose vertices are all ordered pairs (x, e) with $x \in VG$, $e \in E(x)$ and, for any two vertices $(x, e), (y, f)$ in $VP^*(G)$, we draw an edge joining them if and only if one of the following holds:

(i) $x = y$ and e, f are neighbouring;

(ii) $x \ne y$ and $e = f$.

We shall call $P^*(G)$ the *polygon extension* of G with respect to the planar embedding $P(G)$. Intuitively speaking, $P^*(G)$ is obtained from $P(G)$ by replacing each vertex of degree n in $P(G)$ with an n-gon and joining corresponding vertices as indicated in the figure below (for $n = 4$). Note that $P^*(G)$ is always planar and if each vertex of G has degree greater than or equal to 3 , then $P^*(G)$ is always a cubic graph.



The purpose of this paper is to make use of the concept of polygon extensions of planar graphs to provide a simple proof of the following nice characterization of planar graphs due to Holton and Little [3].

MAIN THEOREM. *A graph G is planar if and only if every strict elegant ring in G is even.*

REMARK. Throughout this paper, whenever S denotes a ring in a graph G, we assume without loss of generality that each edge of G is also an edge of a circuit C_i of S, since we are concerned only with the subgraph of G which is the union of all circuits of S.

2. Basic lemmas

The following results will be useful in the sequel.

LEMMA 1 ([1], Kuratowski's Theorem). *G is planar if and only if no subgraph of G is homeomorphic to K_5 or $K_{3,3}$.*

LEMMA 2 ([2]). *There exists an odd strict elegant ring in K_5 and in $K_{3,3}$.*

Let C be a circuit in the plane. We shall write $I(C)$ to mean the open region in the plane bounded by C, $\overline{I(C)}$ to mean the closed region bounded by C, and $O(C)$ to mean the region consisting of all points not in $\overline{I(C)}$.

LEMMA 3. *Let $S = (C_0, C_1, \ldots, C_{n-1})$ be a strict elegant ring in a planar graph G and $P(G)$ a planar embedding of G. Then, for each i, either $I(C_i) \cap VP(G) = \emptyset$ or $O(C_i) \cap VP(G) = \emptyset$.*

Proof. Without loss of generality, we need only to consider the case $i = 0$. Since S is elegant, either $I(C_0) \cap VC_1 = \emptyset$ or

$O(C_0) \cap VC_1 = \emptyset$ (say the former).

CLAIM. $I(C_0) \cap VC_2 = \emptyset$.

Indeed, let $e = \{a, b\}$ be a common edge of C_1 and C_2. Since S is strict, either $a \notin VC_0$ or $b \notin VC_0$ (say the former). Hence, we must have $a \in O(C_0) \cap VC_2$. Suppose to the contrary that $I(C_0) \cap VC_2 \neq \emptyset$, say $c \in I(C_0) \cap VC_2$. Then $C_2(c, a) \cap VC_0 \neq \emptyset$ and $C_2(a, c) \cap VC_0 \neq \emptyset$. This however contradicts the fact that S is strict and elegant. Thus, we must have $I(C_0) \cap VC_2 = \emptyset$, as claimed.

By exactly the same argument as that for the above claim, we have $I(C_0) \cap VC_i = \emptyset$ for all $i = 1, 2, \ldots, n-1$. Therefore $I(C_0) \cap VP(G) = \emptyset$, completing the proof.

LEMMA 4. *Let* $S = (C_0, C_1, \ldots, C_{n-1})$ *be a strict elegant ring in a planar graph* G *and* $P(G)$ *be a planar embedding of* G. *Then, for each* i, *all adjacent edges in* C_i *are also neighbouring edges in* $P(G)$.

Proof. Let $e_1 = \{a, b_1\}$, $e_2 = \{a, b_2\}$ be two adjacent edges of C_i. By Lemma 3, either $I(C_i) \cap VP(G) = \emptyset$ or $O(C_i) \cap VP(G) = \emptyset$. In the first case, we let C be any curve lying entirely within $I(C_i)$ except for the two ends b_1, b_2 of C; where in the second case, we let C be any curve lying entirely within $O(C_i)$ except for the two ends b_1, b_2 of C. Then the open region in the plane bounded by C, e_1, e_2 is disjoint from $N(a)$. Thus, by definition, e_1 and e_2 are neighbouring edges, as required.

LEMMA 5 ([3]). *Let* $S = (C_0, C_1, \ldots, C_{n-1})$ *be a perfect ring in a planar graph* G. *Then* S *is even.*

Proof. Here as in [3], for each i, we shall denote the origin of the unique $\overline{C}_i C_{i+1}$-path by v_i and the terminus by u_i. We shall also denote the path $C_i(u_i, v_i)$ by P_i. Note that P_i is a $C_i C_{i+1}$-path. Suppose to the contrary that S is not even. Then we have the following two cases to consider.

CASE 1. $n = 3$.

In this case, let e_1 be the edge of P_0 incident on v_0 , let e_2 be the other edge of C_0 incident on v_0 and let e_3 be the other edge of C_1 incident on v_0 . Thus $e_1 \notin EC_2$, and e_2, e_3 cannot both belong to EC_2 . Thus if $v_0 \in VC_2$ then the degenerate path with vertex set $\{v_0\}$ is either a C_0C_2-path or a C_1C_2-path. Since there must be a non-degenerate such path, the elegance of S is contradicted. Thus $v_0 \notin VC_2$ and similarly $u_0 \notin VC_2$. It is now immediate that $C_0 \cup C_1 \cup C_2$ is a subdivision of $K_{3,3}$, a contradiction to the planarity of G .

CASE 2. $n \geq 5$.

In this case, the graph

$$\bigcup_{k=0}^{n-2} \left[C_{k+1}(v_k, u_{k+1}) \cup C_{k+1}(v_{k+1}, u_k)\right] \cup C_0(v_{n-1}, u_0) \cup C_0(v_0, u_{n-1}) \cup P_0 \cup P_1 \cup P_2$$

is a subdivision of $K_{3,3}$, again a contradiction to the planarity of G .

The proof of Lemma 5 is therefore complete.

3. The proof

If a graph G is non-planar, by Lemma 1, it contains a subgraph homeomorphic to K_5 or $K_{3,3}$. Hence, by Lemma 2, G contains an odd strict elegant ring.

Conversely, assume that G is planar. Let $S = (C_0, C_1, \ldots, C_{n-1})$ be any strict elegant ring in G . We need only to prove that S is even. First, let $P(G)$ be a planar embedding of G and $P^*(G)$ the polygon-extension of G with respect to $P(G)$. For each i and each x in VC_i , we denote by x^* the set $\{(x, e), (x, f)\}$ where e, f are the two edges of C_i incident on x . Let $C_i^* = \bigcup(x^* \mid x \in VC_i)$. Then, by Lemma 4, C_i^* forms a circuit in $P^*(G)$. Let $S^* = (C_0^*, C_1^*, \ldots, C_{n-1}^*)$.

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Evidently, S^* is a ring in $P^*(G)$. The elegance of S^* follows immediately from that of S. We shall show that S^* is a perfect ring. Indeed, let C_i^*, C_j^* be any two circuits in S^* with $EC_i^* \cap EC_j^* = \emptyset$. Then $EC_i \cap EC_j = \emptyset$. If $VC_i \cap VC_j = \emptyset$, then it follows from our constructions that $VC_i^* \cap VC_j^* = \emptyset$. On the other hand, if $VC_i \cap VC_j$ is not empty, then, by the strictness of S, it must be a singleton. Let $VC_i \cap VC_j = \{x\}$. Now if $VC_i^* \cap VC_j^* \neq \emptyset$, let $(y, e) \in VC_i^* \cap VC_j^*$. Then, $y \in VC_i \cap VC_j$ and so $y = x$. Also $e \in E(x) \cap EC_i \cap EC_j \subseteq EC_i \cap EC_j = \emptyset$, a contradiction. Thus in any case, we must have $VC_i^* \cap VC_j^* = \emptyset$, showing the S^* is a perfect ring. Since $P^*(G)$ is planar, by Lemma 5, S^* is even (that is, n is even). Hence S is even, as required.

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