# THE ENDOMORPHISM RING OF THE ADDITIVE GROUP OF A RING

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## 1. Introduction

One of the still unsolved problems posed by Fuchs in his well-known book "Abelian Groups" [2] is Problem 45: characterize the rings $R$ for which $R \cong \mathscr{E}(R^+)$. I present here a partial solution.

In the first part of the paper, several properties of $R$ which are simply due to the existence of an isomorphism onto $\mathscr{E}(R^+)$ are deduced, and I am able to characterize $R$ in case it is torsion, completely decomposable, not reduced, finitely generated, or mixed with no elements of infinite $p$-height for all relevant primes $p$.

In the second part, the properties of the isomorphism of $R$ onto $\mathscr{E}(R^+)$ are considered, and two essentially different approaches are required, depending on whether $R$ is, or is not, commutative. If it is, then $R$ is a relatively uncomplicated ring, and one can hope for a complete characterization, though this paper does not give one. If not, then $R$ must be a complicated ring indeed; for example, the group of units of $R$ contains a copy of every finite group. The best one can hope for in this case is either to exhibit such an $R$, or to prove its non-existence; once again, I am unable in this paper to do either.

I use the standard notation of abelian group theory, as found for example in Fuchs [2]. Sometimes group theoretic properties are assigned to rings; this means that the additive group of the ring has the property. For example, a ring $R$ is called torsion-free if $R^+$ is torsion-free. Some notation which may not be familiar:

If $x \in R$, then $h(x)(p)$ means the *p-height* of $x$ in $R^+$;

$t(R)$ means the *torsion subgroup* of $R^+$;

$R_p$ means the *p-primary component* of $R^+$.

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If $S$ is a set of primes, $S$-*pure* means $p$-pure for all $p \in S$, and $S$-*divisible* means $p$-divisible for all $p \in S$.

$\oplus$ and $\oplus^*$ mean respectively *direct sum* and *direct product*, either group or ring theoretic.

$Z$, $Q$, $Z_p$, $C(n)$ mean the *group* or *ring* (depending on context) of *integers*, *rationals*, *p-adic integers*, and *integers modulo n*.

$c$ is the *cardinality of the continuum*.

## 2. General remarks on rings $\mathbf{R} \cong \mathscr{E}(\mathbf{R}^+)$

Using some well-known invariants of abelian groups, we first characterize the divisible and torsion subgroups of such rings.

LEMMA 1. *Let $R \cong \mathscr{E}(R^+)$, and suppose $R^+ = A \oplus D$, where $A$ is reduced and $D$ is divisible. Then either $D = 0$, or $D \cong Q$ and $A$ is torsion.*

PROOF. The rank $m$ of the maximal torsion-free direct summand of $D$ is an invariant of $R$. Now End$(R^+)$, and hence $R^+$ has a direct summand isomorphic to End $(\oplus_m Q)$, which is torsion free divisible, and so contained in $D$. Hence $m = 0$ or $1$.

If $D$ contains a direct summand isomorphic to $C(p^\infty)$, then End$(R^+)$ has a direct summand isomorphic to End$(C(p^\infty)) \cong Z_p$, so $R^+$ has a direct summand isomorphic to Hom$(Z_p, C(p^\infty))$. Now $Z_p$ has a factor group isomorphic to $\oplus_c Q$, so Hom$(Z_p, C(p^\infty))$ has a subgroup isomorphic to Hom$(\oplus_c Q, C(p^\infty))$, which is torsion-free divisible of infinite rank, contradicting the first paragraph. Hence $D \cong 0$ or $Q$.

Now suppose $D \cong Q$, and let $r$ be the torsion free rank of $R^+$. Then End$(R^+)$ has a direct summand isomorphic to Hom $(R^+, Q) \cong \oplus_r^* Q$, so $r = 1$, and hence $A$ is torsion.

LEMMA 2. *If $R \cong \mathscr{E}(R^+)$, then for each prime $p$, $R_p \cong C(p^{k_p})$ for some $0 \leqq k_p < \infty$.*

PROOF. If $R_p \neq 0$, it is reduced by Lemma 1. The number $r_n$ of cyclic summands of $R_p$ of order $p^n$ is an invariant of $R$ for all positive integers $n$; let $k$ be minimal such that $r_k \neq 0$. Let $B = \oplus_{j \geqq k} B_j$ be a basic subgroup of $R_p$, where $B_j \cong \oplus_{r_j} C(p^j)$. Now $B_k$ is a bounded pure subgroup, and hence a direct summand of $R^+$, so End$(R^+)$ has a direct summand isomorphic to Hom$(B_k, R^+)$. But this is a direct sum of cyclic groups of order $\leqq p^k$, so its rank $r \leqq r_k$. On the other hand, Hom$(B_k, R^+)$ has a subgroup isomorphic to Hom$(B_k, B) \cong \oplus_{rk}^* \oplus_u C(p^k)$, where $u = \sum_{j \geqq k} r_j$. This subgroup has rank $2^{r_k} u \leqq r \leqq r_k$ if $r_k$ is infinite, or $r_k u \leqq r \leqq r_k$ if $r_k$ is finite. In either case, $0 \neq r_k \leqq u \leqq 1$, so $r_k = u = 1$. Hence $r_j = 0$ for all $j > k$ and $R_p = B = B_k \cong C(p^k)$.

LEMMA 3. *If* $R \cong \mathscr{E}(R^+)$, *and* $R_p \neq 0$, *then* $R_p$ *has a unique complement* $R'_p = \{x \in R \mid h(x)(p) = \infty\}$, *which is an ideal in* $R$; *furthermore* $\mathscr{E}(R'_p) \cong R'_p$.

PROOF. Let $H$ be any group such that $R^+ = R_p \oplus H$. If $pH \neq H$, then End$(R^+)$ has a subgroup isomorphic to Hom$(R_p \oplus H/pH, R_p)$. This is a $p$-group of rank $> 1$, contradicting Lemma 2. Hence $pH = H$, so $H \subset R'_p$. Conversely, let $x \in R'_p$, and write $x = x_1 + x_2$, where $x_1 \in R_p$, $x_2 \in H$. Since $h(x)(p) = \infty$, $x_1 = 0$, so $x \in H$.

$R'_p$ is clearly an ideal, and Hom$(R_p, R'_p)$ = Hom $(R'_p, R_p) = 0$, so $\mathscr{E}(R^+) = \mathscr{E}(R_p) \oplus \mathscr{E}(R'_p) \cong R_p \oplus R'_p$, so $R'_p \cong \mathscr{E}(R'_p)$.

LEMMA 4. *Let* $R \cong \mathscr{E}(R^+)$, *let* $S$ *be the set of relevant primes for* $R$, *and let* $U = \oplus^*_{p \in S} R_p$. *Let* $A = \{x \in R \mid h(x)(p) = \infty$ for all $p \in S\}$. *Then* $R$ *is an extension of* $A$ *by a ring* $T$ *such that* $t(R) \subset T \subset U$, *and* $T$ *is an* $S$*-pure subring of* $U$ *containing the identity.*

PROOF. For each relevant prime $p$, we have by Lemma 3 a unique decomposition $R^+ = R_p \oplus R'_p$, where $R'_p = \{x \in R \mid h(x)(p) = \infty\}$. Thus each $x \in R$ can be uniquely expressed as $x = x_p + x'_p$, where $x_p \in R_p$, $x'_p \in R'_p$. Hence the mapping $e: R \to U$ given by $e(x)(p) = x_p$ is a well defined ring homomorphism with kernel $A$. Let $T$ be the image of $e$; clearly $e(1)$ is the identity of $U$, and $e|_{t(R)}$ is the identity map, so $t(R) \subset T \subset U$.

If $T$ is not $p$-pure in $U$ for some $p \in S$, write $T = R_p \oplus T'$; then $T'$ is not $p$-pure in $\oplus_{q \neq p} R_q$, so $pT' \neq T'$. Hence End$(R^+)$ has a subgroup isomorphic to Hom$(R_p \oplus T'/pT', R_p)$, which is a $p$-group of rank $> 1$, a contradiction.

LEMMA 5. *Let* $R \cong \mathscr{E}(R^+)$ *with* $R^+$ *torsion-free and completely decomposable. Then* $R$ *is a direct sum of finitely many rank* 1 *rings of incomparable types.*

PROOF. Suppose $R^+ = \oplus_{i \in I} A_i$ for some index set $I$, where each $A_i$ is a rank 1 torsion-free group. Then End$(R^+)$ contains as a direct summand $\oplus^*_{i \in I}$ End$(A_i)$. Since End$(R^+)$ is completely decomposable and of rank $|I|$, $I$ is finite, and End$(R^+) = \oplus_{i \in I}$ End$(A_i)$. Hence for each $i \neq j$, Hom$(A_i, A_j) = 0$, so the $A_i$ have incomparable types. Since the type of End$(A_i)$ is less than the type of $A_i$, $A_i \cong$ End$(A_i)$ for all $i$, so $A_i$ is a rank 1 ring. Finally since Hom$(A_i, A_j) = 0$ if $i \neq j$, each $A_i$ is an ideal in $R$.

The characterizations promised in the Introduction follow from these lemmas:

THEOREM 1. *If* $R$ *is torsion, then* $R \cong \mathscr{E}(R^+)$ *if and only if* $R$ *is cyclic.*

PROOF. *It is well known that* $C(n) \cong \mathscr{E}(C(n))$ for all positive integers $n$.

Conversely, if $R$ is torsion then by Lemma 2,

$$\bigoplus_{p \in S} C(p^{k_p}) \cong R \cong \mathscr{E}(R^+) \cong \bigoplus_{p \in S}{}^* C(p^{k_p})$$

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for some set $S$ of primes. Hence $S$ is finite, so $R$ is cyclic.

THEOREM 2. *If $R^+$ is completely decomposable, then $R \cong \mathscr{E}(R^+)$ if and only if $R \cong C(n) \oplus A$, where $n$ is a non-negative integer and $A$ is a direct sum of rank 1 rings of incomparable type, and $A$ is divisible by each prime which divides $n$.*

PROOF. A modest calculation shows that $\mathscr{E}(C(n) \oplus A) \cong C(n) \oplus A$.

Conversely, let $R^+ = t(R) \oplus A$, where $A$ is torsion-free and completely decomposable. Then $\mathrm{Hom}(t(R), A) = 0$, and by Lemma 4, $\mathrm{Hom}(A, t(R)) = 0$, so $R$ is the ring direct sum of its ideals $t(R)$ and $A$. Thus

$$\mathscr{E}(R^+) \cong \mathscr{E}(t(R)) \oplus \mathscr{E}(A),$$

so $t(R) \cong \mathscr{E}(t(R))$, $A \cong \mathscr{E}(A)$. The result now follows from Theorem 1, Lemma 5, and Lemma 4.

THEOREM 3. *If $R$ is not reduced, then $R \cong \mathscr{E}(R^+)$ if and only if $R \cong Q \oplus C(n)$ for some non-negative integer $n$.*

PROOF. Certainly $\mathscr{E}(Q \oplus C(n)) \cong Q \oplus C(n)$.

Conversely, by Lemma 1, $R \cong t(R) \oplus Q$ and by Lemma 2,

$$t(R) \cong \bigoplus_{p \in S} C(p^{k_p})$$

for some set $S$ of primes. Then $\mathrm{End}(R^+)$ contains a direct summand $\oplus^*_{p \in S} C(p^{k_p}) \oplus Q$; by Lemma 1 again, $\oplus^*_{p \in S} C(p^{k_p})$ is torsion, so $S$ is finite. Hence $t(R) \cong C(n)$ for some $n$.

THEOREM 4. *If $R^+$ is finitely generated, then $R \cong \mathscr{E}(R^+)$ if and only if $R \cong Z$, or $R \cong C(n)$ for some non-negative integer $n$.*

PROOF. By Theorem 2, $R$ is the direct sum of a cyclic ring and finitely many copies of the integers. But the torsion-free components have incomparable types, so there is at most one. Furthermore, $Z$ is not divisible by any prime, so if $R$ has a non-zero torsion subgroup, then $R$ is torsion.

THEOREM 5. *If $R$ is a mixed ring, $S$ the set of relevant primes, and $R$ has no element of infinite $p$-height for all $p \in S$, then $R \cong \mathscr{E}(R^+)$ if and only if:*

(1) *$R_p \cong C(p^{k_p})$, $0 < k_p < \infty$ for all $p \in S$*
(2) *If $U$ is the ring $\oplus^*_{p \in S} R_p$, then $R$ is a subring with identity of $U$.*
(3) *$R$ is $S$-pure in $U$.*

PROOF. Suppose $R$ is a subring of $U$ satisfying (1), (2) and (3). The exact sequence of rings

$0 \to t(R) \to R \to R/t(R) \to 0$ induces an exact sequence of groups

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$$0 \to \mathrm{Hom}(R^+/t(R), R^+) \to \mathrm{End}(R^+) \to \mathrm{Hom}(t(R), R^+).$$

Since $R$ is $S$-pure in $U$, $R/t(R)$ is $S$-pure in $U/t(R)$, and hence $S$-divisible. Since $R$ has no elements of infinite $p$-height for all $p \in S$, $\mathrm{Hom}(R/t(R), R) = 0$. Hence $\mathrm{End}(R^+)$ is embedded in $\mathrm{Hom}(t(R), R)$ by the mapping $f \mapsto f|_{t(R)}$. But any $f \in \mathrm{Hom}(t(R), R)$ sends $t(R)$ into $t(R)$, and this mapping is a ring homomorphism. Hence $\mathscr{E}(R^+)$ is embedded as a subring of $\mathscr{E}(t(R))$. Now it is well know that $\mathscr{E}(t(R))$ is the ring of all multiplications in $U$, so if $f \in \mathscr{E}(R^+)$, then $f$ is multiplication in $R$ by some $x \in U$; in particular, $x = f(1) \in R$. Thus the mapping $x \mapsto$ multiplication by $x$ is a ring homomorphism of $R$ onto $\mathscr{E}(R^+)$. The kernel is zero, since $1 \in R$.

Conversely, we have by Lemma 4 that $R^+ \cong T$, an $S$-pure subring of $U = \oplus^*_{p \in S} R_p$ containing the identity, and $R_p \cong C(p^{k_p})$ by Lemma 2.

REMARKS. If $S$ is any infinite set of primes, and $k_p$ a positive integer for each $p \in S$, there are $c$ non-isomorphic subrings of

$$\bigoplus^*_{p \in S} C(p^{k_p}) \Big/ \bigoplus_{p \in S} C(p^{k_p}),$$

each of which is $S$-divisible. Hence there are $c$ non-isomorphic rings $R$ with fixed torsion subring $\oplus_{p \in S} C(p^{k_p})$ which satisfy the hypotheses of Theorem 4.

It is well known that if $R$ is a $p$-pure subring of $Z_p$, or an $S$-pure subring of $\oplus^*_{p \in S} Z_p$, where $S$ is any collection of primes, then $R \cong \mathscr{E}(R^+)$.

Other example of rings $R \cong \mathscr{E}(R^+)$ can be constructed by noting that if $I$ is any index set, and $R_i$, $i \in I$, a collection of rings such that $R_i \cong \mathscr{E}(R_i^+)$ and such that $\mathrm{Hom}(\oplus^*_{j \neq i} R_j, R_i) = 0$, then

$$\bigoplus^*_{i \in I} R_i \cong \mathscr{E}\left(\bigoplus^*_{i \in I} R_i^+\right).$$

Finally, we have a slight improvement on Lemma 4.

THEOREM 6. *Let $R$ be a mixed ring with relevant prime set $S$, and let*

$$A = \{x \in R \mid h(x)(p) = \infty \text{ for all } p \in S\}.$$

*If $R \cong \mathscr{E}(R^+)$, then $R_p \cong C(p^{k_p})$, $0 < k_p < \infty$ for all $p \in S$, and $R$ is an extension of $A$ by a ring $T$ such that $T \cong \mathscr{E}(T^+)$ and*

$$t(R) \subset T \subset \bigoplus^*_{p \in S} R_p.$$

*The extension splits if $A \cong \mathscr{E}(A^+)$.*

PROOF. By Lemma 4, it suffices to prove that $T \cong \mathscr{E}(T^+)$ and the remark about splitting. The first statement follows from Theorem 5. If $A \cong \mathscr{E}(A^+)$, then $A$ has an identity $1_A$; multiplication by $1_A$ is a retraction of $R$ onto $A$.

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### 3. The nature of the isomorphism: the commutative case

Let $R$ be any ring with identity 1. Then the mapping $f \mapsto f(1)_L$ is a retraction of $\mathscr{E}(R^+)$ onto $R_L$, the ring of left multiplications in $R$. This mapping is a group homomorphism with kernel $K = \{f : f(1) = 0\}$. Hence $\mathscr{E}(R^+)$ is a group theoretic direct sum of the ring $R_L$ and the left ideal $K$, so we have the following:

LEMMA 6. *For a ring $R$ with identity 1, the following statements are equivalent*:

(1) *$R$ is commutative and $R \cong \mathscr{E}(R^+)$.*

(2) *The mapping $x \mapsto x_L$, left multiplication by $x$, is an isomorphism of $R$ onto $\mathscr{E}(R^+)$, with inverse $f \mapsto f(1)$.*

(3) *Every endomorphism of $R^+$ is a left multiplication in $R$.*

PROOF. $(1) \to (2)$ By the preceding remarks, $R_L$, which is isomorphic to $R$, is a direct summand of $\mathrm{End}(R^+)$. But a group with commutative endomorphism ring cannot have a proper isomorphic direct summand, so $R_L = \mathscr{E}(R^+)$, and the ring isomorphism $x \mapsto x_L$ maps $R$ onto $\mathscr{E}(R^+)$.

$(2) \to (3)$ is trivial.

$(3) \to (1)$ Clearly $\mathscr{E}(R^+) = R_L \cong R$. In particular, the right multiplications in $R$ are left multiplications; let $x_R$ be a right multiplication, and suppose $x_R = y_L$. Then $x = x_R(1) = y_L(1) = y$, so $x_R = x_L$; hence $R$ is commutative.

DEFINITION. Let us call a ring $R$ which satisfies the conditions of Lemma 6 an *E-ring*, and the additive group of an $E$-ring an *E-group*. This definition of $E$-group coincides with that in [3], where an $E$-group was defined by condition (3). In Corollary 4 below, we see that every ring with identity over an $E$-group is an $E$-ring.

COROLLARY 1. *Every endomorphic image of an E-group $R^+$ is the additive group of a principal ideal of $R$.*

COROLLARY 2. *A group direct summand of an E-ring $R$ is a ring direct summand, and hence $R^+$ cannot be decomposed as an infinite direct sum.*

PROOF. Let $p \in \mathscr{E}(R^+)$ be the projection of $R^+$ onto the direct summand $eR^+$. Then $e = p(1)$, so

$$e^2 = p(1)^2 = e \cdot p(1) = p(p(1)) = p(1) = e.$$

Hence $e$ is an idempotent, so $eR$ is a ring direct summand. Since $R$ has identity 1, $R$ cannot be an infinite direct sum of ideals.

COROLLARY 3. *If $R$ is any ring with identity, then $\mathscr{E}(R^+)$ is commutative if and only if $R$ is an E-ring.*

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PROOF. If $\mathscr{E}(R^+)$ is commutative, let $f \in \mathscr{E}(R^+)$. Then for all $x \in R$,

$$f(x) = f \cdot x_L(1) = x_L \cdot f(1) = x_R \cdot f(1) = f(1)x,$$

so $f = f(1)_L$. Condition (3) of Lemma 6 is satisfied.

The converse follows from the definition of $E$-ring.

COROLLARY 4. *If $R$ is an $E$-ring, and $S$ any ring with identity over $R^+$, then $R \cong S$.*

PROOF. Since $\mathscr{E}(R^+) = \mathscr{E}(S^+)$ is commutative, $S$ is an $E$-ring by Corollary 3. Hence $S \cong \mathscr{E}(S^+) \cong R$.

LEMMA 7. *Let $R$ be a ring with identity* 1, *and let $F$ be an isomorphism of $R$ onto $\mathscr{E}(R^+)$. Then for every $* \in Mult(R^+)$, there exists a unique $a \in R$ such that $x * y = F(F(a)(x))(y)$ for all $x, y \in R$.*

PROOF. There is a chain of well-known group isomorphisms

$$R^+ \xrightarrow{F} \text{End}\,(R^+) \xrightarrow{F^*} \text{Hom}\,(R^+, \text{End}\,(R^+)) \rightarrow \text{Mult}\,(R^+),$$

where for all $a \in R^+$, $a \mapsto F(a) \mapsto f_a \mapsto *_a$, where $f_a(x)(y) = F(F(a)(x))(y) = x *_a y$.

NOTE. The left multiplications in $R$ correspond to elements of $\text{Mult}(R^+)$ of the form $x * y = F(ax)(y)$ for some $a \in R$, and in particular, the identity map on $R^+$ corresponds to the multiplication $x *_1 y = F(x)(y)$. Clearly 1 is a left identity for $*_1$, and we have the following criteria for commutativity of $R$:

LEMMA 8. *Let $R$ be a ring with identity* 1, *let $F$ be an isomorphism of $R$ onto $\mathscr{E}(R^+)$; let $a \mapsto *_a$ be the mapping defined in Lemma 7. Then the following conditions are equivalent:*

(1) *$R$ is an $E$-ring*
(2) *Every multiplication $*_a$ is associative*
(3) *Every multiplication $*_a$ is commutative*
(4) *$*_1$ is associative*
(5) *$*_1$ is commutative*
(6) 1 *is a right identity for $*_1$.*

PROOF. (1) $\mapsto$ (2) and (3).

Since $R$ is an $E$-ring, we may replace $F$, if necessary, by the isomorphism $x \rightarrow x_L$. Then every multiplication in $R^+$ has the form $x *_a y = axy$, so is necessarily associative and commutative.

(2) $\rightarrow$ (4) and (3) $\rightarrow$ (5) are trivial

(4) $\rightarrow$ (6) For all $x, y, z \in R$,

$$F(F(x)(y))(z) = (x_1 * y) *_1 z = x *_1 (y *_1 z) = F(x)\,(F(y)(z)) = F(xy)(z),$$

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so $F(F(x)(y) = F(xy)$. Hence $x *_1 y = F(x)(y) = xy$ for all $x$, $y \in R$ and 1 is a right identity.

$(5) \to (6)$ For all $x \in R$, $x *_1 1 = 1 *_1 x = F(1)(x) = x$.

$(6) \to (1)$ If $x = x *_1 1 = F(x)(1)$ for all $x$, then

$$K = \{f \in \mathscr{E}(R) \,|\, f(1) = 0\} = 0,$$

so by the Remarks preceding Lemma 6, $\mathscr{E}(R^+) = R_L$ and $R$ is an $E$-ring.

COROLLARY 5. *If $R$ is an $E$-ring, there is a* 1-1 *correspondence between elements $a$ of $R$, and rings $R_a$ over $R^+$. This correspondence maps* 1 *into $R$, and units $u$ of $R$ into rings $R_u$ with identity $u^{-1}$ such that $R_u \cong R$.*

PROOF. By Lemma 8, every element $*_a$ of $\mathrm{Mult}(R^+)$ is associative, and so defines a ring $R_a$ over $R^+$, whose multiplication is given by $x * y = axy$; since every ring $R'$ over $R^+$ gives rise to some multiplication in $R^+$, this correspondence is 1-1. The original ring $R = R_1$, and clearly $u^{-1}$ is an identity for $R_u$. Of course all rings with identity over $R^+$ are isomorphic by Corollary 4.

LEMMA 9. *Any endomorphic image of an $E$-group is an $E$-group; every endomorphism of an endomorphic image of an $E$-group $R^+$ can be extended to an endomorphism of $R^+$.*

PROOF. If $R$ is an $E$-ring, any endomorphic image of $R^+$ has the form $aR^+$ for some $a \in R$ by Corollary 1. Define a multiplication $*$ on $aR^+$ by $ax * ay = axy$. Then $*$ induces a commutative ring $S$ over $aR^+$ with identity $a$. Let $f \in \mathscr{E}(aR^+)$; then $f \cdot a_L \in \mathscr{E}(R^+)$, so for all $ax \in aR$,

$$f(ax) = (f \cdot a_L)(1) \cdot x = f(a) \cdot x = ayx,$$

where $ay = f(a) \in aR^+$. Hence $f(ax) = ay * ax$, so $f$ is multiplication by $ay$ in $S$. By Lemma 6, $S$ is an $E$-ring, and $aR^+$ an $E$-group.

Let $f \in \mathscr{E}(aR^+)$, say $f(a) = ay$. Then $y_L \in \mathscr{E}(R^+)$ is an extension of $f$, for if $ax \in aR$,

$$f(ax) = ay * ax = axy.$$

Unfortunately, no such nice property seems to be true in general for rings $R \cong \mathscr{E}(R^+)$. We do know that $\mathscr{E}(R^+) = R_L \oplus K$, where $K = \{f \,|\, f(1) = 0\}$. If $R$ is not commutative, then by Lemma 5, $K \neq 0$, so $R^+$ is isomorphic to a proper direct summand, the inverse image of $R_L$. Clearly, none of the rings described in Theorem 1–5 have this property, so they are all $E$-rings. In addition, we have the following partial result, corresponding to Theorem 6.

THEOREM 7. *Let $R$ be a mixed ring with identity with a set $S$ of relevant primes. Let*

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$$A = \{x \in R \mid h(x)(p) = \infty \text{ for all } p \in S\},$$

*and let* $U = \oplus^*_{p \in S} R_p$. *Then R is an E-ring if and only if*:

(1) $R_p \cong (Cp^{k_p})$, $0 < k_p < \infty$ *for all* $p \in S$

(2) *R is an extension of A by an E-ring T such that* $t(R) \subset T \subset U$

(3) *If* $f \in \mathscr{E}(R^+)$, *then the restriction of* $f - f(1)_L$ *to A is the zero map.*

(4) *If* $f \in \mathscr{E}(R^+)$, *then the unique homomorphism* $\bar{f}: T/t(R) \to A$ *induced by* $f - f(1)_L$ *is the zero map.*

Proof. Suppose $R$ is an $E$-ring. Condition (1) follows from Lemma 2. By Theorem 6, $R$ is an extension of $A$ by a ring $T \cong \mathscr{E}(T^+)$ such that $t(R) \subset T \subset U$. Since $T$ is commutative, it is an $E$-ring. Conditions (3) and (4) are trivial, since $= f(1)_L$.

Now assume that $R$ satisfies conditions (1)–(4), let $f \in \mathscr{E}(R^+)$, and consider the endomorphism $f' = f - f(1)_L$. Since $f'|_{R_p} \in \mathscr{E}(R_p)$, it is multiplication in $R_p$ by $f'(1_p)$, where $1_p$, the $p$-component of 1, is the identity of $R_p$. Hence $f'(1_p) = 0$ for all $p \in S$, so $f'|_{t(R)} = 0$. By condition (3), $f'|_A = 0$, so $A \oplus t(R)$ is contained in the kernel of $f'$; thus $f'$ induces a unique homorphism $\bar{f}$ on

$$R^+/A \oplus t(R) \cong T/t(R),$$

whose image must be a subgroup of $A$, since $T/t(R)$ is $S$-divisible. Hence by condition (4) $\bar{f} = 0$ so $f = f(1)_L$. Thus $R$ is an $E$-ring.

Remark. Conditions (3) and (4) are of little use in either constructing $E$-rings, or deciding whether a ring is an $E$-ring. It might be conjectured that they could be replaced by stronger conditions, for example:

(3′) *A is an E-ring*

(4′) *Hom* $(T, A) = 0$.

However, (3′) is true if (by Theorem 6) and only if (since $\mathscr{E}(R^+)$ is commutative) the extension of Theorem 7 splits, so (3′) implies (4′). While no non-splitting extension has been constructed, there seems no reason to believe they do not exist. Condition (4′) seems more reasonable, and I conjecture that it is a necessary condition for $R$ to be an $E$-ring.

## 4. The nature of the isomorphism: the non-commutative case

Let $R \cong \mathscr{E}(R^+)$, with $R$ not commutative. Then we know from the remarks preceding Lemma 6 that:

$$\mathscr{E}(R^+) = R_L \oplus K, \text{ where } K = \{f \in \mathscr{E}(R) \mid f(1) = 0\} \neq 0.$$

Hence End $(R^+)$, and consequently $R^+$, is an *ID-group*, that is, a group which is isomorphic to a proper direct summand. (It is not difficult to find elements of $K$: for example, if $x$ is not in the centre of $R$, then $x_L - x_R \in K$). Now if $R^+$ is an

$ID$-group, there exist monomorphisms $\phi \in \mathscr{E}(R^+)$ such that $R^+ = \phi(R^+) \oplus H$, where $H \neq 0$. Beaumont and Pierce [1] have proved the following structure theorem for $ID$-groups $R^+$:

Let $M = \bigcap_{n<\omega} \phi^n(R^+)$, let $P = \oplus^*_{n<\omega} H_n$, where $H_n = \phi^n(H) \cong H$, and let $S = \oplus_{n<\omega} H_n$. Then $R^+$ is an extension of $M$ by a group $T$ such that $S \subset T \subset P$, and $\phi \mid M$ is an automorphism of $M$.

In our case, let $F: R \to \mathscr{E}(R^+)$ be the isomorphism, and define $\phi$ by $\phi(x) = F^{-1}(x_L)$ for $x \in R^+$. Then $F(\phi(R^+)) = R_L$ so $F(H) = K$ and by Beaumonts and Pierce's result, $R^+$ is an extension of $\bigcap_{n<\omega} \varphi^n(R^+)$ by a subgroup of $\oplus^*_{\aleph_0} K$. If $R$ contains a fully invariant $E$-ring $A$, then clearly $F$ can be modified so that $F(x) = x_L$ for $x \in A$, and in this case, $\phi\,|_A$ is the identity, so $A^+ \subset \bigcap_{n<\omega} \phi^n(R^+)$. In general however, little else is known.

LEMMA 10. *Let $R \cong \mathscr{E}(R^+)$, $R$ not commutative. Then the group $U$ of units of $R$ contains a copy of every finite group.*

PROOF. It will suffice to show that $U$ contains a copy of $S(n)$, the symmetric group on $n$ symbols for every positive integer $n$. Now

$$R^+ = H \oplus \phi(H) \oplus \cdots \oplus \phi^{n-1}(H) \oplus \phi^n(R^+),$$

where $\phi^i(H) \cong H$, so each $x \in R^+$ can be expressed as

$$x = (x_1, \phi(x_2), \cdots, \phi^{n-1}(x_n), y),$$

with $x_i \in H$ for $i = 1, 2, \cdots n$. For $\sigma \in S(n)$, define $f_\sigma \in \mathscr{E}(R^+)$ by

$$f_\sigma(x_1, \phi(x_2), \cdots, \phi^{n-1}(x_n), y) = (x_{\sigma(1)}, \phi(x_{\sigma(2)}), \cdots, \phi^{n-1}(x_{\sigma(n)}), y).$$

$f_\sigma$ is clearly an automorphism of $R^+$, so the set $\{f_\sigma : \sigma \in S(n)\}$ is a group of units of $\mathscr{E}(R^+)$ isomorphic to $S(n)$.

## References

[1] R. A. Beaumont and R. S. Pierce, 'Isomorphic Direct Summands of Abelian Groups', *Math. Annalen* 153 (1964), 21–37.

[2] L. Fuchs, *Abelian Groups* (Hungarian Academy of Science, Budapest, 1958).

[3] P. Schultz, 'Periodic Homomorphism Sequences of Abelian Groups', *Arch. Math.* 21 (1970), 132–135.

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