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$\mathcal{L}$-INVARIANTS ARISING FROM CONJUGATE MEASURES OF $\mathrm{Sym}^2 E$

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Abstract. We construct three p-adic L-functions attached to the symmetric square of a modular elliptic curve. Following a calculation of Perrin-Riou for one of these functions, we compute the derivative of the p-adic L-function associated to the square of the non-unit root of Frobenius at p. This generalises Greenberg's notion of $\mathcal{L}$-invariant to these three-dimensional Galois representions.

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0. Introduction. The study of adjoint modular forms has proven to be a fruitful area of number theory. Let $\mathrm{Sym}^2 E$ denote the symmetric square of a modular elliptic curve E defined over the rationals. Central to our understanding of the Iwasawa theory of $\mathrm{Sym}^2 E$ is a predicted link between certain arithmetic Iwasawa modules (in the p-ordinary case Selmer groups over the $\mathbb{Z}_p$-extension of $\mathbb{Q}$) and the p-adic L-functions attached to the motive.

Assume that E has good ordinary reduction at a prime $p \neq 2$. Then the local Euler factor at p is of the form

$$(1 - \alpha_p^2 X)(1 - pX)(1 - \overline{\alpha_p}^2 X),$$

where α_p is a p-adic unit. A conjecture of Perrin-Riou [**12**] predicts the existence of a map $\mathbf{L}^p(\mathrm{Sym}^2 E(2))$ interpolating Dirichlet twists of the complex L-function $L(\mathrm{Sym}^2 E, s)$ at $s = 2$; the map is parametrized by removing exactly one of the linear factors above and so should really be thought of as three p-adic L-functions rather than just a single one.

In §1–§4 we construct three analytic L-series corresponding to the three components of $\mathbf{L}^p$ (Existence Theorem, p. 54), except that for the factor $(1 - pX)$ our L-series interpolates the square of the special values. For example, the element obtained by removing $(1 - \alpha_p^2 X)$ is essentially the Iwasawa L-function constructed by Coates and Schmidt [**2**]. Whilst we prove the existence of the components of $\mathbf{L}^p$ we cannot prove uniqueness as our p-adic distributions are only 3-admissible.

Two out of the three L-functions vanish at zero even though $L(\mathrm{Sym}^2 E, 2)$ is non-zero. Perrin-Riou [**13**] has calculated the derivative of the Coates-Schmidt L-function under the assumption that $\mathbf{L}^p$ comes from a norm-compatible system. In §5–§7 we calculate the derivative for the component obtained by removing $(1 - \overline{\alpha_p}^2 X)$ (Derivative Theorem, p. 62); the corresponding formula thus generalizes Greenberg's notion of $\mathcal{L}$-invariant [**8**] to the conjugate p-adic measure.

1. Preliminaries. We begin by recalling some well-known properties of the symmetric square. Suppose that E denotes a modular elliptic curve defined over $\mathbb{Q}$. By the symmetric square $\mathrm{Sym}^2 E$ we mean the pure motive over $\mathbb{Q}$ whose l-adic realisations are $\mathrm{Sym}^2 H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_l)$. As usual we define its L-series by

$$L(\mathrm{Sym}^2 E, s) := \prod_p \mathfrak{D}_p(p^{-s})^{-1} \qquad (\mathrm{Re}(s) > 2),$$

with

$$\mathfrak{D}_p(X) := \det\Big(1 - \mathrm{Frob}_p^{-1} X \big| \big(\mathrm{Sym}^2 H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_l)\big)^{I_p}\Big) \quad \text{for any prime } l \neq p,$$

and where we have fixed a decomposition group $G_{\mathbb{Q}_p}$ with inertial subgroup I_p (note that this definition is independent of the choice of l). If χ denotes a Dirichlet character we write $L(\mathrm{Sym}^2 E, \chi, s)$ for the twisted series $\prod_p \mathfrak{D}_p(\chi(p)p^{-s})^{-1}$.

As a consequence of the work of Gelbart and Jacquet [**7**], $\mathrm{Sym}^2 E$ can be identified with a cuspidal automorphic representation of GL_3 via a base-change lift from GL_2. They prove that for all twists $\mathrm{Sym}^2 E \otimes \chi$ the completed L-function

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{R}}(s - \nu) L(\mathrm{Sym}^2 E \otimes \chi, s)$$

has analytic continuation to the whole s-plane and satisfies a functional equation relating the value at s to the value at $3 - s$.

Central to our interpolation method is the following result of Sturm. Recall for an arbitrary Dirichlet character χ that its Gauss sum is given by

$$G(\chi) := \sum_{n=1}^{\mathrm{cond}(\chi)} \chi(n) \exp\left(\frac{2\pi i n}{\mathrm{cond}(\chi)}\right).$$

We shall write Ω_E^+ (resp. Ω_E^-) for the real (resp. imaginary) period of a Néron differential associated to a minimal Weierstrass equation for E over $\mathbb{Z}$. In [**18**,**19**] Sturm demonstrates that at the critical points the special values

$$\frac{G(\chi)\, L(\mathrm{Sym}^2 E, \overline{\chi}, 1)}{(2\pi i)^{-1}\Omega_E^+ \Omega_E^-} \quad \text{and} \quad \frac{G(\chi)^2\, L(\mathrm{Sym}^2 E, \overline{\chi}, 2)}{(2\pi i)\Omega_E^+ \Omega_E^-}$$

are algebraic numbers lying in the field generated over $\mathbb{Q}$ by the values of χ. Hence we can consider these values as p-adic numbers via some fixed embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$.

Unfortunately the point of symmetry in the functional equation lies between $s = 1$ and $s = 2$, which prevents us from interpolating at both critical points simultaneously. However, the properties of the Kubota-Leopoldt p-adic L-functions enable us to extend our distributions outside of the critical strip and thus check the admissibility of the associated measures.

For the rest of this article we assume that $p \neq 2$. If $\mathbb{Q}(\mu_{p^\infty})$ denotes the field obtained by adjoining all p-power roots of unity to $\mathbb{Q}$, then $G_\infty := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) = \Gamma \times \Delta$, where $\Gamma \cong \mathbb{Z}_p$ and $\Delta \cong (\mathbb{Z}/p\mathbb{Z})^\times$; we define $G_\infty^+ := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})^+/\mathbb{Q})$ to be the Galois group of its maximal real subfield.

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Let γ_0 be a topological generator of Γ. We write $\mathbb{Z}_p[\![\Gamma]\!]$ for the Iwasawa algebra of Γ, which is isomorphic to the power series ring $\mathbb{Z}_p[\![T]\!]$ via the map $\gamma_0 \mapsto 1 + T$. In general this is too small to contain all p-adic L-functions that arise from interpolation problems. For $r \in \mathbb{N}$ let us define

$$\mathcal{H}_r(T) := \left\{ h(T) \in \mathbb{Q}_p[\![T]\!] \text{ such that } h(T) \text{ is } o\big(\log_p^r(1+T)\big) \right\}.$$

Whilst this is not a ring we can easily form one by putting $\mathcal{H}(T) := \cup_r \mathcal{H}_r(T)$; we set

$$\mathcal{H}(G_\infty) := \mathcal{H}(T) \otimes_{\mathbb{Z}_p[\![\Gamma]\!]} \mathbb{Z}_p[\![G_\infty]\!].$$

Let $\mathbb{C}_p$ denote the Tate field, i.e. the completion of the algebraic closure of $\mathbb{Q}_p$. The group of continuous characters $\mathfrak{X}_p := \mathrm{Hom}_{\mathrm{cont}}(G_\infty, \mathbb{C}_p^\times)$ acts naturally on G_∞ and this action extends by linearity and continuity to both $\mathbb{Z}_p[\![G_\infty]\!]$ and $\mathcal{H}(G_\infty)$.

We fix once and for all an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$. Under this embedding we can identify Dirichlet characters of p-power conductor with elements of $\mathfrak{X}_p^{\mathrm{tors}}$. We shall denote the p^{th}-cyclotomic character by κ; this gives the Galois action on the p-power roots of unity.

Finally, if X denotes any module with an action of complex conjugation, then we write X_+ (resp. X_-) for the part on which complex conjugation acts by $+1$ (resp. -1).

2. Properties of the map $\mathbf{L}^p$. In the monograph [**12**] Perrin-Riou outlines a beautiful theory for the p-adic L-function $\mathbf{L}^p(M)$ associated to a motive M with good reduction at p. She predicts that such functions originate as norm-compatible elements in the inverse limits of certain Galois cohomology groups, which can then be transformed into $\mathbf{L}^p(M)$ via an interpolating homomorphism, LOG_∞, say.

In particular $\mathbf{L}^p(M)$ is parametrized by a suitable exterior power of the Dieudonné module associated to the p-adic representation, and is defined by its special values on a certain set of Tate twists $J \subset \mathbb{Z}$. The motive is called "J-admissible" if this set of twists is large enough to uniquely determine the p-adic L-function. As an example consider the Tate motive $\mathbb{Q}(1)$; the norm-compatible elements are the cyclotomic units, $\mathbf{L}^p(M)$ is (up to normalisation) the Kubota-Leopoldt p-adic zeta-function, and the map LOG_∞ is none other than the power series construction of Coleman.

Before specialising to the case of the symmetric square, we recall from [**6**] the definition of the topological $G_{\mathbb{Q}_p}$-modules B_{crys} and B_{dR}: B_{dR} is a discrete valuation field with residue field $\mathbb{C}_p$ and decreasing filtration B_{dR}^i for $i \in \mathbb{Z}$. The subring B_{crys} has in addition a Frobenius operator and a filtration induced from that of B_{dR}. If V is a finite-dimensional p-adic representation of $G_{\mathbb{Q}_p}$, then we define vector spaces by

$$\mathbf{D}_{\mathrm{cr}}(V) := \big(V \otimes B_{\mathrm{crys}}\big)^{G_{\mathbb{Q}_p}} \qquad \text{and} \qquad \mathbf{D}_{\mathrm{dR}}(V) := (V \otimes B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}.$$

The space V is said to be *crystalline* (resp. *de Rham*) if $\dim_{\mathbb{Q}_p}\mathbf{D}_{\mathrm{cr}}(V) = \dim_{\mathbb{Q}_p} V$ (resp. $\dim_{\mathbb{Q}_p}\mathbf{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$). Both spaces have decreasing exhaustive filtrations induced from B_{dR}, and $\mathbf{D}_{\mathrm{cr}}(V)$ has a Frobenius operator we shall denote by φ; moreover, if V is crystalline, then $\mathbf{D}_{\mathrm{cr}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$.

From now on $M = \mathrm{Sym}^2 E(2)$ and $V = \mathrm{Sym}^2 T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $T_p E$ is the p-adic Tate module of E. If M_{dR} is the de Rham realisation of M, let $\mathbf{e}$ denote a base over $\mathbb{Q}$ of the space $\det M_{\mathrm{dR}}$ and fix a generator γ_B of $\det\big(\mathrm{Sym}^2 H_1(E, \mathbb{Z})_+\big)$ over $\mathbb{Z}$. Then the complex period $\Omega_{\infty,\omega_{\mathbb{Q}}}$ is defined by $\Omega_{\infty,\omega_{\mathbb{Q}}}\mathbf{e} = \omega_{\mathbb{Q}} \wedge \gamma_B$, where $\omega_{\mathbb{Q}}$ is a chosen base of $\mathrm{Fil}^0 M_{\mathrm{dR}}$. Multiplying $\omega_{\mathbb{Q}}$ by an element of $\mathbb{Q}^\times$ if necessary, we shall assume that

$$\Omega_{\infty,\omega_{\mathbb{Q}}} = (2\pi i)\Omega_E^+\Omega_E^- .$$

Analogously there is a p-adic period map

$$\begin{aligned} \Omega_{p,\omega_{\mathbb{Q}}} : \wedge^2 \mathbf{D}_{\mathrm{dR}}(V) &\to \mathbb{Q}_p \otimes \det M_{\mathrm{dR}} \\ n &\mapsto \omega_{\mathbb{Q}} \wedge n. \end{aligned}$$

Whilst $\Omega_{p,\omega_{\mathbb{Q}}}(n)$ clearly depends on the choice of parameter $n \in \wedge^2 \mathbf{D}_{\mathrm{dR}}(V)$, the complex period $\Omega_{\infty,\omega_{\mathbb{Q}}}$ is fixed.

FORMULA VAL.SP (M, χ)—[**12**]. *Assume that E has good reduction at $p \neq 2$. There should exist a map $\mathbf{L}^p = \mathbf{L}^p(M)$ in* $\mathrm{Hom}(\wedge^2 \mathbf{D}_{\mathrm{cr}}(V), \mathcal{H}(G_\infty^+))$ *satisfying*

$$\wedge^2(\varphi)^{-m_\chi}\chi^{-1}(\mathbf{L}^p)\mathbf{e} = \frac{G(\chi)^2\ L(\mathrm{Sym}^2 E, \overline{\chi}, 2)}{\Omega_{\infty,\omega_{\mathbb{Q}}}}\Omega_{p,\omega_{\mathbb{Q}}}$$

for all non-trivial even characters $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$ of conductor p^{m_χ}, with constant term

$$(1 - p^{-1}\varphi^{-1})\mathbf{1}(\mathbf{L}^p)\mathbf{e} = \frac{L_{\{p\}}(\mathrm{Sym}^2 E, 2)}{\Omega_{\infty,\omega_{\mathbb{Q}}}}(1 - \varphi)\Omega_{p,\omega_{\mathbb{Q}}} .$$

In down-to-earth terms, the above formulae predict how the Euler factor at p should be modified in order to yield admissible p-adic functions. Thus the dimension of $\wedge^2 \mathbf{D}_{\mathrm{cr}}(V)$ reflects the number of (linearly-independent) L-functions that can be interpolated by hand.

REMARK. The Frobenius $\wedge^2(\varphi)$ acts on $\wedge^2 \mathbf{D}_{\mathrm{cr}}(V)$ while χ^{-1} should be viewed as a specialisation from $\mathcal{H}(G_\infty^+)$ to $\overline{\mathbb{Q}_p}$. In fact the formula at the trivial character $\mathbf{1}$ was omitted from [**12**] because p^{-1} is an eigenvalue of φ on $\mathbf{D}_{\mathrm{cr}}(V)$, so that the operator $(1 - \varphi)$ is not invertible. (Note that the action of φ on $f \in \mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{D}_{\mathrm{cr}}(V), \mathbb{Q}_p)$ is given by $\varphi(f)(x) = p^{-1} f(\varphi^{-1}x)$.)

It should be pointed out that by itself VAL.SP(M, χ) *does not* determine these functions uniquely; in fact we need to establish analogous formulae VAL.SP$(M, \chi\kappa^j)$ for a J-admissible subset of $\mathbb{Z}$. At present we are unable to do this due to the lack of an algebraicity result at non-critical Tate twists. In order to construct a function satisfying VAL.SP(M, χ) it is sufficient to find p-adic L-functions corresponding to a suitably chosen eigenbasis for $\wedge^2(\varphi)$ on $\wedge^2 \mathbf{D}_{\mathrm{cr}}(V)$.

From now on we assume that E has good ordinary reduction at $p \neq 2$. Factorizing the characteristic polynomial of Frobenius at p by

$$X^2 - a_p X + p \ = \ (X - \alpha_p)(X - \beta_p)$$

we suppose that $\alpha_p \in \mathbb{Z}_p^\times$ as in the introduction. If $U = T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ then pick generators $u_0 \in \mathbf{D}_{\mathrm{cr}}(U)^{\varphi = \alpha_p^{-1}}$, $u_{-1} \in \mathbf{D}_{\mathrm{cr}}(U)^{\varphi = \beta_p^{-1}}$ and assume $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{cr}}(U) = \mathbb{Q}_p(u_0 + \lambda u_{-1})$ with $\lambda \neq 0$ (so that U does not split). We can then define bases $e_0 = u_0^2$ for $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = \alpha_p^{-2}}$, $e_{-1} = u_0 u_{-1}$ for $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = p^{-1}}$ and $e_{-2} = u_{-1}^2$ for $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = \beta_p^{-2}}$, respectively; moreover e_{-1} is uniquely determined if we specify that $u_0 \wedge u_{-1}$ equals 1 in $\mathbf{D}_{\mathrm{cr}}(\mathbb{Q}_p(1)) = \mathbb{Q}_p$. For the same reasons

$$\mathbf{e} = e_0 \wedge e_{-1} \wedge e_{-2}$$

is independent of the choice of $\{u_0, u_{-1}\}$ and generates $\det \mathbf{D}_{\mathrm{cr}}(V)$.

Finally, we have a φ-eigenbasis $n_{\alpha^2} = e_{-1} \wedge e_{-2}$, $n_p = e_0 \wedge e_{-2}$ and $n_{\beta^2} = e_0 \wedge e_{-1}$ for the space $\wedge^2 \mathbf{D}_{\mathrm{cr}}(V)$ that we shall use to parametrize the map $\mathbf{L}^p$. Each basis element corresponds to choosing a root of $\mathfrak{D}_p(X)$ when p-adically interpolating $L(\mathrm{Sym}^2 E, \overline{\chi}, 2)$.

3. p-adic distributions on the cyclotomic extension. We begin by fixing some notations; let $\mathfrak{H}$ denote the upper half plane. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ the action $\gamma z := \frac{az+b}{cz+d}$ defines an automorphism of $\mathfrak{H} \cup \mathbb{R} \cup \{\infty\}$. If $g : \mathfrak{H} \to \mathbb{C}$ is any function we write the (weight 2) action of γ on g as

$$(g|\gamma)(z) \ := \ (\det \gamma)^{\frac{1}{2}} (cz + d)^{-2} g(\gamma z).$$

If g, f are continuous functions on $\mathfrak{H}$ with transform like modular forms of weight 2 and character ρ under the action of the congruence modular group $\Gamma_0(C)$, then we normalise the Petersson inner product via

$$\langle g, f \rangle_C := \int_{\Gamma_0(C) \backslash \mathfrak{H}} \overline{g(z)} f(z) \, d\,\mathrm{Im}(z) \, d\,\mathrm{Re}(z).$$

Now suppose that g is an eigenform of weight 2, exact level pC and nebentypus character ρ such that $4|C$, $(p, C) = 1$ and $(\mathrm{cond}(\rho), C) = 1$. If g has the q-expansion $\sum_{n \geq 1} \alpha_n q^n$ with $q = e^{2\pi i z}$, then we define the Hecke operator U_p and the involution $\#$ via

$$g|U_p := \sum_{n \geq 1} \alpha_{np} q^n \quad \text{and} \quad g^{\#} := \sum_{n \geq 1} \overline{\alpha_n} q^n, \quad \text{respectively.}$$

In particular $g|U_p \ = \ \alpha_p g$ and $g^{\#}|U_p \ = \ \overline{\alpha_p} g^{\#}$ with $\alpha_p \neq 0$.

It is perhaps easier to phrase everything in terms of p-adic measures. By a p-adic distribution $d\mu$ on $\mathbb{Z}_p^\times$ with values in a ring R we mean a finitely additive function from the compact open subsets of $\mathbb{Z}_p^\times$ whose image lies in R. Recall that $G_\infty \cong \mathbb{Z}_p^\times$ via the cyclotomic character κ, and so we may interchange these two groups as we please.

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Now assume $d\mu$ takes values in $\mathbb{C}_p$. We say that $d\mu$ is a *bounded measure* if

$$\left| \int_{a+p^n\mathbb{Z}_p} d\mu \right|_p \qquad \text{is bounded independently of } a \text{ and } n,$$

for all $n \in \mathbb{N}$ and $(a, p) = 1$. Let h be a positive integer and let x_p denote the inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{C}_p$. We recall from [**20**] that the p-adic distributions $x_p^r d\mu$ extend to an *h-admissible measure* if

$$\left| \int_{a+p^n\mathbb{Z}_p} (x-a)_p^r \, d\mu \right|_p = \left| \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \int_{a+p^n\mathbb{Z}_p} x_p^j \, d\mu \right|_p \qquad \text{is of } o\big(p^{n(h-r)}\big)$$

for all $n \in \mathbb{N}$, $(a, p) = 1$ and $r = 0, \ldots, h-1$. Equivalently the Mellin transform

$$\mathrm{Mel}_\mu := \int_{x \in \mathbb{Z}_p^\times} (1+T)^x \, d\mu$$

is a function of type $o\big(\log_p^h(1+T)\big)$ on the unit disc. In particular, Mel_μ is uniquely determined by its special values at χx_p^r for all Dirichlet characters χ of p-power conductor and all integers $r = 0, \ldots, h-1$.

Definition. Let $d\mu(g)$ denote the p-adic distribution satisfying

$$\int_{\mathbb{Z}_p^\times} \chi \, d\mu(g) = \frac{G(\chi)}{(\alpha_p^2)^{m_\chi}} \frac{\mathrm{cond}(\rho_\chi)\mathrm{cond}(\chi)}{G(\rho_\chi)} \times \frac{(1-\chi(p)\alpha_p^{-2}p)(1-\overline{\rho_\chi}(p))}{(1-\rho_\chi(p)p^{-1})} \frac{L(\mathrm{Sym}^2(g), \overline{\chi}, 2)}{\pi^3\langle g, g\rangle_{pC}}$$

for all Dirichlet characters χ of conductor p^{m_χ}, with $\rho_\chi := (\overline{\chi}\rho)_{\mathrm{prim}}$.

Key Lemma. *The distribution $d\mu(g)$ extends to an even $\big([2\mathrm{ord}_p\alpha_p]+1\big)$-admissible measure. Furthermore, if α_p is a p-adic unit, then $d\mu(g)$ is an even bounded measure.*

Proof. A similar type of result was proven in [**3**] and so we briefly sketch the argument—the major difference here is that p now divides the level of g and may also divide the conductor of ρ. We stick to our previous notation (which was originally developed by Panchishkin in [**11**]).

Choose integers $M, M' \in p^{\mathbb{N}}$ such that $p\,\mathrm{cond}(\chi) \big| M$, $p^2\mathrm{cond}(\chi)^2 \big| M'$ and pM' is a square. Then, for all $s \in \mathbb{C}$ and Dirichlet characters χ such that $\chi(-1) = (-1)^\nu$ with $\nu \in \{0, 1\}$ we define the complex-valued distribution $D_{s,M}(\chi)$ by

$$D_{s,M}(\chi) := \frac{G(\chi)\,\mathrm{cond}(\chi)^{s-1}}{\alpha_p^{2m_\chi}} \Big(1 - \chi(p)\alpha_p^{-2}p^{s-1}\Big) L^\bullet(\mathrm{Sym}^2(g), \overline{\chi}, s) \,.$$

Here the superscript $^\bullet$ indicates that the Euler factors at bad primes l such that $l|C$, $\alpha_l = 0$ have been removed. Of course at the end we shall have to remember to put them back in!

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Now $D_{s,M}(\chi)$ can be written as a Rankin convolution of g with a theta-series, and this convolution has a useful representation as a scalar product. Skipping some tedious algebra which can be found in [**3**, §2.2] we deduce the identity

$$(4\pi)^{-(\frac{s+\nu}{2})}\Gamma\Big(\frac{s+\nu}{2}\Big)D_{s,M}(\chi) \;=\; \frac{i^\nu(pCM')^{\frac{2s-1}{4}}}{\alpha_p^{1+\mathrm{ord}_p M'}}\;\zeta_{pC}(2s-2,\overline{\chi}^2\rho^2)$$
$$\times\Big\langle g^{\#}\big|V_C\;,\;\theta^{(\nu)}(\chi_M)\big|W_{4pCM'}\;E(z,s+\nu-2)\Big\rangle_{4C^2pM'}\;.$$

Here $g^{\#}\big|V_C=\sum_{n\geq 1}\overline{\alpha_n}q^{nC}$, the operator $W_{\cdot}$ denotes the Atkin-Lehner involution acting on modular forms of level $4pCM'$ and half-integral weight, and χ_M is the Dirichlet character modulo M induced from χ. The theta-series defined by

$$\theta^{(\nu)}(\chi_M):=\sum_{n\geq 1}\chi_M(n)n^\nu q^{n^2}$$

has weight $\nu+\frac{1}{2}$ and character $\chi\big(\frac{-1}{\cdot}\big)^\nu$, where $\big(\frac{\cdot}{\cdot}\big)$ denotes the Jacobi symbol. As in [**16**], define the automorphic factor of half-integral weight by

$$j_\theta(\gamma,z)\;:=\;\Big(\frac{c}{d}\Big)\sigma_d^{-1}(cz+d)^{\frac{1}{2}},$$

where $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(4)$ and $\sigma_d=\begin{cases}1 & \text{if } d\equiv 1(4),\\ i & \text{if } d\equiv 3(4).\end{cases}$

On putting $\xi:=\overline{\chi}\rho\big(\frac{-1}{\cdot}\big)^\nu\big(\frac{C}{\cdot}\big)$, the Eisenstein series $E(z,s;\,\frac{3-2\nu}{2},\xi,4C^2pM')$ is given by

$$E(z,s):=\mathrm{Im}(z)^{\frac{s}{2}}\sum_{\gamma\in\mathrm{Stab}_\infty\backslash\Gamma_0(4C^2pM')}\xi(d_\gamma)j_\theta(\gamma,z)^{-(3-2\nu)}\big|j_\theta(\gamma,z)\big|_\infty^{-2s},$$

which is of weight $\frac{3-2\nu}{2}$, character $\overline{\xi}$, level $4C^2pM'$ and is absolutely convergent for $\mathrm{Re}(s)>\nu+\frac{1}{2}$. Let us define gamma factors $\Gamma^{\pm}(s,\chi)$ by

$$\Gamma^+(s,\chi):=\frac{2i^{1-\nu}\Gamma(s-1)\Gamma(s)}{(2\pi)^{2s-1}}\cos\Big(\frac{\pi(s+\nu-2)}{2}\Big)\quad\text{and}\quad\Gamma^-(s,\chi):=\frac{\Gamma(s)}{(2\pi)^s}\;.$$

If we normalise distributions via $D^{\pm}_{s,M}(\chi):=\Gamma^{\pm}(s,\chi)\,\frac{D_{s,M}(\chi)}{\langle g,g\rangle_{pC}}$, then applying the trace map from $X_0(4C^2pM')$ to $X_0(4C^2p)$ we find that

$$D^{\pm}_{s,M}(\chi)\;=\;C^{-(\frac{2s-1}{4})}\,\frac{\gamma(M')}{\langle g,g\rangle_{pC}}$$
$$\times\Big\langle g^{\#}\big|V_C,\,\big(\theta^{(\nu)}(\chi_M)\big|V_C\cdot G^{\pm}(z,s+\nu-2)\big|U_p^{\mathrm{ord}_pM'}\;W_{4C^2p}\Big\rangle_{4C^2p}$$

with $\gamma(M'):=\frac{2i^\nu C^{\frac{2\nu+1}{4}}}{\alpha_p^{1+\mathrm{ord}_pM'}}$. Here the Eisenstein series $G^{\pm}(z,s)$ are (up to a normalisation) the functions considered by Shimura in [**17**] who went on to calculate their Fourier expansions.

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Remark. The series $\big(\theta^{(\nu)}(\chi_M)\big|V_C\cdot G^{\pm}(z,s+\nu-2)\big)$ are only real analytic modular forms but we can compute their holomorphic projections. Firstly, if $s=1$ then $G^{-}(z,s+\nu-2)$ has bounded growth. If Hol denotes the operator of holomorphic projection, then one can prove that

$$\mathrm{Hol}\big(\theta^{(\nu)}(\chi_M)\big|V_C\cdot G^{-}(z,s+\nu-2)\big)\quad (s=1)$$

is a cusp form of weight 2 and character ρ. Similarly, if $s=2$ and $\xi^2\neq\mathbf{1}$, then $G^{+}(z,s+\nu-2)$ also has bounded growth and hence

$$\mathrm{Hol}\big(\theta^{(\nu)}(\chi_M)\big|V_C\cdot G^{+}(z,s+\nu-2)\big)\quad (s=2,\ \xi^2\neq\mathbf{1})$$

is again a cusp form of weight 2 and character ρ. In the exceptional case $\xi^2=\mathbf{1}$ we can only say that $\mathrm{Hol}(\cdot)$ is a holomorphic modular form.

Putting $F^{\pm}(z,s,\chi):=C^{-\left(\frac{2s-1}{4}\right)}\,\mathrm{Hol}\big(\theta^{(\nu)}(\chi_M)\big|V_C\cdot G^{\pm}(z,s+\nu-2)\big)$, at the two critical points we have

$$D^{-}_{1,M}(\chi)\ =\ \gamma(M')\,\ell_g\Big(F^{-}(z,1,\chi)\big|U_p^{\mathrm{ord}_p M'}\ W_{4C^2p}\Big)$$

and

$$D^{+}_{2,M}(\chi)\ =\ \gamma(M')\,\ell_g\Big(F^{+}(z,2,\chi)\big|U_p^{\mathrm{ord}_p M'}\ W_{4C^2p}\Big)\,,$$

where Hida's linear functional ℓ_g sends a modular form h (of weight 2, level $4C^2p$ and character $\overline{\rho}$) to the algebraic number $\frac{\langle g^{\#}|V_C,h\rangle_{4C^2p}}{\langle g,g\rangle_{pC}}$. Note that the non-holomorphic part of $\big(\theta^{(\nu)}(\chi_M)\big|V_C\cdot G^{\pm}(z,s+\nu-2)\big)$ is killed off by the operator $\ell_g\circ W_{4C^2p}\circ U_p^{\mathrm{ord}_p M'}\circ\mathrm{Hol}$.

For the moment we focus on the value at $s=2$; under our embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}_p$ define the p-adic distribution dD^{+} by

$$\int_{\mathbb{Z}_p^\times}\chi\,dD^{+}:=\frac{\mathrm{cond}(\rho_\chi)}{G(\rho_\chi)}\,\frac{1-\overline{\rho_\chi}(p)}{1-\rho_\chi(p)p^{-1}}\ \times\ D^{+}_{2,M}(\chi)\,.$$

By Atkin-Lehner theory the functional ℓ_g degenerates into a finite $\overline{\mathbb{Q}}$-linear combination of the Fourier coefficients of $F^{+}(z,2,\chi)\big|U_p^{\mathrm{ord}_p M'}$, and so to prove that dD^{+} extends to an h-admissible measure (with $h=[2\mathrm{ord}_p\alpha_p]+1$) it is enough to establish the h-admissibility of each Fourier coefficient separately (the notation $x_p^j\,dD^{+}$ will be used for the corresponding distributions). We now give a description of these coefficients.

Definition. For $s=2,3,4,\dots$ and $n\in\mathbb{N}_0$ define algebraic numbers

$$v^{+}(M'n,s,\chi)\quad:=\sum_{\substack{M'n=Cn_1^2+n_2\\ p\nmid n_1\\ n_1,n_2\in\mathbb{N}}}\chi(n_1)\,n_1^{\nu}\,\big(M'n-Cn_1^2\big)^{\frac{s-\nu-1}{2}}\,\beta(n_2,s-1,\varepsilon_{n_2}\overline{\chi})$$
$$\times\ \frac{G(\varepsilon_{n_2})}{G(\rho)}\ \frac{C^{-\left(\frac{2s-1}{4}\right)}}{\chi\left(\frac{\mathrm{cond}(\varepsilon_{n_2})}{\mathrm{cond}(\rho)}\right)\cdot\left(\frac{\mathrm{cond}(\varepsilon_{n_2})}{\mathrm{cond}(\rho)}\right)^{s-1}}$$

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$$\times \left\{ \frac{\mathrm{cond}(\varepsilon_{n_2}\overline{\chi})^{s-1}}{G(\varepsilon_{n_2}\overline{\chi})} \frac{1-\overline{\varepsilon_{n_2}}\chi(p)p^{s-2}}{1-\varepsilon_{n_2}\overline{\chi}(p)p^{-(s-1)}} \right.$$
$$\left. \times \frac{2i^{1-\nu}\Gamma(s-1)}{(2\pi)^{s-1}} \cos\left(\frac{\pi(s+\nu-2)}{2}\right) \zeta_{pC}(s-1, \varepsilon_{n_2}\overline{\chi}) \right\},$$

where $\varepsilon_{n_2}(\,\cdot\,) := \left(\left(\frac{-Cn_2}{\cdot}\right)\rho(\,\cdot\,)\right)_{\mathrm{prim}}$ and

$$\beta(n_2, s-1, \varepsilon_{n_2}\overline{\chi}) := \sum_{\substack{a,b \in \mathbb{N} \\ (a,pC)=(b,pC)=1 \\ ab|m}} \mu(a)\, \varepsilon_{n_2}\overline{\chi}(a)\, \left(\varepsilon_{n_2}\overline{\chi}\right)^2(b)\, a^{1-s}\, b^{3-2s}$$

with m denoting the largest integer such that $n_2/m^2 \in \mathbb{N}$.

The reason behind this (painful!) definition of the $v^+(M'n, s, \chi)$'s is that at $s = 2$ they turn up in the q-expansion of $F^+(z, 2, \chi)\big|U_p^{\mathrm{ord}_p M'}$. Specifically we already calculated in [**3**, p. 593] that

$$\frac{\mathrm{cond}(\rho_\chi)}{G(\rho_\chi)} \frac{1-\overline{\rho_\chi}(p)}{1-\rho_\chi(p)p^{-1}} \times \left(F^+(z, 2, \chi)\Big|U_p^{\mathrm{ord}_p M'}\right) = \sum_n v^+(M'n, 2, \chi)\, q^n \quad ,$$

unless ρ_χ is a real quadratic character, in which case we should modify $v^+(M'n, 2, \chi)$ by a term of type $O\big(|M'n|_p^{\frac{1}{2}}\big)$ (which we ignore as it does not affect admissibility).

The observant reader will have spotted that the $\{\,\cdot\,\}$-expression above is none other than the special value

$$\int_{(\mathbb{Z}/c_{n_2}\mathbb{Z})^\times \times \mathbb{Z}_p^\times} \chi x_p^{s-1} \,.\, d\nu(\zeta_p) \otimes \varepsilon_{n_2},$$

where c_{n_2} is the prime-to-p-part of $\mathrm{cond}(\varepsilon_{n_2})$ and $d\nu(\zeta_p)$ denotes the bounded pseudo-measure associated to the Kubota-Leopoldt p-adic zeta-function interpolating $\zeta_C(s-1, \overline{\chi})$ for $s-1 \in \mathbb{N}$. (We actually avoid its pole because $\varepsilon_{n_2}\overline{\chi}$ is never trivial.)

Examining the precise form of the Fourier coefficients, we see that $v^+(M'n, s, \chi)$ is congruent (modulo M') to a linear combination of terms like

$$\chi(u)u^{s-1} \times \int_{(\mathbb{Z}/c_{n_2}\mathbb{Z})^\times \times \mathbb{Z}_p^\times} \chi x_p^{s-1} \,.\, d\nu(\zeta_p) \otimes \varepsilon_{n_2}, \quad \text{for various } u \in \mathbb{Z}_p^\times \cap \mathbb{Q}.$$

Combining this fact with the degeneracy of the functional ℓ_g, in order to bound the integral

$$\int_{a+M\mathbb{Z}_p} (x-a)_p^{s-2}\, dD^+ = \sum_{j=0}^{s-2} \binom{s-2}{j} \frac{(-a)^{s-2-j}}{\phi(M)} \sum_{\chi \bmod M} \chi^{-1}(a) \int_{\mathbb{Z}_p^\times} \chi x_p^j\, dD^+,$$

for $s = 2, 3, 4, \ldots$ and $(a, p) = 1$, it is enough to bound the expressions

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$$\sum_{j=0}^{s-2}\binom{s-2}{j}\frac{(-a)^{s-2-j}}{\phi(M)}\sum_{\chi \bmod M}\chi^{-1}(a)\,.\,\gamma(M')\int_{x\in\mathbb{Z}_p^\times}\chi(ux)(ux_p)^{j+1}\,.\,d\nu(\zeta_p)\otimes\varepsilon_{n_2}$$
$$= \quad \gamma(M')\,u^{s-1}\int_{x\equiv au^{-1}\ (\mathrm{mod}\ M)}(x-au^{-1})_p^{s-2}\,.\,x_p\,d\nu(\zeta_p)\otimes\varepsilon_{n_2}\quad.$$

This last term has $O\Big(|\gamma(M')|_p|M|_p^{s-2}\Big)$, as $d\nu(\zeta_p)$ is bounded and so choosing $M'=pM^2$ yields a bound of type $O\Big(|M|_p^{s-2-[2\mathrm{ord}_p\alpha_p]}\Big)$ from the definition of $\gamma(M')$. Consequently dD^+ extends to an h-admissible measure (resp. a bounded measure if $\mathrm{ord}_p\alpha_p=0$). Moreover it is an even measure since the $\big(x_p\,d\nu(\zeta_p)\otimes\varepsilon_{n_2}\big)$'s are even.

One can play the same game at $s=1$ using the p-adic distribution dD^- defined by $\int_{\mathbb{Z}_p^\times}\chi\,dD^- := D^-_{1,M}(\chi)$. An identical argument to the one above shows that dD^- extends to an even h-admissible (resp. bounded if $\mathrm{ord}_p\alpha_p=0$) measure, except that the Fourier coefficients of $F^-(z,1,\chi)\big|U_p^{\mathrm{ord}_pM'}$ are now combinations of p-adic zeta-functions interpolating "$\zeta_C(s-1,\overline{\chi})$" for $s=1,0,-1,\ldots$ instead.

Finally, we must replace the missing Euler factors in $L^\bullet(\mathrm{Sym}^2(g),\overline{\chi},s)$ whilst retaining our admissibility conditions. The (imprimitive) functional equation between $L^\bullet(\mathrm{Sym}^2(g),\overline{\chi},2)$ and $L^\bullet(\mathrm{Sym}^2(g),\chi,1)$ means that the distributions dD^+ and dD^- are contragredient and so it is enough to prove that the Euler factors we are replacing are coprime to the corresponding dual Euler factors as elements of $\mathbb{Z}_p[\![T]\!][\Delta]$. This can be accomplished by applying the Weiestrass Preparation Theorem and showing that as functions on the open disc

$$\big\{T\in\mathbb{C}_p\ \big|\ |T|_p<1\big\}$$

their zeros are disjoint. (See [**3**, p. 603].)

The proof of our lemma is therefore complete.

4. Existence of the map $\mathbf{L}^p$. We now give the main result of this article. We state the result only in terms of the motive $\mathrm{Sym}^2E(2)$ although one can easily formulate the corresponding version of this theorem for $\mathrm{Sym}^2E(1)$ via the functional equation.

EXISTENCE THEOREM. *Assume that E has good ordinary reduction at $p\neq 2$.*
(a) *There exists a unique element $\mathbf{L}^p(n_{\alpha^2})\in\mathbb{Z}_p[\![G_\infty^+]\!]\otimes\mathbb{Q}$ satisfying*

$$\chi^{-1}(\mathbf{L}^p(n_{\alpha^2}))=\frac{G(\chi)^2\mathrm{cond}(\chi)}{(\alpha_p^2)^{m_\chi}}\,\frac{L(\mathrm{Sym}^2E,\overline{\chi},2)}{(2\pi i)\Omega_E^+\Omega_E^-}$$

for all non-trivial characters $\chi\in\mathfrak{X}_p^{\mathrm{tors}}$, with a trivial zero at **1**.
(b) *There exists an element $\boldsymbol{L}^p(n_{\beta^2})\in\mathcal{H}(G_\infty^+)$ of type $O(log_p^2)$ satisfying*

$$\chi^{-1}\big(\mathbf{L}^p(n_{\beta^2})\big)=\frac{G(\chi)^2\mathrm{cond}(\chi)}{(\beta_p^2)^{m_\chi}}\,\frac{L(\mathrm{Sym}^2E,\overline{\chi},2)}{(2\pi i)\Omega_E^+\Omega_E^-}$$

for all non-trivial characters $\chi\in\mathfrak{X}_p^{\mathrm{tors}}$, with a trivial zero at **1**.

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(c) *There exists an element* $\widetilde{\mathbf{L}^p(n_p)} \in \mathcal{H}(G_\infty^+)$ *of type* $O(\log_p^2)$ *satisfying*

$$\chi^{-1}\Big(\widetilde{\mathbf{L}^p(n_p)}\Big) \;=\; \left(\frac{G(\chi)^2\mathrm{cond}(\chi)}{(p)^{m_\chi}}\;\frac{L(\mathrm{Sym}^2E,\overline{\chi},2)}{(2\pi i)\Omega_E^+\Omega_E^-}\right)^2$$

for all non-trivial characters $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$, *with leading term*

$$\mathbf{1}\Big(\widetilde{\mathbf{L}^p(n_p)}\Big) \;=\; \left(\left(1-\frac{1}{p}\right)\left(1-\frac{\alpha_p}{\beta_p}\right)\left(1-\frac{\beta_p}{\alpha_p}\right)\frac{L(\mathrm{Sym}^2E,2)}{(2\pi i)\Omega_E^+\Omega_E^-}\right)^2.$$

Remark. Unfortunately only $\mathbf{L}^p(n_{\alpha^2})$ is uniquely determined by this data, both $\mathbf{L}^p(n_{\beta^2})$ and $\widetilde{\mathbf{L}^p(n_p)}$ requiring further information at an extra two Tate twists. The notation $\widetilde{\mathbf{L}^p(n_p)}$ indicates that this should be related to the *square* of the true component $\mathbf{L}^p(n_p)$ of type $O(\log_p)$(?) predicted by Formula VAL.SP(M, χ).

Proof. Nothing changes if we twist Sym^2E by the quadratic character of conductor 4 and so without loss of generality we may assume that N_E is divisible by 4 (where N_E denotes the conductor of E).

Let f_E denote the newform associated to a strong Weil parametrization of E. We put $g(z) = f_E(z) - \beta_p f_E(pz)$, which is an eigenform of weight 2, level pN_E and trivial character $\rho = \mathbf{1}$; in particular

$$L(\mathrm{Sym}^2(g),s) \;=\; \Big(1-\beta_p^2p^{-s}\Big)\big(1-p^{1-s}\big)L(\mathrm{Sym}^2E,s).$$

Consequently the distribution $\frac{\pi^3\langle g,g\rangle_{pN_E}}{(2\pi i)\Omega_E^+\Omega_E^-}\,d\mu(g)$ is bounded by our Key Lemma since $\mathrm{ord}_p\alpha_p = 0$. Taking Mellin transforms, this measure corresponds to a bounded power series $\mathbf{L}^p(n_{\alpha^2})$. Moreover Sturm's algebraicity result at $s = 2$ and the evenness of $d\mu(g)$ implies that the element $\mathbf{L}^p(n_{\alpha^2})$ lies in $\mathbb{Z}_p[\![G_\infty^+]\!]\otimes\mathbb{Q}$, so that part (a) is proved.

The proof of (b) is identical except that we use instead the conjugate newform $g^\#(z) = f_E(z) - \alpha_p f_E(pz)$ so that

$$L(\mathrm{Sym}^2(g^\#),s) \;=\; \Big(1-\alpha_p^2p^{-s}\Big)\big(1-p^{1-s}\big)L(\mathrm{Sym}^2E,s)\;.$$

This time our Key Lemma implies that the distribution $\frac{\pi^3\langle g^\#,g^\#\rangle_{pN_E}}{(2\pi i)\Omega_E^+\Omega_E^-}\,d\mu(g^\#)$ extends to an $h^\#$-admissible measure, where $h^\# = [2\mathrm{ord}_p\beta_p]+1 = 3$. Its Mellin transform $\mathbf{L}^p(n_{\beta^2})$ will thus be of type $o(\log_p^3)$ or more accurately $O(\log_p^2)$.

Finally, the product of $\mathbf{L}^p(n_{\alpha^2})$ and $\mathbf{L}^p(n_{\beta^2})$ yields a power series, $\mathcal{G}$ say, of type $O(\log_p^2)$, which has the same special values at non-trivial $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$ as the element predicted in part (c); (this follows from the identity $\alpha_p^2\beta_p^2 = p^2$). However $\mathcal{G}$ has at least a double zero at $\mathbf{1}$ because both $\mathbf{L}^p(n_{\alpha^2})$ and $\mathbf{L}^p(n_{\beta^2})$ have trivial zeros. Fortunately $\mathcal{H}(\Gamma)$ contains some very useful elements; for example the function $\frac{\log(\gamma_0)}{\gamma_0-1}$, which is zero on the whole of $\mathfrak{X}_p^{\mathrm{tors}}$ except at the trivial character where it equals 1. This allows us to modify the value of $\mathcal{G}$ at $\mathbf{1}$ as we please whilst preserving the $O(\log_p^2)$ condition. In particular this implies the existence of $\widetilde{\mathbf{L}^p(n_p)}$.

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Remark. The method even works at bad primes. If we assume that E has potential good ordinary reduction at $p > 3$ and E is not the quadratic twist of a curve with good reduction, then there exists a character ψ of Δ such that the new-form $g = f_E \otimes \psi$ has level $\widetilde{N} = p^{-1}N_E$. Consequently we can use our Key Lemma to produce measures $\frac{\pi^3 \langle g,g\rangle_{\tilde{N}}}{(2\pi i)\Omega_E^+\Omega_E^-}\, d\mu(g) \otimes \psi^2$ (resp. $\frac{\pi^3 \langle g^\#,g^\#\rangle_{\tilde{N}}}{(2\pi i)\Omega_E^+\Omega_E^-}\, d\mu(g^\#) \otimes \psi^{-2}$) which are the analogues of $\mathbf{L}^p(n_{\alpha^2})$ (resp. $\mathbf{L}^p(n_{\beta^2})$) in the bad reduction case.

Taking the Mellin transform of the convolution of these two measures and then computing its special values, we prove the following result.

Theorem. *Assume that E has potential good ordinary reduction at $p > 3$. Then there exists an element $\widetilde{\mathbf{L}^p(?)} \in \mathcal{H}(G_\infty^+)$ of type $O(\log_p^2)$ satisfying*

$$\chi^{-1}\left(\widetilde{\mathbf{L}^p(?)}\right) = \left(\frac{G(\chi)^2 \mathrm{cond}(\chi)}{(p)^{m_\chi}}\, \frac{L(\mathrm{Sym}^2 E, \overline{\chi}, 2)}{(2\pi i)\Omega_E^+\Omega_E^-}\right)^2$$

for all Dirichlet characters $\chi \neq \mathbf{1}, \psi^2, \psi^{-2}$ of conductor p^{m_χ}.

5. Local Iwasawa theory. In the next three sections we use Perrin-Riou's local Iwasawa theory to obtain a formula for the derivative of $\mathbf{L}^p(n_{\beta^2})$. The calculation for the component $\mathbf{L}^p(n_{\alpha^2})$ has already been done in [**13**, §2.3] but we include it as it is very interesting to compare the two. All these formulae rely upon the hypothesis that there exists a norm-compatible family in the global Galois cohomology that yields the map $\mathbf{L}^p$.

For this section V will denote any crystalline representation of $G_{\mathbb{Q}_p}$. If K is a field and $i \in \mathbb{N}_0$, then we write $H^i(K, \cdot)$ for the Galois cohomology groups $H^i_{\mathrm{cont}}(G_K, \cdot)$ defined using continuous cochains. Recall that Bloch and Kato [**1**] define subspaces of $H^1(\mathbb{Q}_p, V)$ by

$$H^1_f(\mathbb{Q}_p, V) := \mathrm{Ker}\big(H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}})\big),$$

$$H^1_g(\mathbb{Q}_p, V) := \mathrm{Ker}\big(H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}})\big),$$

and an exponential map

$$\exp_{f,V} : \mathbf{D}_{\mathrm{cr}}(V)\big/\mathrm{Fil}^0 \longrightarrow H^1_f(\mathbb{Q}_p, V).$$

In particular, if $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi=1} = 0$, then $\exp_{f,V}$ is an isomorphism and we denote the inverse map by $\log_{f,V}$. Under the cup product pairing the quotient map $\exp_{f/e,V^*(1)}$: $\mathbf{D}_{\mathrm{cr}}(V^*(1))\big/(1-\varphi) \to H^1_{f/e}(\mathbb{Q}_p, V^*(1))$ induces a dual exponential map

$$\exp_V^* : H^1_g(\mathbb{Q}_p, V) \longrightarrow \mathbf{D}_{\mathrm{cr}}(V)^{\varphi=p^{-1}}$$

with $H^1_f(\mathbb{Q}_p, V)$ as the kernel.

Let ϵ denote a generator of the Tate module $\mathbb{Z}_p(1)$, and fix a positive integer h such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cr}}(V) = \mathbf{D}_{\mathrm{cr}}(V)$. It is the main result of [**14**] that there exists a unique $\mathcal{H}(G_\infty)$-homomorphism

$$\Omega^{\epsilon}_{V,h} : \mathcal{H}(G_\infty) \otimes \mathbf{D}_{\mathrm{cr}}(V) \to \mathcal{H}(G_\infty) \otimes \varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})/\mathbf{T}^{G_{K_\infty}}$$

such that for all integers j satisfying $h + j \geq 1$ and $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = p^{-j}} = 0$, we have

$$\exp_{f, V(j)}\big((1 - p^{-j-1}\varphi^{-1})(1 - p^j\varphi)^{-1}\kappa^j(g)\big) = (-1)^j\, \Gamma(h+j)\, \pi_0\big(\Omega^{\epsilon}_{V,h}(g) \otimes \epsilon^{\otimes j}\big)$$

for all $g \in \mathcal{H}(G_\infty) \otimes \mathbf{D}_{\mathrm{cr}}(V)$. Here $\mathbf{T}$ is a Galois-stable lattice in V and π_0 is the natural projection from $\varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}(j))$ to $H^1(\mathbb{Q}_p, V(j))$.

The map $\Omega^{\epsilon}_{V,h}$ depends on the choice of h and ϵ but, if $h' > h$ are sufficiently large then

$$\Omega^{\epsilon}_{V,h} = \prod_{j=h}^{h'-1}\left(j - \frac{\log_p \gamma_0}{\log_p \kappa(\gamma_0)}\right)^{-1} \Omega^{\epsilon}_{V,h'}.$$

Definition. For $h \geq 1$ define $\mathrm{LOG}_\infty : \varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}) \to \mathrm{Frac}(\mathcal{H}(G_\infty)) \otimes \mathbf{D}_{\mathrm{cr}}(V)$ by

$$\mathrm{LOG}_\infty(x) := \prod_{j=0}^{h-1}\left(j - \frac{\log_p \gamma_0}{\log_p \kappa(\gamma_0)}\right)\big(\Omega^{\epsilon}_{V,h}\big)^{-1}(x)\,.$$

Let $\varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})_f$ (resp. $\varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})_g$) denote all the elements in $\varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})$ that lie in $H^1_f(\mathbb{Q}_p, V)$ (resp. $H^1_g(\mathbb{Q}_p, V)$) under the map π_0.

Proposition. **[13]** *There exists a section S^ϵ from $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = p^{-1}} \otimes \mathrm{Frac}(\mathcal{H}(G_\infty))$ onto $\varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})_g \otimes \mathrm{Frac}(\mathcal{H}(G_\infty))$ such that if either one of $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = p^{-1}}$ or $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cr}}(V)$ is non-zero, then*

$$(1 - p^{-1}\varphi^{-1})(1 - \varphi)^{-1}\partial \mathrm{LOG}_\infty(x) \equiv \log_{g, V} \pi_0(x) \bmod \mathrm{Fil}^0\mathbf{D}_{\mathrm{cr}}(V),$$

where

$$\log_{g, V}(y) := \log_{f, V}\big(y - \pi_0(S^\epsilon(\exp^*_V(y)))\big) \ \text{ for } y \in H^1_g(\mathbb{Q}_p, V)$$

and ∂ denotes the differential operator $\lim_{s \to 0} \frac{d\kappa^s}{ds}(\,\cdot\,)$.

The section S^ϵ is designed to split the sequence

$$0 \to \varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})_f \to \varprojlim H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T})_g \to \mathbf{D}_{\mathrm{cr}}(V)^{\varphi = p^{-1}}$$

after tensoring with $\mathrm{Frac}(\mathcal{H}(G_\infty))$. The construction of S^ϵ depends upon an embedding of the field B_{st} into B_{dR} which in terms of Iwasawa theory, is equivalent to picking a branch of the logarithm satisfying $\log_p p = 0$. In fact $\log_{g, \mathbb{Q}_p(1)}$ equals $\log_p$ upon identifying $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ with the completed tensor product $\mathbb{G}_m(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Q}_p = \varprojlim\left(\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ via Kummer theory.

We also mention that the proof of the above proposition requires the explicit reciprocity laws recently proved (independently) by Benois, Colmez and Kato-Tsuji-Kurihara. (See [**13**, §1.3] for details of the construction of S^{ϵ}.)

6. $\mathcal{L}$-invariants via Selmer groups. From now on $\mathbf{T} = \mathrm{Sym}^2 T_p E$ and $V = \mathbf{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We shall apply the proposition of the last section to calculate the value of $\partial \mathrm{LOG}_\infty(x)$ in terms of the dual exponential map $\exp_V^* : H^1(\mathbb{Q}_p, V) \longrightarrow \mathbf{D}_{\mathrm{cr}}(V)^{\varphi = p^{-1}}$. We want formulae of the type

$$\partial \mathrm{LOG}_\infty(x) \wedge n \;=\; (\text{Euler factor}) \times (\mathcal{L}\text{-invariant}) \,.\, e_0 \wedge \exp_V^*(\pi_0(x)) \wedge e_{-2},$$

where $x \in \varprojlim H^1(\mathbb{Q}(\mu_{p^n}), \mathbf{T})_+ \otimes \mathbb{Q}_p$, and the "$\mathcal{L}$-invariant" is a p-adic number depending on the $G_{\mathbb{Q}}$-representation V and the parameter $n \in \wedge^2 \mathbf{D}_{\mathrm{cr}}(V)$. In order to define the $\mathcal{L}$-invariants we must choose coordinates (μ_0, μ_1, μ_2) on the cohomology group $H^1(\mathbb{Q}_p, V)$ (remember that $\dim_{\mathbb{Q}_p} H^1(\mathbb{Q}_p, V) = 3$). Since $H^1(\mathbb{Q}_p, V) = H^1_g(\mathbb{Q}_p, V)$ we use the map $\log_{g,V}$ of the previous section to aid us.

We assumed E had good ordinary reduction at p, so that as a $G_{\mathbb{Q}_p}$-representation V has the ordinary filtration

$$0 \subset \mathrm{F}^2 V \subset \mathrm{F}^1 V \subset V, \quad \text{where } I_p \text{ acts on } \mathrm{gr}^i(V) \text{ via } \kappa^i$$

with $\mathbf{D}_{\mathrm{cr}}(\mathrm{F}^2 V) = \mathbb{Q}_p e_{-2}$ and $\mathbf{D}_{\mathrm{cr}}(\mathrm{F}^1 V) = \mathbb{Q}_p e_{-2} \oplus \mathbb{Q}_p e_{-1}$. The short exact sequence $0 \to \mathrm{F}^2 V \stackrel{j}{\to} \mathrm{F}^1 V \stackrel{\delta}{\to} \mathbb{Q}_p(1) \to 0$ induces an exact sequence on cohomology

$$0 \to H^1(\mathbb{Q}_p, \mathrm{F}^2 V) \stackrel{j}{\to} H^1(\mathbb{Q}_p, V) \stackrel{\delta}{\to} H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$$

since $H^0(\mathbb{Q}_p, \mathbb{Q}_p(1)) = 0$ and $H^1(\mathbb{Q}_p, \mathrm{F}^1 V) = H^1(\mathbb{Q}_p, V)$. In fact this sequence must be right-exact because $\dim_{\mathbb{Q}_p} H^1(\mathbb{Q}_p, \mathrm{F}^2 V) = 1$ and $\dim_{\mathbb{Q}_p} H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) = 2$.

Remark. If Pr_{-2} denotes the natural projection $\mathbf{D}_{\mathrm{cr}}(V) \twoheadrightarrow \mathbf{D}_{\mathrm{cr}}(V)^{\varphi = \beta_p^{-2}}$ then we have a well-defined section

$$\Sigma^{\epsilon} := \exp_{f,V} \circ \mathrm{Pr}_{-2} \circ \log_{g,V}$$

from $H^1(\mathbb{Q}_p, V)$ onto $H^1(\mathbb{Q}_p, \mathrm{F}^2 V)$, because the space $\mathbf{D}_{\mathrm{cr}}(V)^{\varphi = \beta_p^{-2}}$ is isomorphic to $H^1(\mathbb{Q}_p, \mathrm{F}^2 V) = H^1_f(\mathbb{Q}_p, \mathrm{F}^2 V)$ via $\exp_{f,V}$. Clearly Σ^{ϵ} depends on the choice of ϵ as $\log_{g,V}$ is constructed using the section S^{ϵ}.

We thus get our first coordinate $\mu_2 : H^1(\mathbb{Q}_p, V) \twoheadrightarrow \mathbb{Q}_p$ given by

$$\mu_2(\,\cdot\,) e_{-2} := \log_{f,V} \circ \; \Sigma^{\epsilon}(\,\cdot\,) \;=\; \mathrm{Pr}_{-2} \circ \log_{g,V}(\,\cdot\,)\,.$$

To define the other two coordinates μ_0, μ_1 we use a little Kummer theory. To begin with

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) = \mathbb{G}_m(\mathbb{Q}_p) \widehat{\otimes} \mathbb{Q}_p & \stackrel{\sim}{\to} & \mathbb{Q}_p \oplus \mathbb{Q}_p \\ \mathbf{q} & \mapsto & \log_p \mathbf{q} \oplus \mathrm{ord}_p\, \mathbf{q}\,. \end{array}$$

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Now $H^1_f(\mathbb{Q}_p, \mathrm{F}^1V) = H^1_f(\mathbb{Q}_p, V)$ implies that $\mathrm{Im}(\log_{g,V}) = \mathrm{Im}(\log_{f,V}) = \mathbf{D}_{\mathrm{cr}}(\mathrm{F}^1V)$; moreover, under the projection map $\mathrm{Pr}_{-1} : \mathbf{D}_{\mathrm{cr}}(V) \twoheadrightarrow \mathbf{D}_{\mathrm{cr}}(V)^{\varphi=p^{-1}}$ we have

$$\mathrm{Pr}_{-1} \circ \log_{g,V} = \left(\log_{g,\mathbb{Q}_p(1)} \circ \delta\right)e_{-1} = \left(\log_p \circ \delta\right)e_{-1} .$$

Consequently $\mathrm{ord}_p \circ \delta$ maps the kernel of $\log_{g,V}$ bijectively onto $\mathbb{Q}_p$. The function $\mathrm{ord}_p \circ \delta$ is closely related to the dual exponential map. In fact

$$\exp^*_V = \left(\mathrm{ord}_p \circ \delta\right)e_{-1} \quad \text{as elements of } \mathbf{D}_{\mathrm{cr}}(V)^{\varphi=p^{-1}},$$

because $\exp^*_{\mathbb{Q}_p(1)}$ is simply the valuation map on $\mathbb{G}_m(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Q}_p$. In view of this we define $\mu_0, \mu_1 : H^1(\mathbb{Q}_p, V) \twoheadrightarrow \mathbb{Q}_p$ by

$$\mu_0 := \mathrm{ord}_p \circ \delta \qquad \text{and} \qquad \mu_1 := \log_p \circ \delta .$$

Summarizing, we have the commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \mathbf{D}_{\mathrm{cr}}(\mathrm{F}^2V) & \xrightarrow{\text{incl.}} & \mathbf{D}_{\mathrm{cr}}(\mathrm{F}^1V) & \xrightarrow{\mathrm{Pr}_{-1}} & \mathbf{D}_{\mathrm{cr}}(V)^{\varphi=p^{-1}} & \longrightarrow 0 \\
 & \uparrow{\scriptstyle \log_{f,V}} & & \uparrow{\scriptstyle \log_{g,V}} & & \uparrow{\scriptstyle \log_p(\cdot)e_{-1}} & \\
0 \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^2V) & \xrightarrow{j} & H^1(\mathbb{Q}_p, V) & \xrightarrow{\delta} & H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \longrightarrow 0 \\
 & \uparrow & & \uparrow & & \exists\uparrow{\scriptstyle \text{a map}} & \\
 & 0 & \longrightarrow & \mathrm{Ker}(\log_{g,V}) & \xrightarrow{\mathrm{ord}_p \circ \delta} & \mathbb{Q}_p & \longrightarrow 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

with exact rows and exact columns. We have proved the following result.

LEMMA. *There is a (non-canonical) isomorphism of $\mathbb{Q}_p$-vector spaces given by*

$$\mu_0 \oplus \mu_1 \oplus \mu_2 : H^1(\mathbb{Q}_p, V) \xrightarrow{\sim} \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus \mathbb{Q}_p .$$

The (weak) Selmer group attached to the representation V is defined by

$$\mathcal{S}'(V/\mathbb{Q}) := \mathrm{Ker}\left(H^1(\mathbb{Q}, V) \longrightarrow \bigoplus_{l \neq p} H^1(I_l, V)\right),$$

where I_l is the inertia subgroup in $G_{\mathbb{Q}_l}$ and the maps above denote restriction. Flach, Wiles and Diamond [**5**,**21**,**4**] have shown under various hypotheses that the Bloch-Kato Selmer group

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$$H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q}, V) := \mathrm{Ker}\left(\mathcal{S}'(V/\mathbb{Q}) \longrightarrow \frac{H^1(\mathbb{Q}_p, V)}{H^1_f(\mathbb{Q}_p, V)}\right)$$

is zero; (e.g. if E_p is an absolutely irreducible $G_{\mathbb{Q}}$-module then $H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q}, V) = 0$ by [**4**]).

Assuming the triviality of $H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q}, V)$, the weak Leopoldt conjecture for V and $V(-1)$ holds, and so the space $\mathcal{S}'(V/\mathbb{Q})$ is one-dimensional over $\mathbb{Q}_p$. Let $\mathfrak{z}'$ denote the image of a generator of $\mathcal{S}'(V/\mathbb{Q})$ in $H^1(\mathbb{Q}_p, V)$.

Definition. Assume that $H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q}, V)$ is zero and $\exp^*_V(\mathfrak{z}') \neq 0$. Define $\mathcal{L}$-invariants by

$$\mathcal{L}^{\mathrm{Gr}} := \frac{\mu_1(\mathfrak{z}')}{\mu_0(\mathfrak{z}')} \qquad \text{and} \qquad \mathcal{L}^{\mathrm{conj}} := \frac{\mu_1(\mathfrak{z}')}{\mu_0(\mathfrak{z}')} - \frac{2}{\lambda}\frac{\mu_2(\mathfrak{z}')}{\mu_0(\mathfrak{z}')},$$

with $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cr}}(U) = \mathbb{Q}_p(u_0 + \lambda u_{-1})$, $\lambda \neq 0$, as before.

The quantity $\mathcal{L}^{\mathrm{Gr}} = \frac{\log_p(\delta(\mathfrak{z}'))}{\mathrm{ord}_p(\delta(\mathfrak{z}'))}$ is none other than Greenberg's $\mathcal{L}$-invariant in [**8**]; the number $\mathcal{L}^{\mathrm{conj}}$ can thus be viewed as a generalization of this to the conjugate measure.

A priori it is not clear that these really are invariant under the choices made. First of all $\exp^*_V(\mathfrak{z}')$ is non-zero if and only if $\mathrm{ord}_p(\delta(\mathfrak{z}'))$ is non-zero, so at least we are not dividing by zero! Now changing $\mathfrak{z}'$ by an element of $\mathbb{Q}_p^\times$ will not affect the ratios $\frac{\mu_1}{\mu_0}$ and $\frac{\mu_2}{\mu_0}$ by the previous lemma and so it remains to show independence from our given basis of $\mathbf{D}_{\mathrm{cr}}(V)$.

Recalling that $\mathbf{e} = e_0 \wedge e_{-1} \wedge e_{-2}$ set $\omega_{\mathbf{e}} := \frac{1}{2\lambda}e_0 + e_{-1} + \frac{\lambda}{2}e_{-2}$ (which generates $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cr}}(V)$). Since $\frac{\mu_1(\mathfrak{z}')}{\mu_0(\mathfrak{z}')}$ does not depend on $\{e_0, e_{-1}, e_{-2}\}$ it suffices to demonstrate the same of $\frac{2}{\lambda}\frac{\mu_2(\mathfrak{z}')}{\mu_0(\mathfrak{z}')}$. Observing that $\mathrm{Pr}_{-2}(\omega)$ is proportional to $\mathrm{Pr}_{-1}(\omega)$ as we vary generators ω of $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cr}}(V)$, clearly $\frac{2}{\lambda}\frac{\mu_2(\mathfrak{z}')}{\mu_0(\mathfrak{z}')}$ is well-defined if and only if the ratio "$\frac{2}{\lambda}\frac{\mu_2(\mathfrak{z}')}{\mu_0(\mathfrak{z}')} \cdot \frac{\mathrm{Pr}_{-2}(\omega_{\mathbf{e}})}{\mathrm{Pr}_{-1}(\omega_{\mathbf{e}})}$" is too. However $\frac{2}{\lambda}\mathrm{Pr}_{-2}(\omega_{\mathbf{e}}) = e_{-2}$ and $\mathrm{Pr}_{-1}(\omega_{\mathbf{e}}) = e_{-1}$, so that

$$\frac{2}{\lambda}\frac{\mu_2(\mathfrak{z}')}{\mu_0(\mathfrak{z}')} \cdot \frac{\mathrm{Pr}_{-2}(\omega_{\mathbf{e}})}{\mathrm{Pr}_{-1}(\omega_{\mathbf{e}})} = \frac{\mu_2(\mathfrak{z}')e_{-2}}{\mu_0(\mathfrak{z}')e_{-1}} = \frac{\log_{f,V}(\Sigma^{\epsilon}(\mathfrak{z}'))}{\exp^*_V(\mathfrak{z}')}$$

which is independent of our original choice of $\{e_0, e_{-1}, e_{-2}\}$.

Proposition. *Assume that E has good ordinary reduction at $p \neq 2$, $H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q}, V)$ is zero and $\exp^*_V(\mathfrak{z}')$ is non-zero. If $x \in \varprojlim H^1(\mathbb{Q}(\mu_{p^n}), \boldsymbol{T})_+ \otimes \mathbb{Q}_p$ then*

$$(1 - p^{-1}\varphi^{-1})(1 - \varphi)^{-1}\partial\mathrm{LOG}_\infty(x) \wedge n_{\alpha^2} = -\left(\frac{1}{2\lambda}\right)\mathcal{L}^{\mathrm{Gr}} \cdot e_0 \wedge \exp^*_V(\pi_0(x)) \wedge e_{-2}$$

and

$$(1 - p^{-1}\varphi^{-1})(1 - \varphi)^{-1}\partial\mathrm{LOG}_\infty(x) \wedge n_{\beta^2} = -\left(\frac{\lambda}{2}\right)\mathcal{L}^{\mathrm{conj}} \cdot e_0 \wedge \exp^*_V(\pi_0(x)) \wedge e_{-2}\,.$$

Proof. Let us assume that $\exp_V^*(\pi_0(x))$ is non-trivial; we begin by proving the second statement. Put $d = (1 - p^{-1}\varphi^{-1})(1-\varphi)^{-1}\partial\mathrm{LOG}_\infty(x)$ which lies in $\mathbb{Q}_p e_0 \oplus \mathbb{Q}_p e_{-2}$. Then

$$d \wedge e_{-1} \wedge e_0 \;=\; d \wedge \omega_{\mathbf{e}} \wedge e_0 \;=\; \log_{g,V}(\pi_0(x)) \wedge \omega_{\mathbf{e}} \wedge e_0$$

by the proposition of the previous section. However $\log_{g,V}(\pi_0(x))\mu_1(\pi_0(x))e_{-1} \oplus\mu_2(\pi_0(x))e_{-2}$, which implies that

$$\begin{aligned} d \wedge e_{-1} \wedge e_0 &= \mu_1(\pi_0(x))e_{-1} \wedge \omega_{\mathbf{e}} \wedge e_0 \;\oplus\quad \mu_2(\pi_0(x))e_{-2} \wedge \omega_{\mathbf{e}} \wedge e_0 \\ &= \left(\frac{\lambda}{2}\right)\mu_1(\pi_0(x))e_0 \wedge e_{-1} \wedge e_{-2} \;\oplus\; -\mu_2(\pi_0(x))e_0 \wedge e_{-1} \wedge e_{-2} \,. \end{aligned}$$

Moreover $n_{\beta^2} = -(e_{-1} \wedge e_0)$ and $\exp_V^*(\pi_0(x)) = \mu_0(\pi_0(x))e_{-1} \neq 0$. Hence

$$d \wedge n_{\beta^2} = -\left(\frac{\lambda}{2}\right)\frac{\mu_1(\pi_0(x)) - \frac{2}{\lambda}\mu_2(\pi_0(x))}{\mu_0(\pi_0(x))} \,.\, e_0 \wedge \exp_V^*(\pi_0(x)) \wedge e_{-2} \,.$$

Thus the second assertion follows upon observing that

$$\mathbb{Q}_p\pi_0(x) \;=\; \pi_0\Big(\lim_{\leftarrow} H^1(\mathbb{Q}(\mu_{p^n}), \mathbf{T})_+ \otimes \mathbb{Q}_p\Big) \;=\; \mathrm{res}_p(\mathcal{S}'(V/\mathbb{Q})) = \mathbb{Q}_p\mathfrak{F}',$$

where $\mathrm{res}_p : H^1(\mathbb{Q}, V) \to H^1(\mathbb{Q}_p, V)$, as we then have

$$\frac{\mu_1(\pi_0(x)) - \frac{2}{\lambda}\mu_2(\pi_0(x))}{\mu_0(\pi_0(x))} = \frac{\mu_1(\mathfrak{F}')}{\mu_0(\mathfrak{F}')} - \frac{2}{\lambda}\frac{\mu_2(\mathfrak{F}')}{\mu_0(\mathfrak{F}')} \;=\; \mathcal{L}^{\mathrm{conj}} \,.$$

The proof of the first assertion is very similar. We just remark that

$$d \wedge e_{-1} \wedge e_{-2} \;=\; \log_{g,V}(\pi_0(x)) \wedge \omega_{\mathbf{e}} \wedge e_{-2} \;=\; -\left(\frac{1}{2\lambda}\right)\mu_1(\pi_0(x))e_0 \wedge e_{-1} \wedge e_{-2}$$

and then proceed as above.

7. Norm-compatible families. To make any further progress we must now assume that the function $\mathbf{L}^p$ is the image of a norm-compatible element under the map LOG_∞; c.f. [**12**, Conjecture 4.4.3]. Implicit in this assumption is that there should be some trick for relating the non-Iwasawa components $\mathbf{L}^p(n_p)$ and $\mathbf{L}^p(n_{\beta^2})$ to the complex L-values at $s = 3, 4$.

Hypothesis (*ES*).
There exists an element $\mathbf{z}_\infty$ *in* $\lim_{\leftarrow} H^1(\mathbb{Q}(\mu_{p^n}), \mathbf{T})_+ \otimes \mathbb{Q}_p$ *satisfying*

(A) $\quad \mathbf{L}^p(n)\mathbf{e} \;=\; \mathrm{LOG}_\infty(\mathbf{z}_\infty) \wedge n \quad$ for all $n \in \wedge^2\mathbf{D}_{\mathrm{cr}}(V)$;

(B) $\quad \exp_V^*(\pi_0(\mathbf{z}_\infty)) \;=\; -\left(1 - \frac{\alpha_p}{\beta_p}\right)\left(1 - \frac{\beta_p}{\alpha_p}\right)\frac{L(\mathrm{Sym}^2 E, 2)}{\Omega_{\infty,\omega_{\mathbb{Q}}}}\,\mathrm{Pr}_{-1}(\omega_{\mathbb{Q}}).$

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The value $\exp_V^*(\pi_0(\mathbf{z}_\infty))$ is automatically non-zero since the complex function $L(\mathrm{Sym}^2 E, s)$ does not vanish at $s = 2$; in particular if (ES) holds then

$$\mathbf{1}\big(\mathbf{L}^p(n_p)\big)\mathbf{e} = \left(1 - \frac{1}{p}\right)\left(1 - \frac{\alpha_p}{\beta_p}\right)\left(1 - \frac{\beta_p}{\alpha_p}\right)\frac{L(\mathrm{Sym}^2 E, 2)}{\Omega_{\infty,\omega_{\mathbb{Q}}}}\ \Omega_{p,\omega_{\mathbb{Q}}}(n_p)$$

as predicted by Formula VAL.SP(M, χ). Theoretically the dual exponential map should contain the congruence L-values as, for example, with the Kato-Beilinson Euler system.

Derivative Theorem. *Assume that E has good ordinary reduction at $p \neq 2$, the Selmer group $H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q}, V)$ is zero and there exists an element $\mathbf{z}_\infty$ satisfying Hypothesis (ES). Then*

(a) $$\partial\mathbf{L}^p(n_{\alpha^2}) = \mathcal{L}^{\mathrm{Gr}}\left(1 - \alpha_p^{-2}\right)\left(1 - p\alpha_p^{-2}\right)\frac{L(\mathrm{Sym}^2 E, 2)}{(2\pi i)\Omega_E^+\Omega_E^-},$$

(b) $$\partial\mathbf{L}^p(n_{\beta^2}) = \mathcal{L}^{\mathrm{conj}}\left(1 - \beta_p^{-2}\right)\left(1 - p\beta_p^{-2}\right)\frac{L(\mathrm{Sym}^2 E, 2)}{(2\pi i)\Omega_E^+\Omega_E^-}.$$

Proof. We start with part (a). Clearly all the conditions of the proposition in §6 are satisfied, since $\exp_V^*(\pi_0(\mathbf{z}_\infty)) \neq 0$; hence (ES) implies that

$$\begin{aligned}&\left(1 - p^{-1}\alpha_p^2\right)\left(1 - \alpha_p^{-2}\right)^{-1}\partial\mathbf{L}^p(n_{\alpha^2})\,\mathbf{e}\\ &\quad = \left(\frac{1}{2\lambda}\right)\mathcal{L}^{\mathrm{Gr}}\left(1 - \frac{\alpha_p}{\beta_p}\right)\left(1 - \frac{\beta_p}{\alpha_p}\right)\frac{L(\mathrm{Sym}^2 E, 2)}{\Omega_{\infty,\omega_{\mathbb{Q}}}}\ e_0 \wedge \mathrm{Pr}_{-1}(\omega_{\mathbb{Q}}) \wedge e_{-2}\,.\end{aligned}$$

Choosing $u \neq 0$, so that $\omega_{\mathbb{Q}} = u\omega_{\mathbf{e}}$ and $\mathrm{Pr}_{-1}(\omega_{\mathbb{Q}}) = ue_{-1}$, we obtain

$$\Omega_{p,\omega_{\mathbb{Q}}}(n_{\alpha^2}) \ = \ \omega_{\mathbb{Q}} \wedge e_{-1} \wedge e_{-2} \ = \ \left(\frac{u}{2\lambda}\right)\mathbf{e}\,.$$

Combining these two equations we find that

$$\partial\mathbf{L}^p(n_{\alpha^2})\,\mathbf{e} \ = \ \mathcal{L}^{\mathrm{Gr}}\left(1 - \alpha_p^{-2}\right)\left(1 - p\alpha_p^{-2}\right)\frac{L(\mathrm{Sym}^2 E, 2)}{\Omega_{\infty,\omega_{\mathbb{Q}}}}\ \Omega_{p,\omega_{\mathbb{Q}}}(n_{\alpha^2}),$$

and so (a) is proved. The proof of (b) follows identical lines.

An obvious question to ask is whether $\mathcal{L}^{\mathrm{Gr}}$ and $\mathcal{L}^{\mathrm{conj}}$ are non-zero. In unpublished work, Greenberg and Tilouine have shown that when $p \| N_E$ the analogue of (a) above is true with $\mathcal{L}^{\mathrm{Gr}} = \frac{\log_p \mathbf{q}_E}{\mathrm{ord}_p \mathbf{q}_E}$ ($\mathbf{q}_E$ being the Tate period of E). Furthermore, the fact that $\log_p \mathbf{q}_E \neq 0$ was recently proved by Barré-Sirieix, Diaz, Gramain and Philibert.

To establish a similar result in the good ordinary case we need three things.

(i) An explicit construction of a generator $\mathfrak{s}'$ of the space $\mathrm{res}_p(\mathcal{S}'(V/\mathbb{Q}))$.

(ii) An analytic description of the map $\log_{f,V} : H^1_f(\mathbb{Q}_p, \mathrm{F}^2 V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cr}}(\mathrm{F}^2 V)$.

(iii) The calculation of the image of $\mathfrak{s}'$ under $\log_{f,V} \circ \Sigma^\epsilon$ and $\log_p \circ\, \delta$.

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Addressing (i) first, Flach [**5**] constructs via K-theory elements $c(l) \in H^1(\mathbb{Q}, \mathbf{T})$, $l \nmid pN_E$ which are unramified outside p and l, but unfortunately for us have trivial image in $H^1_{/f}(\mathbb{Q}_p, \mathbf{T})$. As Kato suggested, a better place to look for such a generator might be in $K_3^{\mathrm{Mil}}\left(X_0(N_E) \times X_0(N_E) \otimes \mathbb{Z}\left[\frac{1}{N_E}\right]\right)$, where this is Milnor K-theory of the rational functions on $X_0(N_E) \times X_0(N_E)$ (the tricky part is the right choice of divisors for the modular units occuring in the cup-product).

Turning our attention to (ii), the space $\mathrm{F}^2 V$ is none other than the representation associated to the Tate module of the formal group of $E_{/\mathbb{Z}_p}$ tensored with itself. It is tempting to hope that $\mathrm{F}^2 V$ has an associated p-divisible group, but Tate has shown that such a representation must have Hodge-Tate weights in $\{0, 1\}$ yet $\mathrm{F}^2 V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \mathbb{C}_p(2)$.

All may not be lost—under our assumptions the Abelian surface $E \times E$ has good ordinary reduction over $\mathbb{Q}_p$. Consequently the formal group attached to the Néron model for $E \times E$ over $\mathbb{Z}_p$ has height $2 = \dim(E \times E)$. We write $\mathrm{Log}_{E\times E}$ for the extension to $E \times E$ of the formal group logarithm, so that

$$\mathrm{Log}_{E\times E} : \big(E \times E\big)(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Q}_p \longrightarrow \text{tangent space of } E \times E \,\big/\mathbb{Q}_p.$$

Now $H^1_f(\mathbb{Q}_p, \mathrm{F}^2 V)$ is contained in $H^1_f(\mathbb{Q}_p, V)$ which is itself a direct summand of

$$H^1_f\big(\mathbb{Q}_p, H^2_{\text{ét}}(\overline{E} \times \overline{E}, \mathbb{Q}_p(2))\big) \;\cong\; \big(E \times E\big)(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Q}_p\ ,$$

this last isomorphism coming from Bloch-Kato [**1**]. Identifying $\mathbf{D}_{\mathrm{cr}}(\mathrm{F}^2 V)$ within the tangent space of $E \times E$, the map $\log_{f,V}$ will then coincide with the restriction of $\mathrm{Log}_{E\times E}$ to $H^1_f(\mathbb{Q}_p, \mathrm{F}^2 V)$. Interestingly this description of $\mathcal{L}^{\mathrm{conj}}$ mixes up the logarithm map on a formal group of height 1 (i.e. $\mathbb{G}_m$) with the logarithm map on a formal group of height 2.

Finally, we have no idea at all how to attack (iii). Essentially we need to know "the shape" of $\mathrm{res}_p(\mathcal{S}'(V/\mathbb{Q}))$ inside $H^1(\mathbb{Q}_p, V)$. In the bad multiplicative case it turns out that $\mathbf{q}_E$ is a universal norm for the $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$ cut out by the image of the map

$$H^1(\mathbb{Q}_p, \mathrm{Sym}^2 T_p E) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{Z}_p) \;=\; \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\mathbf{F}_\infty/\mathbb{Q}_p), \mathbb{Z}_p) \;\cong\; \mathbb{Z}_p^2$$

induced by quotienting $\mathrm{Sym}^2 T_p E$ by its sublattice of strictly positive Hodge-Tate weight; (here $\mathbf{F}_\infty$ is the compositum of all the $\mathbb{Z}_p$-extensions of $\mathbb{Q}_p$). However in the good ordinary case there is no such easy local description for $\mathcal{S}'$.

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