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Hurwitz on Hadamard designs

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An $n \times n$-matrix on n signed variables is called *Hadamard of Williamson type* if each variable occurs exactly once in each row, and the inner product of any pair of distinct rows is zero. We show here that these matrices correspond in a natural way to rational formulas for products of sums of n squares, shown by Hurwitz to exist only for $n = 1, 2, 4$, and 8. Hurwitz' arguments contain an implicit proof that this correspondence is one-to-one (we show this directly) and hence that Hadamard matrices of Williamson type exist for orders 1, 2, 4 and 8 only.

An *Hadamard design* [1, 2] is an $n \times n$ array $H(n; k)$ of k signed variables ("letters") with the property that the inner product of any two distinct rows of the array is zero. Such a design has been said to be of *Williamson type* (after [6]) if $k = n$ and each letter occurs exactly once in each row of $H^{(n;n)} \triangleq H^{(n)}$, and it has recently been shown [5] that $H^{(n)}$ exists if and only if $n = 1, 2, 4$, or 8. The purpose of this note is to show that this result was known to Hurwitz [3, 4].

If

$$(1) \quad \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)\left(y_1^2 + y_2^2 + \ldots + y_n^2\right) = P_1^2 + P_2^2 + \ldots + P_n^2 ,$$

where the P_i are bilinear forms in the x_j, y_k, we may form an $n \times n$ matrix $H = [H_{ij}]$ whose ij-entry H_{ij} is the (signed) coefficient of x_i in P_j. In the special case $n = 4$, for example,

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$P_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, $P_2 = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$,

$P_3 = x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2$, $P_4 = x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1$,

and

$$H = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & y_4 & -y_3 \\ y_3 & -y_4 & -y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix}$$

is Hadamard of Williamson type. Hurwitz essentially showed that a formula of type (1) exists for a given n if and only if $n = 1, 2, 4$, or 8, and that each of these formulas leads to an $H^{(n)}$ (which he exhibits in [3], p. 570) of the corresponding order. Thus, if it could be shown that to each $H^{(n)}$ there corresponds a formula of type (1), the admissible orders n for $H^{(n)}$ would also be known. This is implicit in Hurwitz' discussion, being, in fact, almost trivial. For, given an Hadamard matrix $H^{(n)} \triangleq H = [H_{ij}]$, define bilinear forms $P_1, P_2, \ldots, P_n$ by the previous correspondence; further define $\hat{H} = [\hat{H}_{ij}]$ where, if $H_{ik} = \pm y_j$, then $\hat{H}_{ij} = \pm x_k$. Clearly $\hat{H}$ is Hadamard of Williamson type and, if

$$X = x_1^2 + x_2^2 + \ldots + x_n^2 , \quad Y = y_1^2 + y_2^2 + \ldots + y_n^2 ,$$

then

$$H^T H \hat{H} \hat{H}^T = (H\hat{H}^T)(H\hat{H}^T)^T = XY.I \quad (I = n \times n \text{ identity matrix}) .$$

But

$$(H\hat{H}^T)_{ij} = \sum_{k=1}^{n} H_{ik}\hat{H}_{jk} \not\equiv 0 , \quad (H\hat{H}^T)^T_{ij} = \sum_{k=1}^{n} H_{jk}\hat{H}_{ik} \not\equiv 0$$

and

$$H_{ik}\hat{H}_{jk} = (\text{coeff. of } x_k \text{ in } P_i).(\text{coeff. of } y_k \text{ in } P_j) .$$

If, now, $y_s = (\text{coeff. of } x_k \text{ in } P_i)$, let

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$x_{\sigma(k)}$ = (coeff. of y_s in P_j) ; this defines a permutation σ of the n-set. Further, from the definition of H , we have (coeff. $x_{\sigma(k)}$ in P_i) = ± (coeff. x_k in P_j) ; thus

$$|(\text{coeff. of } x_k \text{ in } P_i).(\text{coeff. of } y_k \text{ in } P_j)| = |(\text{coeff. of } x_{\sigma(k)} \text{ in } P_j).(\text{coeff. of } y_{\sigma(k)} \text{ in } P_i)|$$

and hence $\dot{H} = \left[\left|(HH^T)_{ij}\right|\right]$ is symmetric. This implies that XY is a sum of n squares of bilinear forms in the x_i, y_j . Continuing with the case $n = 4$, for example, we find

$$H\hat{H}^T = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ P_2 & -P_1 & P_4 & -P_3 \\ P_3 & -P_4 & -P_1 & P_2 \\ P_4 & P_3 & -P_2 & -P_1 \end{bmatrix}$$

which is, again, Hadamard of Williamson type, as expected.

As a final interesting note, Hurwitz' closing remark in [3] implies that an Hadamard design of order 16 on p letters, each of which must occur exactly once in each row, must satisfy

$$p < \frac{2\log 16}{\log 2} + 2 \sim 10.1 ,$$

that is $p \leq 10$.

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