J. Austral. Math. Soc. (*Series A*) 32 (1982), 369–379

DUALITY THEOREMS AND AN OPTIMALITY CONDITION FOR NON-DIFFERENTIABLE CONVEX PROGRAMMING

P. KANNIAPPAN and SUNDARAM M. A. SASTRY

(Received 22 August 1980; revised 23 March 1981)

Communicated by H. Lausch

Abstract

Necessary and sufficient optimality conditions of Kuhn-Tucker type for a convex programming problem with subdifferentiable operator constraints have been obtained. A duality theorem of Wolfe's type has been derived. Assuming that the objective function is strictly convex, a converse duality theorem is obtained. The results are then applied to a programming problem in which the objective function is the sum of a positively homogeneous, lower-semi-continuous, convex function and a continuous convex function.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 90 C 25, 90 C 30, 90 C 48.

0. Introduction

In this paper, we study the following pair of problems:
Problem (P). Minimize $f(x)$ subject to

$$G(x) \leqslant 0 \quad \text{and} \quad x \in A.$$

Problem (D). Maximize $f(x) + \langle z^*, G(x)\rangle$ subject to

$$z^* \geqslant 0, \qquad x \in A \quad \text{and}$$
$$0 \in \partial f(x) + z^* \circ \partial G(x) + N(x/A).$$

The first author is on leave from Gandhigram Rural Institute (Deemed University), Gandhigram-624302, India and his research is supported by U.G.C. of India (V plan).

This paper was presented at the Annual Conference of the Indian Mathematical Society, held in December 1980, at Bangalore, India.

Here f is a continuous convex functional defined on a locally convex space X and G is a continuous convex operator, which is regularly subdifferentiable on A, a convex subset of X, defined on X into another locally convex space Z having a closed convex cone defining a partial ordering in Z. $N(x/A)$ denotes the *normal cone to A at x* defined by

$$N(x/A) = \{w^* \in X' \colon \langle w^*, y - x\rangle \leqslant 0 \text{ for all } y \in A\},$$

where X' is the dual space of X.

$N(x/A)$ is the subdifferential of the *indicator function* of the set A at x, $\delta(x/A)$ defined by

$$\delta(x/A) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

If X and Z are finite dimensional, f and G are differentiable and $A = X$, then this is the problem studied by Wolfe and he has proved a duality theorem in [9]. M. Schechter [7] has derived a duality theorem in Wolfe's problem without assuming the differentiability of the objective function and the constraint functions. If $A = X$, the authors have proved a duality theorem, assuming that f is strictly convex, between the problems (P) and (D) in [5].

In this paper, we shall derive, in Section 2, a set of necessary and sufficient conditions of Kuhn-Tucker type for a point to be optimal for problem (P). We shall use this generalized Kuhn-Tucker theorem to prove a duality and a converse duality theorem between the problems (P) and (D) in Section 3. In Section 4, we apply these theorems in the case of the objective function is the sum of a continuous convex function and a positively homogeneous, lower-semi-continuous, convex function.

1. Preliminaries

In this paper X and X', as well as Z and Z', shall be pairs of real vector spaces in duality, with their respective weak topologies. Thus all the spaces will be locally convex spaces. We denote by $\langle \cdot , \cdot \rangle$ the canonical bilinear form of the dualities between the spaces X and X', as well as Z and Z'. We let $H \subset Z$ be a closed convex cone with non-empty interior defining a partial order in Z—for $x, y \in Z$; $x \leqslant y$ if $y - x \in H$. For $x, y \in Z$, $x < y$ is equivalent to $y - x$ is an interior point of H. Let H^* stand for the conjugate cone, namely,

$$H^* = \{z^* \in Z' \colon \langle z^*, z\rangle \geqslant 0 \text{ for every } z \in H\}.$$

Then, H^* defines a partial order in Z'.

Let $G: X \to Z$ be an operator. G is said to be *convex* if

$$G(tx + (1-t)y) \leqslant tG(x) + (1-t)G(y),$$

for all $x, y \in X$ and $0 \leqslant t \leqslant 1$.

A continuous linear operator $T: X \to Z$ is said to be a *subgradient* of G at a point $x_0 \in X$ if

$$T(x - x_0) \leqslant G(x) - G(x_0)$$

for every $x \in X$. The set of all subgradients of G at x_0 is called the *subdifferential* of G at x_0 and is denoted by $\partial G(x_0)$.

The operator $G: X \to Z$ is said to be *regularly subdifferentiable* at x_0 if

$$\partial(z^* \circ G)(x_0) = z^* \circ \partial G(x_0)$$

for every $z^* \in H^*$ [1]. If G is regularly subdifferentiable at every point of a subset A of X, then G is said to be regularly subdifferentiable on A.

We need the following proposition, whose proof can be found in [4].

PROPOSITION 1.1. *Let F be a positively homogeneous, lower-semicontinuous, convex function defined on a locally convex space V; and let $u \neq 0$. Then*

$$\partial F(u) = \{u^* \in \partial F(0) \colon F(u) = \langle u, u^* \rangle\}.$$

We shall also need the following definition and a lemma, which can be proved easily.

DEFINITION. Let $f: X \to R$ be a function, and let $a \in X$. f is said to be *strictly convex at a* if

$$f(ta + (1-t)b) < tf(a) + (1-t)f(b)$$

for every $a \neq b \in X$, $0 < t < 1$.

LEMMA 1.2. *Let $f: X \to R$ be convex. If f is strictly convex at $a \in X$, then for every $u^* \in \partial f(a)$, we have*

$$f(x) - f(a) > \langle u^*, x - a \rangle$$

for every $x \in X$, $x \neq a$.

2. Necessary and sufficient conditions

Before establishing a necessary and sufficient condition of Kuhn-Tucker type, we shall prove a theorem of Fritz-John type.

THEOREM 2.1. *Let X be a locally convex space and let f be a convex function, continuous at a point of the convex set A and let Z be a locally convex space with a positive cone H with non-empty interior. Let G be a continuous convex operator from X to Z, which is regularly subdifferentiable on A. If x_0 is an optimal solution of the problem* (P), *then there exists $\lambda_0 \geqslant 0$, $z_0^* \in H^*$, not both zero, such that*

$$0 \in \lambda_0 \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A)$$

and $\langle z_0^*, G(x_0)\rangle = 0$.

PROOF. Consider the set C in $Z \times R$ defined as follows:

$$C = \{(z, a) \in Z \times R\colon \text{there exists } x \in A \text{ such that } f(x) - f(x_0) < a,\ G(x) \leqslant z\}$$

Since C contains $H \times R^+$ and H has non-empty interior, C has non-empty interior.

The set C is convex, since f and G are convex. Further $(0, 0) \notin C$, for if $(0, 0) \in C$, then there exists $x \in A$ such that $f(x) - f(x_0) < 0$, and $G(x) \leqslant 0$, which is a contradiction to the assumption that x_0 is an optimal solution of the problem (P). Hence by separation theorem, there exists $(0, 0) \neq (z_0^*, \lambda_0) \in Z' \times R$ such that

(1) $$\langle z_0^*, z\rangle + \lambda_0 a \geqslant 0 \quad \text{for every } (z, a) \in C.$$

In particular, for every $a > 0$, $(G(x_0), a) \in C$ and hence we have

(2) $$\langle z_0^*, G(x_0)\rangle + \lambda_0 a \geqslant 0.$$

Letting $a \to 0^+$, we obtain

(3) $$\langle z_0^*, G(x_0)\rangle \geqslant 0.$$

From (2) and (3), we have, by contradiction,

(4) $$\lambda_0 \geqslant 0.$$

Also for every $h \in H$, $(G(x_0) + h, 1) \in C$, so that (1) gives

$$\langle z_0^*, G(x_0)\rangle + \lambda_0 + \langle z_0^*, h\rangle \geqslant 0.$$

That is, $\langle z_0^*, h\rangle \geqslant -[\langle z_0^*, G(x_0)\rangle + \lambda_0]$ for every $h \in H$. Again from (3) and (4), we have by contradiction $z_0^* \in H^*$. But since $G(x_0) \in -H$ and $z_0^* \in H^*$, we have

(5) $$\langle z_0^*, G(x_0)\rangle \leqslant 0.$$

Putting (3) and (5) together, we get

(6) $$\langle z_0^*, G(x_0)\rangle = 0$$

as desired.

Now $(G(x), f(x) - f(x_0) + \varepsilon) \in C$, for all $\varepsilon > 0$ and for all $x \in A$. Then by (1), we have

$$\langle z_0^*, G(x) \rangle + \lambda_0 (f(x) - f(x_0) + \varepsilon) \geqslant 0 \quad \text{for all } x \in A.$$

Combining with (6), we have

$$\langle z_0^*, G(x) - G(x_0) \rangle + \lambda_0 (f(x) - f(x_0) + \varepsilon) \geqslant 0 \quad \text{for all } x \in A.$$

As $\varepsilon \to 0$, we have

$$\langle z_0^*, G(x) - G(x_0) \rangle + \lambda_0 (f(x) - f(x_0)) \geqslant 0 \quad \text{for all } x \in A.$$

That is

(7) $$\lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle \leqslant \lambda_0 f(x) + \langle z_0^*, G(x) \rangle \quad \text{for all } x \in A.$$

Hence x_0 minimizes the function $\lambda_0 f(x) + \langle z_0^*, G(x) \rangle$ on A. That is x_0 is a solution of the problem:

$$\underset{x \in X}{\text{minimize}}\, \lambda_0 f(x) + \langle z_0^*, G(x) \rangle + \delta(x/A).$$

Therefore, by Proposition 1, page 81 in [3], we have

$$0 \in \partial(\lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle + \delta(x_0/A)).$$

Since, f and G are continuous and G is regularly subdifferentiable on A, by the Moreau-Rockafeller theorem [6],

$$0 \in \lambda_0 \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A).$$

Hence the theorem.

We shall now prove a theorem of Kuhn-Tucker type.

THEOREM 2.2. *In addition to the assumptions of Theorem* 2.1, *if we further assume that Stater's constraint qualification is satisfied* (*that is, there exists* $x' \in A$ *such that* $G(x') < 0$), *then* $\lambda_0 \neq 0$ *and one can set* $\lambda_0 = 1$. *In this case, the necessary and sufficient condition for* x_0 *to be an optimal solution of the problem* (P) *is that there exists an* $z_0^* \in H^*$ *such that*

(8) $$0 \in \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A) \quad \textit{and} \quad \langle z_0^*, G(x_0) \rangle = 0.$$

PROOF. Suppose Slater's constraint qualification is satisfied. Then there exists $x' \in A$ such that $G(x') < 0$.

Since all the conditions of Theorem 2.1 are satisfied, we have by (7) in the proof of Theorem 2.1, there exists $\lambda_0 \geqslant 0$, $z_0^* \in H^*$, not both zero such that

$$\lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle \leqslant \lambda_0 f(x) + \langle z_0^*, G(x) \rangle$$

for all $x \in A$ and $\langle z_0^*, G(x_0) \rangle = 0$.

https://doi.org/10.1017/S1446788700024927 Published online by Cambridge University Press

If $\lambda_0 = 0$, then $z_0^* \neq 0$, $z_0^* \in H^*$ and we have

$$\lambda_0 f(x') + \langle z_0^*, G(x') \rangle = \langle z_0^*, G(x') \rangle < 0 = \lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle$$

and this contradicts (7). Therefore $\lambda_0 \neq 0$. Hence we can set $\lambda_0 = 1$ and the relations (8) are satisfied.

Conversely, suppose $x_0 \in A$ such that $G(x_0) \leqslant 0$, $z_0^* \in H^*$ satisfy relations (8). Now (8) implies by the Moreau-Rockafellar theorem [6]

$$0 \in \partial(f + z_0^* \circ G + \delta(\cdot / A))(x_0).$$

Then by Proposition 1, page 81 in [3], we have x_0 is an optimal solution of the problem

$$\underset{x \in X}{\text{minimize}}\, f(x) + z_0^* \circ G(x) + \delta(x/A).$$

This implies

$$f(x_0) + z_0^* \circ G(x_0) \leqslant f(x) + z_0^* \circ G(x) + \delta(x/A)$$

for every $x \in X$, as $x_0 \in A$. Hence,

$$f(x_0) + z_0^* \circ G(x_0) \leqslant f(x) + z_0^* \circ G(x) \tag{9}$$

for every $x \in A$. Then for any $x \in A$ satisfying $G(x) \leqslant 0$, we have

$$\begin{aligned} f(x_0) = f(x_0) + \langle z_0^*, G(x_0) \rangle &\leqslant f(x) + \langle z_0^*, G(x) \rangle, \quad \text{by (9)} \\ &\leqslant f(x). \end{aligned}$$

This means that x_0 is an optimal solution of problem (P).

REMARK. If $Z = R^m$, then Theorems 2.1 and 2.2 reduce to Theorems 1.1 and 1.2 in [8] proved by M. Schechter using the theory of Dubovitski-Milyutin [2]. If $A = X$, then Theorem 2.2 becomes Theorem 2 in [4].

3. Duality and converse duality theorems

Using the necessary conditions of the previous section, we prove a duality theorem and a converse duality theorem between the problems (P) and (D). We assume that the Slater's constraint qualification is satisfied.

THEOREM 3.1 (Duality). *If x_0 is an optimal solution of* (P), *then there exists an z_0^* such that (x_0, z_0^*) is optimal for* (D). *Further, the two problems have the same extremal values.*

PROOF. Since x_0 is an optimal solution of (P), Theorem 2.2 guarantees the existence of feasible solutions to problem (D).

Let (x, z^*) be a feasible solution for problem (D). Then $z^* \geqslant 0$ and $0 \in \partial f(x) + z^* \circ \partial G(x) + N(x/A)$. This implies that there exist $x^* \in \partial f(x)$, $T \in \partial G(x)$ and $y^* \in N(x/A)$ such that $0 = x^* + z^* \circ T + y^*$. Now,

$$\begin{aligned} f(x_0) - [f(x) + \langle z^*, G(x)\rangle] &= [f(x_0) - f(x)] - \langle z^*, G(x)\rangle \\ &\geqslant \langle x^*, x_0 - x\rangle - \langle z^*, G(x)\rangle, \quad \text{since } x^* \in \partial f(x) \\ &= -\langle z^* \circ T + y^*, x_0 - x\rangle - \langle z^*, G(x)\rangle \\ &= -\langle z^*, T(x_0 - x)\rangle - \langle y^*, x_0 - x\rangle - \langle z^*, G(x)\rangle \\ &\geqslant -\langle z^*, G(x_0) - G(x)\rangle - \langle y^*, x_0 - x\rangle - \langle z^*, G(x)\rangle \\ &= -\langle z^*, G(x_0)\rangle - \langle y^*, x_0 - x\rangle \\ &\geqslant 0 \quad (\text{since } z^* \geqslant 0,\ G(x_0) \leqslant 0 \text{ and } y^* \in N(x/A)). \end{aligned}$$

Thus,

$$f(x_0) \geqslant f(x) + \langle z^*, G(x)\rangle \tag{1}$$

for any feasible solution (x, z^*) for problem (D). Since x_0 is an optimal solution of (P), we have from Theorem 2, that there exists $z_0^* \in H^*$ such that $\langle z_0^*, G(x_0)\rangle = 0$ and $0 \in \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A)$. In other words, (x_0, z_0^*) is a feasible solution for (D). Hence

$$f(x_0) = f(x_0) + \langle z_0^*, G(x_0)\rangle. \tag{2}$$

Thus, from (1) and (2), (x_0, z_0^*) is an optimal solution of problem (D), and that the two problems have the same extremal value.

THEOREM 3.2 (Converse Duality). *Let us assume that the primal problem* (P) *has a solution* $\bar{x}$. *If* (x_0, z_0^*) *is an optimal solution of the dual problem* (D), *and if* f *is strictly convex at* x_0, *then* $x_0 = \bar{x}$. *Hence* x_0 *solves the problem* (P). *Furthermore, the extremal values of the two problems are same.*

PROOF. Suppose $x_0 \neq \bar{x}$. Since $\bar{x}$ is a solution of (P), it follows from the duality Theorem 3.1, there exists $\bar{z}^* \in H^*$ such that $(\bar{x}, \bar{z}^*)$ is optimal for (D).

Let $L(x, z^*) = f(x) + \langle z^*, G(x)\rangle$ be the Lagrangian of (P). Then,

$$L(\bar{x}, \bar{z}^*) = L(x_0, z_0^*) = \max_{(x, z^*) \in K} L(x, z^*)$$

where $K = \{(x, z^*)\colon x \in A,\ z^* \in H^* \text{ and } 0 \in \partial f(x) + z^* \circ \partial G(x) + N(x/A)\}$. Note that $(\bar{x}, \bar{z}^*) \in K$.

https://doi.org/10.1017/S1446788700024927 Published online by Cambridge University Press

Since $(x_0, z_0^*) \in K$, we have $0 \in \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A)$. Hence there exist $x^* \in \partial f(x_0)$, $T \in \partial G(x_0)$ and $y^* \in N(x_0/A)$ such that $0 = x^* + z_0^* \circ T + y^*$. Now,

$$\begin{aligned} L(\bar{x}, z_0^*) - L(x_0, z_0^*) &= f(\bar{x}) + \langle z_0^*, G(\bar{x})\rangle - f(x_0) - \langle z_0^*, G(x_0)\rangle \\ &= f(\bar{x}) - f(x_0) + \langle z_0^*, G(\bar{x}) - G(x_0)\rangle \\ &> \langle x^*, \bar{x} - x_0\rangle + \langle z_0^*, G(\bar{x}) - G(x_0)\rangle, \quad \text{by Lemma 1.2,} \\ &\geqslant \langle x^*, \bar{x} - x_0\rangle + \langle z_0^*, T(\bar{x}) - T(x_0)\rangle, \quad \text{since } T \in \partial G(x_0) \\ &= \langle x^*, \bar{x} - x_0\rangle + \langle z_0^* \circ T, \bar{x} - x_0\rangle \\ &= +\langle x^* + z_0^* \circ T, \bar{x} - x_0\rangle \\ &= -\langle y^*, \bar{x} - x_0\rangle \quad \text{by (1)} \\ &\geqslant 0, \quad \text{since } y^* \in N(x_0/A). \end{aligned}$$

It follows that, $L(\bar{x}, z_0^*) > L(x_0, z_0^*) = L(\bar{x}, \bar{z}^*)$. That is,

$$f(\bar{x}) + \langle z_0^*, G(\bar{x})\rangle > f(\bar{x}) + \langle \bar{z}^*, G(\bar{x})\rangle. \tag{3}$$

By hypothesis, since $\bar{x}$ is a solution of (P), it follows from Theorem 2, $\langle \bar{z}^*, G(\bar{x})\rangle = 0$. Hence, by (3), $\langle z_0^*, G(\bar{x})\rangle > 0$, which is a contradiction to the fact that $z_0^* \in H^*$, $G(\bar{x}) \leqslant 0$. Hence, $\bar{x} = x_0$ and x_0 solves the problem (P).

Further, we have, $f(x_0) = f(\bar{x}) = f(\bar{x}) + \langle \bar{z}^*, G(\bar{x})\rangle = L(\bar{x}, \bar{z}^*) = L(x_0, z_0^*) = f(x_0) + \langle z_0^*, G(x_0)\rangle$. Hence, the extremal values of the two problems are equal.

4. Applications

We shall now specialize the theorems derived in Section 3 to the case where the objective function is the sum of a positively homogeneous, lower-semi-continuous convex function and a continuous convex function.

Let the objective function $f\colon X \to R$ be of the form $f = f_1 + f_2$, where f_1 is a continuous convex function and f_2 is a positively homogeneous lower-semi-continuous convex function. Then the problem (P) becomes

$$(\mathrm{P}_1)\colon \text{Minimize} \quad f_1(x) + f_2(x) \quad \text{subject to}$$
$$G(x) \leqslant 0, \quad \text{and} \quad x \in A.$$

Let us now construct the dual problem (D_1) using the above argument.

$$(\mathrm{D}_1)\colon \text{Maximize} \quad f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle \quad \text{subject to}$$
$$s^* \geqslant 0, u^* \in \partial f_2(0), \langle u^*, x\rangle = f_2(x), x \in A \quad \text{and}$$
$$0 \in \partial f_1(x) + u^* + z^* \circ \partial G(x) + N(x/A).$$

We will now show that the duality theorem still holds even if one of the constraints is removed from the dual problem (D_1).

$$(D_2)\text{: Maximize } f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle \quad \text{subject to}$$
$$z^* \geqslant u^* \in \partial f_2(0),\ x \in A \quad \text{and}$$
$$0 \in \partial f_1(x) + u^* + z^* \circ \partial G(x) + N(x/A).$$

THEOREM 4.1. *If x_0 is an optimal solution of (P_1), then there exist z_0^*, u_0^* and w_0^* such that $(x_0, z_0^*, u_0^*, w_0^*)$ is optimal for (D_2). Further, the two problems have the same extremal values.*

PROOF. Since x_0 is optimal for (P_1), by Theorem 2.2 there exists an $z^* \in H^*$ such that $\langle z^*, G(x_0)\rangle = 0$ and $0 \in \partial(f_1 + f_2)(x_0) + z^* \circ \partial G(x_0) + N(x_0/A)$. But $\partial(f_1 + f_2)(x_0) = \partial f_1(x_0) + \partial f_2(x)$ by the Moreau-Rockafellar theorem [6]. Also, $\partial f_2(x_0) = \{u^* \in \partial f_2(0)\colon f_2(x_0) = \langle u^*, x_0\rangle\}$, by Proposition 1.1. Therefore,

$$0 \in \partial f_1(x_0) + \{u^* \in \partial f_2(0)\colon f_2(x_0) = \langle u^*, x_0\rangle\} + z^* \circ \partial G(x_0) + N(x_0/A).$$

Hence, there is $u^* \in \partial f_2(0)$ satisfying $f_2(x_0) = \langle u^*, x_0\rangle$ such that $0 \in \partial f_1(x_0) + u^* + z^* \circ \partial G(x_0) + N(x_0/A)$. Thus feasible solutions to problem (D_2) exist.

Let (x, z^*, u^*, w^*) be any feasible solution for (D_2). Then $z^* \in H^*$, $u^* \in \partial f_2(0)$ and there exist $x^* \in \partial f_1(x)$, $T \in \partial G(x)$ and $w^* \in N(x/A)$ such that

$$0 = x^* + u^* + z^* \circ T + w^*. \tag{1}$$

Now, using the idea of subdifferential calculus, the definition of normal cone and the relation (1), we can easily prove

$$f_1(x_0) + f_2(x_0) \geqslant f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle$$

for every feasible solution (x, z^*, u^*, w^*) of (D_2). Now, since x_0 is optimal for (P_1), then there are $z_0^* \in H^*$, $u_0^* \in \partial f_2(0)$ satisfying $f_2(x_0) = \langle u_0^*, x_0\rangle$ such that $0 \in \partial f_1(x_0) + u_0^* + z_0^* \circ \partial G(x_0) + N(x_0/A)$ and such that $\langle z_0^*, G(x_0)\rangle = 0$. Hence $f_1(x_0) + f_2(x_0) + \langle z_0^*, G(x_0)\rangle \geqslant f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle$ for every feasible solution (x, z^*, u^*, w^*) of (D_2). That is, $(u_0, z_0^*, u_0^*, w_0^*)$ is optimal for (D_2). Further, it is clear that the extremal values of the two problems are the same.

REMARK. The $(u_0, z_0^*, u_0^*, w_0^*)$ which optimizes D_2, in fact, also optimizes D_1.

THEOREM 4.2. *Let $\bar{x}$ be an optimal solution of (P_1). If $(x_0, z_0^*, u_0^*, w_0^*)$ is optimal for (D_1) and if f_1 is strictly convex at x_0, then $x_0 = \bar{x}$. Hence x_0 solves (P_1). Further, the extremal values of the two problems are equal.*

PROOF. Suppose $x_0 \neq \bar{x}$. Since $\bar{x}$ is a solution of (P_1), it follows from the duality Theorem 4.1, there exist $\bar{z}^* \in H^*$, $\bar{u}^* \in \partial f_2(0)$ satisfying $f_2(\bar{x}) = \langle \bar{u}^*, \bar{x} \rangle$ and $\bar{w}^* \in N(\bar{x}/A)$ such that $0 \in \partial f_1(\bar{x}) + \bar{u}^* + \bar{z}^* \circ \partial G(\bar{x}) + \bar{w}^*$. That is, $(\bar{x}, \bar{z}^*, \bar{u}^*, \bar{w}^*)$ is optimal for (D_1).

Let $\phi(x, z^*, u^*) = f_1(x) + \langle u^*, x \rangle + \langle z^*, G(x) \rangle$. Hence,

$$\Phi(\bar{x}, \bar{z}^*, \bar{u}^*) = \phi(x_0, z_0^*, u_0^*) = \max_{(x, z^*, u^*) \in N} \phi(x, z^*, u^*)$$

where $N = \{(x, z^*, u^*): x \in A,\ z^* \in H^*,\ u^* \in \partial f_2(0)$ satisfying $f_2(x) = \langle u^*, x \rangle$ such that $0 \in \partial f_1(x) + u^* + z^* \circ \partial G(x) + N(x/A)\}$. Note that $(x_0, z_0^*, u_0^*) \in N$.

Since $(x_0, z_0^*, u_0^*) \in N$, we have $0 \in \partial f_1(x_0) + u_0^* + z_0^* \circ \partial G(u_0) + N(x_0/A)$. Hence, there exist $x^* \in \partial f_1(x_0)$, $T \in \partial G(u_0)$ and $w^* \in N(x_0/A)$ such that

$$0 = x^* + u_0^* + z_0^* \circ T + w^*. \tag{3}$$

Using the idea of subdifferential calculus, definition of normal cone and using the Lemma 1.2 and relation (3), we can prove,

$$\begin{aligned}\phi(\bar{x}, z_0^*, \bar{u}^*) - \phi(x_0, z_0^*, u_0^*) &> -\langle u_0^*, \bar{x} \rangle + \langle \bar{u}^*, \bar{x} \rangle - \langle w^*, \bar{x} - x_0 \rangle \\ &\geqslant -f_2(\bar{x}) + f_2(\bar{x}) - \langle w^*, \bar{x} - x_0 \rangle,\end{aligned}$$

since $\bar{u}^* \in \partial f_2(0)$ satisfying $f_2(\bar{x}) = \langle \bar{u}^*, \bar{x} \rangle$ and $u_0^* \in \partial f_2(0)$ which implies that $f_2(\bar{x}) \geqslant \langle u_0^*, \bar{x} \rangle = -\langle w^*, \bar{x} - x_0 \rangle \geqslant 0$, since $w^* \in N(x_0/A)$. Therefore, $\phi(\bar{x}, z_0^*, \bar{u}^*) > \phi(x_0, z_0^*, u_0^*) = \phi(\bar{x}, \bar{z}^*, \bar{u}^*)$. That is,

$$f_1(\bar{x}) + \langle \bar{u}^*, \bar{x} \rangle + \langle z_0^*, G(\bar{x}) \rangle > f_1(\bar{x}) + \langle \bar{u}^*, \bar{x} \rangle + \langle \bar{z}^*, G(\bar{x}) \rangle. \tag{4}$$

Since $\bar{x}$ is an optimal solution of (P_1), it follows from Theorem 2.2, $\langle \bar{z}^*, G(\bar{x}) \rangle = 0$. Hence $\langle z_0^*, G(\bar{x}) \rangle > 0$, from (4) which is not possible because $z_0^* \in H^*$, $G(\bar{x}) \leqslant 0$. Hence $\bar{x} = x_0$, and x_0 solves the problem (P_1).

Further, we have,

$$\begin{aligned}f_1(x_0) + f_2(x_0) &= f_1(\bar{x}) + f_2(\bar{x}) \\ &= f_1(\bar{x}) + \langle \bar{u}^*, \bar{x} \rangle + \langle \bar{z}^*, G(\bar{x}) \rangle \\ &= \phi(\bar{x}, \bar{z}^*, \bar{u}^*) = \phi(x_0, z_0^*, u_0^*) \\ &= f_1(x_0) + \langle u_0^*, x_0 \rangle + \langle z_0^*, G(x_0) \rangle.\end{aligned}$$

Hence, the extremal values of the two problems are equal.

REMARK. We are not able to prove a converse duality between (P_1) and (D_2).

Special cases of problems of type (P_1) with finite dimensional applications have been discussed in [7, 8, 5].

The authors wish to thank the referee for his useful comments and helpful suggestions.

References

[1] V. Barbu and Th. Precupanu, *Convexity and optimization in Banach spaces* (Sijthoff and Noordhoff, The Netherlands, 1978).
[2] I. V. Girsenov, *Lectures on mathematical theory of extremum problems*, (Springer-Verlag, New York, 1972).
[3] A. D. Ioffe and V. M. Tihomirov, *Theory of extremal problems* (Studies in Mathematics and its Applications, 6, North-Holland, Amsterdam, New York, Oxford, 1979).
[4] P. Kanniappan and Sundaram M. A. Sastry, 'A duality theorem for non-differentiable convex programming with operatorial constraints', *Bull. Austral. Math. Soc.* **22** (1980), 145–152.
[5] P. Kanniappan and Sundaram M. A. Sastry, 'A subgradient converse duality theorem for a convex programming', communicated to *J. Math. Anal. Appl.*
[6] R. T. Rockafellar, 'Extension of Fenchel's duality theorem for convex functions,' *Duke Math. J.* **33** (1966), 81–89.
[7] M. Schechter, 'A subgradient duality theorem', *J. Math. Anal. Appl.* **61** (1977), 850–855.
[8] M. Schechter, 'More on subgradient duality', *J. Math. Anal. Appl.* **71** (1979), 251–262.
[9] P. Wolfe, 'A duality theorem for non-linear programming', *Quart. Appl. Math.* **19** (1961), 239–244.

School of Mathematics
Madurai Kamaraj University
Madurai-625 021
Tamil Nadu
India