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EXTREMAL GRAPHS OF DIAMETER 3

L. CACCETTA

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Abstract

This paper is concerned with graphs of order n and diameter at most 3 having the property that by deleting any s or fewer vertices (edges) the resulting subgraphs (partial graphs) have diameter at most λ. A graph satisfying the above constraints and having minimum number of edges is said to be extremal. A characterization of extremal graphs is presented for the case $s = 1$.

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1. Introduction

We consider only undirected graphs without loops or multiple edges. The terminology of Bondy and Murty (1976) will be adopted throughout unless stated otherwise.

Denote by $G_V(n,k,\lambda,s)$ the class of graphs with n vertices and diameter at most k having the property that by deleting any s or fewer vertices the resulting subgraphs have diameter at most $\lambda \geqslant k$. A graph with minimum number of edges within the class $G_V(n,k,\lambda,s)$ is denoted by $\operatorname{Min} G_V(n,k,\lambda,s)$ and the minimum possible number of edges is denoted by $M_V(n,k,\lambda,s)$. Similarly the class $G_E(n,k,\lambda,s)$ consists of those graphs with n vertices and diameter at most k having the property that by deleting any s or fewer edges the resulting partial graphs have diameter at most $\lambda \geqslant k$. A graph with the minimum possible number of edges within the class $G_E(n,k,\lambda,s)$ is denoted by $\operatorname{Min} G_E(n,k,\lambda,s)$, and the minimum possible number of edges is denoted by $M_E(n,k,\lambda,s)$. The graphs with the minimal number of edges within these classes will be called extremal graphs.

Several papers have appeared in the literature on this problem. In Caccetta (1978) we presented a comprehensive bibliography of all pertinent developments.

An earlier paper by Bollobás and Harary (1976) also provides an exposition of some of the known results. In this paper we are concerned only with the case of diameter 3, that is $k = 3$. For $s > 1$ some asymptotic results are known (see Bollabás (1976) and Caccetta (1978)); in particular

$$\lim_{n\to\infty} \frac{M_V(n,3,n-1,s)}{n} = \frac{1}{2}\left\{(s+1)+\frac{1}{s+1}\right\},$$

and for $\lambda \geqslant 4$

$$\lim_{n\to\infty} \frac{M_E(n,3,\lambda,s)}{n} = \tfrac{1}{2}(s+2).$$

For $s = 1$ the functions $M_V(n,3,\lambda,1)$, $\lambda \geqslant 4$, and $M_E(n,3,\lambda,1)$, $\lambda \geqslant 6$ were determined by Bollobás (1968a, b). Apart from determining $M_E(n,3,\lambda,1)$ for $\lambda = 4$ and 5, which was previously unknown, the main contribution of this paper is the characterization of the extremal graphs. The case $\lambda = 3$ was studied in Caccetta (1976a) and so in the following we consider only the case $\lambda \geqslant 4$.

2. Preliminaries

Throughout this paper G always denotes an extremal graph with M edges; the class of graphs for which G is a member will be clear by the context in which G is used. The *diameter* of G is denoted by $D(G)$. We denote by $V(G)$ ($E(G)$) the vertex (edge) set of G. For any set S of vertices or edges we denote by $|S|$ the number of elements in S. The minimum degree of G is denoted by $\mu(G)$. We denote by α any vertex of G having minimum degree. P, Q and R denote the subgraphs of $G-\alpha$ whose vertices are at distance 1, 2 and 3, respectively, from α in G. If $R \neq \emptyset$ we let $R_1, R_2, \ldots, R_p$ denote the components of R. Denote by:

A: those vertices of Q that are adjacent to at least two vertices of P or Q.

W: those vertices of $Q-A$ that are adjacent to exactly one other vertex of Q.

U: the vertices of $Q-A-W$.

Q_i: those vertices of Q that are adjacent to vertices of R_i.

A_i, W_i and U_i: those vertices of Q_i that are in A, W and U, respectively.

Q^*: the subgraph spanned by the vertices of Q that are not adjacent to any vertex of R.

e_i: the number of edges in R_i.

f_i: the number of edges connecting R_i to Q.

$[x]$: integer part of x.

For each set of vertices defined above the corresponding lower case letter will always denote the number of vertices in that set, for example a denotes the number of vertices in A.

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We conclude this section with the following rather obvious observations:

(1) A $G_V(n,k,\lambda,s)$ ($G_E(n,k,\lambda,s)$) graph is also a $G_V(n,k',\lambda',s')$ ($G_E(n,k',\lambda',s')$) graph, whenever $k' \geqslant k$, $\lambda' \geqslant \lambda$ and $s' \leqslant s$. Consequently the functions $M_V(n,k,\lambda,s)$ and $M_E(n,k,\lambda,s)$ are monotonic non-decreasing in s, and monotonic non-increasing in k and λ.

(2) In a $G_V(n,k,\lambda,s)$ ($G_E(n,k,\lambda,s)$) there will be at least $s+1$ vertex (edge) disjoint paths of length $\leqslant \lambda$ between any two non-adjacent (any two) vertices, at least one of which has length $\leqslant k$.

(3) The degree of every vertex of G is at least $s+1$, that is, $\mu(G) \geqslant s+1$.

(4) If $\mu(G) = s+1$, then every vertex of G which is not adjacent to α must be connected to each of the $s+1$ vertices adjacent to α by a path of length $\leqslant \lambda - 1$ (from observation (2)).

3. The structures of Min $G_V(n,3,4,1)$

Denote by K^{2m} the class of graphs on $2m$ ($\geqslant 6$) vertices that are obtained by connecting two vertices x_1 and x_4 by m disjoint paths of length 3. The class K^{2m+1} is obtained from the class K^{2m} by adding one vertex and connecting it to any two non-adjacent vertices of K^{2m}. Clearly the graphs of K^{2m} and K^{2m+1} belong to the class $G_V(n,3,4,1)$ and they have $[\frac{1}{2}(3n-5)]$ edges (see Figure 1; we have labelled the vertex x_i as i). Note that there are 3 members K_j^{2m+1}, $j = 1,2,3$, of the class K^{2m+1}. Bollobás (1968a) conjectured that these are the only extremal graphs of $G_V(n,3,4,1)$. We shall prove this is indeed the case.

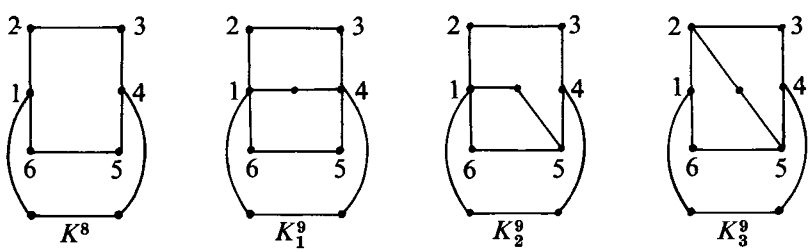

FIGURE 1.

The graphs of K^{2m} and K^{2m+1} have one common feature, namely two adjacent vertices of degree 2. We shall prove that every G has two adjacent vertices of degree 2 for all $n \geqslant 6$ except for $n = 9$, and furthermore for $n = 9$ the only other possible graph is the graph H^9 of Fig. 2. Once this is established we can easily obtain the extremal structures because of the following lemma.

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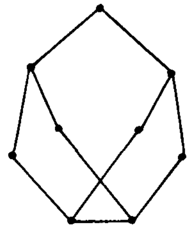

H^9

FIGURE 2.

LEMMA 1. *If G possesses two adjacent vertices of of degree* 2, *then*

$$M_V(n,3,4,1) = [\tfrac{1}{2}(3n-5)],$$

and the graphs of K^{2m} *and* K^{2m+1} *are the only elements of* $\operatorname{Min} G_V(n,3,4,1)$ for $n = 2m$ *and* $n = 2m+1$, *respectively.*

PROOF. Let α and β denote any two adjacent vertices of G having degree 2 and suppose α (β) is adjacent to x (y). We adopt the notation of the previous section with one minor modification. Here we let Q denote the subgraph of $G-\alpha-\beta$ whose vertices are adjacent to at least one of x and y, and denote by R the subgraph formed by the remaining vertices. Every vertex of $R \cup Q$ must be connected by a path of length $\leqslant 2$ to each of x and y (by observation (4) of Section 2). Hence $U = \varnothing$ and every vertex of R is adjacent to at least two vertices of Q. Therefore simply by counting the number of edges:

$$(1) \qquad M \geqslant 3+2a+\tfrac{3}{2}w+2r+\varepsilon = \tfrac{1}{2}(3n-6)+\tfrac{1}{2}(a+r+2\varepsilon),$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\varepsilon = 0$ and $a+r \leqslant 1$ for otherwise $M > (3n-5)/2$. Now $M = \frac{1}{2}(3n-6)$ only if $A \cup R = \varnothing$ and n is even, giving the structures K^{2m}, $n = 2m$. If n is odd, $M = \frac{1}{2}(3n-5)$ only if $a+r \leqslant 1$, hence the structures K^{2m+1}, $n = 2m+1$. This completes the proof of Lemma 1.

Before proceeding to prove the existence of two adjacent vertices by degree 2 in G we introduce some further notation. Denote by:

$R^{(1)}$: those vertices of R that are adjacent to at least two vertices of Q.

$R^{(2)}$: the vertices of $R - R^{(1)}$.

$R_i^{(1)}$ and $R_i^{(2)}$: the vertices of $R_i \cap R^{(1)}$ and $R_i \cap R^{(2)}$, respectively.

V_i: those vertices of U_i that are adjacent to at least one vertex of $R_i^{(1)}$.

For each of the above sets the corresponding lower case letter will denote the number of elements in that set.

The following observations are easily made.

(1) $\mu(G) = 2$ since $M < 3n/2$ (by the above constructions). We let $P = \{x, y\}$.

(2) Every vertex of $Q \cup R$ should be connected by a path of length $\leqslant 3$ to each of x and y (by observation (4) of Section 2). Hence every vertex of U must be adjacent to at least one vertex of $R^{(1)}$.

(3) If $R = \varnothing$ then G possesses two adjacent vertices of degree 2 for otherwise $Q = A$ and hence $M \geqslant 2n-4 > \frac{1}{2}(3n-5)$.

As the aim is to prove the existence of two adjacent vertices of degree 2, let us suppose that no two adjacent vertices in G have degree 2. Then from observation (3) above $R \neq \varnothing$.

For a connected component R_i, we have $e_i \geqslant r_i - 1$, $f_i \geqslant \max\{q_i, r_i + r_i^{(1)}\}$ and hence

$$e_i + f_i \geqslant \tfrac{3}{2} r_i + \tfrac{1}{2} v_i + \tfrac{1}{2}\{(q_i - v_i) + r_i^{(1)} + |q_i - r_i - r_i^{(1)}| - 2\}. \tag{2}$$

Thus we have

LEMMA 2. *For every $i = 1, 2, \ldots, p$ we have*

$$d_i = e_i + f_i - \tfrac{3}{2} r_i - \tfrac{1}{2} v_i \geqslant \tfrac{1}{2}\{(q_i - v_i) + r_i^{(1)} + |q_i - r_i - r_i^{(1)}| - 2\} = \delta_i. \tag{3}$$

Observe that $d_i \geqslant -\frac{1}{2}$, since if $r_i^{(1)} = 0$, $v_i = 0$ and $q_i \geqslant 2$. Also, if $d_i > -\frac{1}{2}$ then $d_i \geqslant 0$.

The following are three useful corollaries of Lemma 2.

COROLLARY 1. *If $d_i = -\frac{1}{2}$, then $r_i^{(1)} = r_i = 1$ and $q_i = v_i = 2$.*

COROLLARY 2. *If $d_i = 0$, then either*

(a) *$r_i^{(1)} = 1$ and one of the following conditions is satisfied:*

(i) *$v_i = 2$, $r_i = 2$ and $q_i = 2$ or 3,*

(ii) *$v_i = q_i = 3$ and $r_i = 1$,*

(iii) *$v_i = r_i = w_i = 1$ and $q_i = 2$,*

or

(b) *$r_i^{(1)} = 2$, $q_i = v_i = 4$ and $r_i = 2$.*

COROLLARY 3. *If $d_i = \frac{1}{2}$, then one of the following must hold:*

(a) *$r_i^{(1)} = 0$ and $q_i = r_i = 3$ or $q_i = r_i - 1 = 2$.*

(b) *$r_i^{(1)} = 1$ and one of the following conditions holds:*

(i) *$v_i = 2$, $r_i = 3$ and $q_i = 2, 3$ or 4*

(ii) *$v_i = 2$, $r_i = 1$ and $q_i = 3$*

(iii) *$v_i = w_i = 1$, $r_i = 2$ and $q_i = 3$*

(iv) *$v_i = 0$, $r_i = 1$ and $q_i = 2$.*

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(c) $r_i^{(1)} = 2$ *and one of the following conditions holds:*

(i) $q_i = v_i = 3$ *and* $r_i = 2$

(ii) $q_i = v_i = 4$ *and* $r_i = 3$

(iii) $q_i = 5$, $v_i = 4$ *and* $r_i = 3$

(*d*) $r_i^{(1)} = 3$ *and* $q_i = v_i = 6$.

The above corollaries are easily established from Lemma 2.

LEMMA 3. *G possesses two adjacent vertices of degree* 2 *for every* $n \geqslant 5$ *except possibly* $n = 9$ *for which case the only other possible graph is the graph* H^9 *of Figure* 2.

PROOF. Simply by counting the number of edges:

$$M \geqslant 2 + 2a + \tfrac{3}{2}w + u + \varepsilon + \sum_{i=1}^{p} (e_i + f_i), \tag{4}$$

where ε is defined as in the proof of Lemma 1.

Since every vertex of U is adjacent to at least one vertex of $R^{(1)}$ we must have

$$\sum_{i=1}^{p} v_i \geqslant u. \tag{5}$$

Note that if $V_i \cap V_j \neq \emptyset$ for any $i \neq j$, then strict inequality holds in (5). Equations (2), (4) and (5) give:

$$M \geqslant 2 + \tfrac{3}{2}(n-3) + \tfrac{1}{2}a + \varepsilon + \sum_{i=1}^{p} d_i = \tfrac{1}{2}(3n-5) + \tfrac{1}{2}a + \varepsilon + \sum_{i=1}^{p} d_i. \tag{6}$$

Now if $\frac{1}{2}a + \varepsilon + \sum_{i=1}^{p} d_i \geqslant \frac{1}{2}$, then obviously $M > \frac{1}{2}(3n-5)$. Therefore, since $M \leqslant \frac{1}{2}(3n-5)$ (by above constructions), we may suppose that

$$\tfrac{1}{2}a + \varepsilon + \sum_{i=1}^{p} d_i < \tfrac{1}{2}. \tag{7}$$

Let

$$d_1 = \min_{1 \leqslant i \leqslant p} \{d_i\}.$$

Obviously $d_1 = 0$ or $-\frac{1}{2}$. We consider the two cases separately.

Case (*a*) $d_1 = 0$.

Clearly for (7) to hold we must have $A = \emptyset$, $\varepsilon = 0$, $d_i = 0$ for every $i = 1, 2, \ldots, p$ and $V_i \cap V_j = \emptyset$ for every $i \neq j$. So every component R_i of R must satisfy one of the conditions of Corollary 2 of Lemma 2. If $p = 1$, then condition (a) of that corollary can be discarded, since in all cases one of x or y has degree 2 in which case G would have two adjacent vertices of degree 2. In the other case we obtain the graph H^9. If $p > 1$, then it is easily checked that the diameter constraints are violated under the assumption that $V_i \cap V_j = \emptyset$ for every $i \neq j$.

Case (*b*) $d_1 = -\frac{1}{2}$.

If $p = 1$, then obviously G has adjacent vertices of degree 2, hence $p \geq 2$. By Corollary 1 of Lemma 2 we must have $v_1 = q_1 = 2$ and $r_1 = 1$. For every $i \geq 2$, $Q_1 \cap Q_i \neq \emptyset$ since $D(G) \leq 3$. Hence if $d_i = -\frac{1}{2}$, then $V_1 \cap V_i \neq \emptyset$ since $Q_i = V_i$. Also, if $d_i = 0$, $i \geq 2$, then $V_1 \cap V_i \neq \emptyset$ to provide for an alternative path of length ≤ 4 between the vertices of R_1 and R_i when we delete the vertex of $V_1 \cap Q_i$ (note that $r_i \leq 2$ and $r_i^{(1)} \geq 1$, since $d_i = 0$). If $d_i = \frac{1}{2}$, $i \geq 2$, then either $V_1 \cap V_i \neq \emptyset$ or else $|V_1 \cap Q_i| = 2$, $r_i = 3$ and $r_i^{(1)} = 0$ or 1 for otherwise the diameter constraints are violated.

Let p_1, p_2, p_3 and p_4 denote the number of components of R with $d_i = -\frac{1}{2}, 0, \frac{1}{2}$ and $> \frac{1}{2}$ respectively. For $i \geq 2$ we let $V_i' = V_i - V_i \cap V_1$, and for $i \neq j \geq 2$ we let $b_{ij} = |V_i' \cap V_j'|$. It follows from the above that

$$\sum_{i=1} (v_i + d_i) \geq u + p_2 + p_3 + 2p_4 - 1 + \sum_{2 \leq i \neq j \leq p} b_{ij}. \tag{8}$$

Equations (2), (4) and (8) give:

$$M \geq \tfrac{1}{2}(3n-5) + \tfrac{1}{2}\{a + 2\varepsilon + p_2 + p_3 + 2p_4 - 1 + \sum_{2 \leq i \neq j \leq p} b_{ij}\}. \tag{9}$$

If $A \neq \emptyset$, then $M = \frac{1}{2}(3n-5)$ only if $a = 1$, $\varepsilon = 0$, $p_2 = p_3 = p_4 = 0$ and $\sum_{2 \leq i \neq j \leq p} b_{ij} = 0$, that is, only if G has adjacent vertices of degree 2. Therefore we may suppose that $A = \emptyset$. If n is even, then clearly $M = \frac{1}{2}(3n-6)$ only if G has adjacent vertices of degree 2. So the only case that needs to be considered is the case n odd. Now clearly $M = \frac{1}{2}(3n-5)$ only if

$$\varepsilon = 0, \quad p_4 = 0 \quad \text{and} \quad p_2 + p_3 + \sum_{2 \leq i \neq j \leq p} b_{ij} \leq 1.$$

Since x and y are not adjacent ($\varepsilon = 0$) every component R_i with $d_i = -\frac{1}{2}$ must have one vertex of V_i adjacent to x and the other to y in order to provide for an alternative path of length ≤ 4 between α and the vertex of R_i when we delete one of x or y. Two possibilities arise:

(i) $p_2 + p_3 = 1$. Then $b_{ij} = 0$ for every $i, j \geq 2$, $i \neq j$. Suppose $d_p > -\frac{1}{2}$. Then $d_p = 0$ or $\frac{1}{2}$. If $d_p = 0$, then $r_p \leq 2$ and $q_p \leq 3$ (by Corollary 2 of Lemma 2). The lemma is obvious if $p \geq 3$, hence we suppose $p = 2$. Then $|V_1 \cap Q_p| = 2$ for otherwise we have adjacent vertices of degree 2. But then $q \leq 3$, and hence one of x or y has degree 2. Suppose now that $d_p = \frac{1}{2}$. Clearly $V_p \cap V_i = \emptyset$ for every $i \neq p$ for otherwise strict inequality holds in (8). Therefore $|V_1 \cap Q_p| = 2$. Since $d_p = \frac{1}{2}$, R_p must be a tree and $r_p = r_p^{(2)} = 3$. Consequently at least one vertex, s say, of V_1 must be adjacent to a vertex, t say, of R_p having degree 2. But then there is no (t, x)-path of length ≤ 4 when we delete the vertex s, hence the lemma is proved for this case.

(ii) $p_2 = p_3 = 0$. Then $\sum_{2 \leq i \neq j \leq p} b_{ij} \leq 1$ and $d_i = -\frac{1}{2}$ for every $i = 1, 2, \ldots, p$. $V_i \cap V_j \neq \emptyset$ for every $i \neq j$ since $D(G) \leq 3$. Now $q \geq 4$ for otherwise one of x or y has degree 2. Consequently G must have adjacent vertices of degree 2 as otherwise

$$\sum_{2\leqslant i\neq j\leqslant p} b_{ij} \geqslant 2.$$

This completes the proof of Lemma 3.

Lemmas 1 and 3 give:

THEOREM 1. *For every* $n \geqslant 5$, $M_V(n,3,4,1) = [\frac{1}{2}(3n-5)]$ *and the graphs of* H^9 K^{2m} *and* K^{2m+1} *are elements of* $\text{Min}\, G_V(n,3,4,1)$ *for* $n = 9$, $n = 2m$ *and* $n = 2m+1$, *respectively.*

4. The structures of $\text{Min}\, G_V(n,3,\lambda,1)$, $\lambda \geqslant 5$

Bollobás (1968a) showed that $M_V(n,3,\lambda,1) = [\frac{1}{2}(3n-6)]$, $\lambda \geqslant 5$. In this section we confirm this result and the structures of $\text{Min}\, G_V(n,3,\lambda,1)$.

Denote by L_1^{2m+1} the class of graphs on $2m+1$ vertices ($m>3$) obtained from the class K^{2m} (defined in the previous section) by inserting a vertex x_7 along the edge x_1x_6. The class L_2^{2m+1} is obtained from the class $L_1^{2m'+1}$ ($m'<m$) by adding $m-m'$ disjoint (x_1,x_5)-paths of length 3. We form the class L_1^{2m} from the class L_1^{2m-3} by adding 3 vertices, x_{2m-2}, x_{2m-1} and x_{2m} together with the edges x_1x_{2m-2}, $x_{2m-2}\,x_{2m-1}$, $x_{2m-1}\,x_{2m}$, $x_{2m}\,x_4$ and $x_{2m-1}x_6$ or $x_{2m-1}x_5$. The class L_2^{2m} is formed from the class L_1^{2m-1} or L_2^{2m-1} by adding one vertex x_{2m} and connecting it to x_1 and any one of the vertices x_3, x_4, x_5 or x_6. The class L_3^{2m} is obtained from the class L_1^{2m-1} by adding a vertex x_{2m} and connecting it to x_7 and any one of the vertices x_3, x_4 or x_5. For $m\geqslant 4$ put $L_4^{2m} = K^{2m}$. We illustrate some of these constructions in Figure 3.

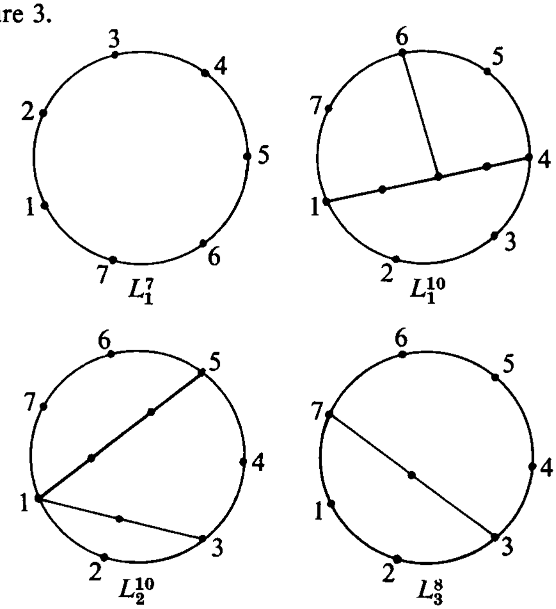

FIGURE 3.

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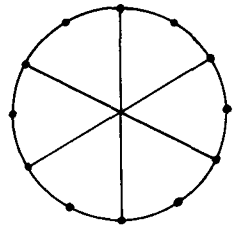
It is easily seen that the graphs of L_j^{2m+1} ($j = 1,2$) and L_j^{2m} ($j = 1,2,3,4$) belong to the class $G_V(n,3,\lambda,1)$, $\lambda \geqslant 5$, for $n = 2m+1$ and $n = 2m$, respectively, and they have $[\frac{1}{2}(3n-6)]$ edges (see Figure 3). We shall prove that these graphs together with the graph H^{12} of Figure 4 are the elements of $\operatorname{Min} G_V(n,3,\lambda,1)$, $\lambda \geqslant 5$, for every $n \geqslant 6$. To do this we need the following lemma.



H^{12}

FIGURE 4.

LEMMA 4. *G possesses two adjacent vertices of degree 2 for every $n \geqslant 6$, except possibly for $n = 12$ for which case the only other possible graph is the graph H^{12} of Figure 4.*

Replacing v_i by u_i in (3) and noting that d_i may now take on the value -1 (unlike the case $\lambda = 4$ it is not necessary that every vertex of U be adjacent to a vertex of $R^{(1)}$), we can establish Lemma 4 by using arguments similar to those in the proof of Lemma 3. He omit the details here, but full details can be found in Caccetta (1976b).

THEOREM 2. *If $\lambda \geqslant 5$ and $n \geqslant 6$, then $M_V(n,3,\lambda,1) = [\frac{1}{2}(3n-6)]$ and the graphs of L_j^{2m+1} ($j = 1,2$) and L_j^{2m} ($j = 1,2,3,4$) are the elements of $\operatorname{Min} G_V(n,3,\lambda,1)$ for $n = 2m+1$ and $n = 2m$, respectively.*

PROOF. By Lemma 4, G has two adjacent vertices, α and β say, of degree 2. We adopt the notation in the proof of Lemma 1. Since $D(G) \leqslant 3$ every vertex of R_i must be adjacent to at least two vertices of Q, hence

$$f_i \geqslant \max\{2r_i, q_i\} = \tfrac{1}{2}(2r_i+q_i)+\tfrac{1}{2}|q_i-2r_i|.$$

If $U = \varnothing$, then simply by counting the number of edges

$$M \geqslant 3+2a+\tfrac{3}{2}w+2r = \tfrac{1}{2}(3n-6)+\tfrac{1}{2}(a+r).$$

Therefore $M = [\frac{1}{2}(3n-6)]$ only if n is even and $A \cup R = \varnothing$, giving the structures L_4^{2m}, $n = 2m$.

We now consider the case $U \neq \emptyset$ and hence $R \neq \emptyset$. Clearly each component R_i of R satisfies the inequality,

$$(10) \qquad e_i + f_i \geqslant \tfrac{3}{2} r_i + \tfrac{1}{2} u_i + \delta_i$$

where

$$(11) \qquad \delta_i = \tfrac{1}{2}\{(q_i - u_i) + |q_i - 2r_i| + r_i - 2\}.$$

Let

$$\delta_1 = \min_{1 \leqslant i \leqslant p} \{\delta_i\}.$$

Simply by counting the number of edges:

$$(12) \qquad M \geqslant 3 + 2a + \tfrac{3}{2} w + u + \sum_{i=1}^{p} (e_i + f_i).$$

If $\delta_1 > 0$, then (10) and (12) give $M \geqslant \frac{1}{2}(3n-5)$, hence $\delta_1 \leqslant 0$. Two cases arise.

Case (*a*). $\delta_1 = 0$.

Then

$$(13) \qquad \sum_{i=1}^{p} (u_i + \delta_i) \geqslant u.$$

Note that strict inequality holds in (13) if either $U_i \cap U_j \neq \emptyset$ for some $i \neq j$ or $\delta_i > 0$ for some $i \geqslant 2$. Now equations (10), (12) and (13) give $M \geqslant \frac{1}{2}(3n-6)$, with strict inequality holding if $A \neq \emptyset$ or one of (12) or (13) holds with strict inequality. Therefore, as $D(G) \leqslant 3$, the only possibility is $p = 1$ and n even, giving the structures L_1^{2m} (when $r_1 = 2$ and $q_1 = u_1 = 4$) and L_2^{2m} (when $r_1 = 1$ and $q_1 = u_1 = 3$).

Case (*b*). $\delta_1 = -\frac{1}{2}$.

Then $r_1 = 1$ and $q_1 = u_1 = 2$. $U_1 \cap U_i \neq \emptyset$ for every $i \geqslant 2$ since $D(G) \leqslant 3$. Consequently

$$(14) \qquad \sum_{i=1}^{p} (u_i + \delta_i) \geqslant u - 2 + p_1 + 2p_2 + 3p_3,$$

where p_1, p_2 and p_3 denote the number of components with $\delta_i = 0$, $\frac{1}{2}$ and $\geqslant 1$, respectively. Clearly $p_2 = p_3 = 0$ and $p_1 \leqslant 1$ for otherwise $M \geqslant \frac{1}{2}(3n-5)$. Therefore $\delta_i = -\frac{1}{2}$ for all except possibly one i, $i = 1, 2, \ldots, p$.

If $p_1 = 0$, then equality in (14) holds only if $|U_1 \cap U_i| = 1$ for every $i \geqslant 2$ and $U'_i \cap U'_j = \emptyset$ for every $i, j \geqslant 2$, $i \neq j$, where $U'_i = U_i - U_i \cap U_1$. Equations (10), (12) and (14) give:

$$M \geqslant \tfrac{1}{2}(3n-7) + \tfrac{1}{2} a.$$

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Now $M = \frac{1}{2}(3n-7)$ only if $A = \varnothing$ and strict equality holds in (12) and (13). Hence when n is odd the only possible structures are the graphs L_1^{2m+1} (if $p = 1$) and L_2^{2m+1} (if $p > 1$), $n = 2m+1$.

Clearly $M = \frac{1}{2}(3n-6)$ only if $a+p_1 \leqslant 1$, and if $a+p_1 = 1$ then $U_i' \cap U_j' = \varnothing$ for every $i \neq j \geqslant 2$ and $|U_1 \cap U_i| = 1$ for every $i \geqslant 2$. When $a+p_1 = 1$ we get the structures L_2^{2m}, $n = 2m$. When $a = p_1 = 0$ we simply have two vertices of R connected to the same two vertices of Q thus giving the structures L_2^{2m} (if $W = \varnothing$) and L_3^{2m} (if $W \neq \varnothing$). This completes the proof of Theorem 3.

5. The Structures of Min $G_E(n, 3, 4, 1)$

Consider the graphs displayed in Figure 5 below. For $m > 4$ we obtain the class M_j^{2m} ($j = 1, 2, \ldots, 6$) from the class M_j^{2m-2} by adding an (x_1, x_2)-path of length 3. The class M_7^{2m} is obtained from the class $M_1^{2m'}$, $m' < m$, by adding $m-m'$ triangles with x_1 being a common vertex in all triangles and the other two being new vertices. It is easily seen that these graphs belong to the class $G_E(n, 3, 4, 1)$ and have $[\frac{1}{2}(3n-4)]$ edges. It turns out that these graphs together with the graphs K_1^{2m+1} will exhaust all the extremal graphs in the class $G_E(n, 3, 4, 1)$ except for $n = 8$, 10 and 12 for which the graphs of Figure 6 also belong to the class $G_E(n, 3, 4, 1)$ and have $[\frac{1}{2}(3n-4)]$ edges.

We note that the graphs of M_7^{2m} are the only one of the above graphs having a cut vertex. Suppose that G has a cut vertex, c say, and $G = G_1 \cup G_2$, where

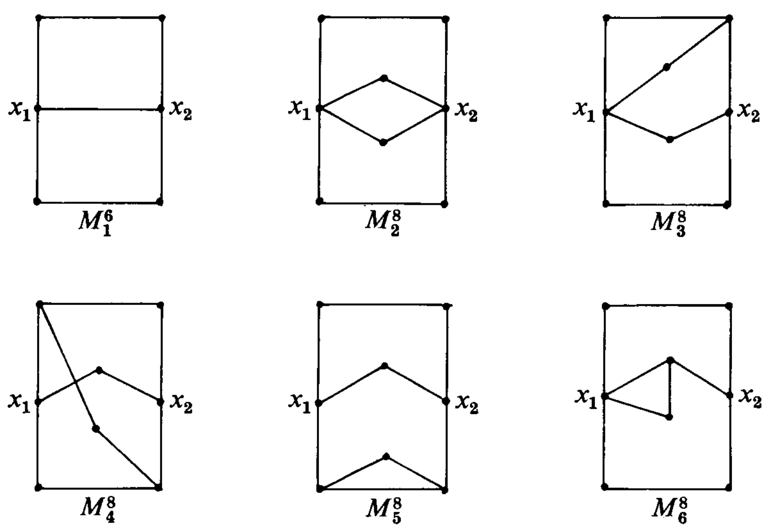

FIGURE 5.

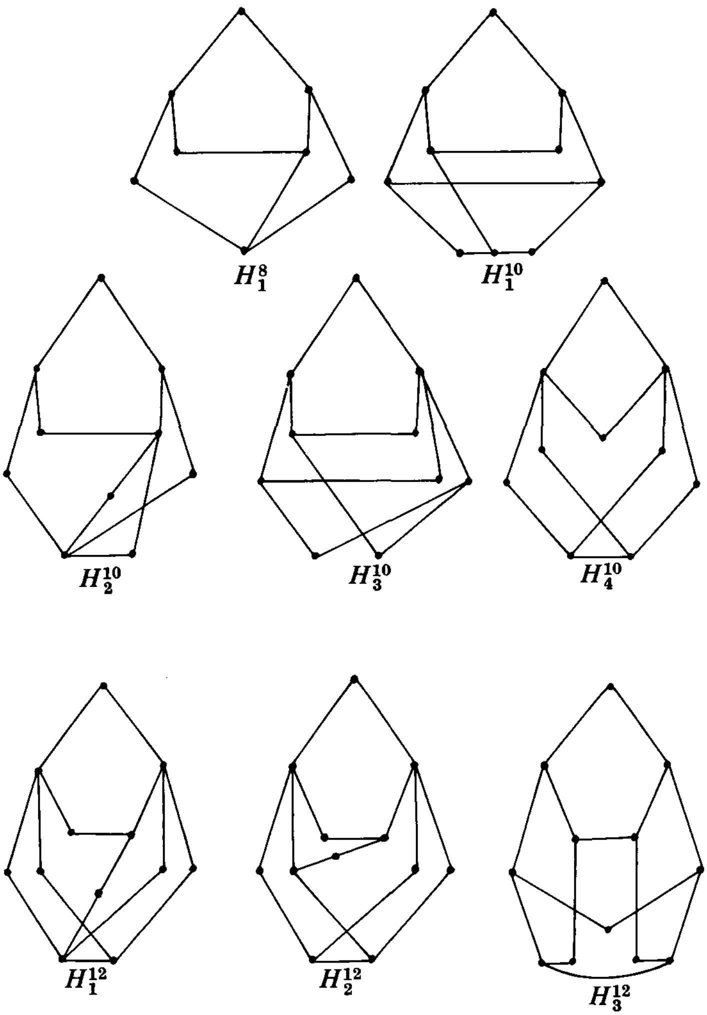

FIGURE 6.

$V(G_1) \cap V(G_2) = c$. Let $n_1 = |V(G_1 - c)|$ and $n_2 = |V(G_2)|$ so that $n = n_1 + n_2$. Since $D(G) \leqslant 3$, it can be supposed that every vertex of $G_1 - c$ is adjacent to c. Hence the subgraph spanned by G_1 has $\geqslant \frac{3}{2} n_1$ edges. Bollobás (1968b) proved that for $\lambda \geqslant 6$, $M_E(n, 3, \lambda, 1) = [\frac{1}{2}(3n - 6)]$. This together with observation (1) of Section 2 gives

$$[\tfrac{1}{2}(3n-6)] \leqslant M_E(n, 3, 4, 1) \leqslant [\tfrac{1}{2}(3n-4)].$$

Therefore $M_E(n, 3, 4, 1) = [\frac{1}{2}(3n - k)$, where k is an integer taking on values between 4 and 6. Now the subgraph obtained by deleting the n_1 vertices of $G_1 - c$ belongs to the class $G_E(n,3,4,1)$ and has at least

$$M-\tfrac{3}{2}n_1 = [\tfrac{1}{2}(3n-k)]-\tfrac{3}{2}n_1 = [\tfrac{1}{2}(3n_2-k)] \text{ edges.}$$

We can therefore restrict our structural study of $\operatorname{Min} G_E(n,3,4,1)$ to the case where G has no cut vertex, since if it had, then we can delete vertices and once we have a basic graph then we can add vertices to it so as to form a cut vertex.

We notice that Lemma 2 with its three corollaries holds for the edge case. Also, equations (4)–(9) (with equality possible in (7)) hold. It seems natural to expect that the proof of Lemma 3 can be used to establish an edge analogue of that lemma. This is in fact the case. However, here the problem becomes much more complex as there are more cases to be considered (for example, we need to consider the case $d_1 = \frac{1}{2}$). Consequently we do not give the details here but simply state the results, for full details we refer to Caccetta (1976b).

LEMMA 5. *G possesses two adjacent vertices of degree 2 for every $n \geq 5$ except possibly $n = 8$, 10 and 12. Furthermore, for these values of n the only other possible structures are the graphs H_1^8, H_j^{10} $(j = 1,2,3,4)$ and H_j^{12} $(j = 1,2,3)$ of Figure 6.*

With only minor modification the proof of Lemma 1 can be used to prove,

LEMMA 6. *If G has no cut vertex then $M_E(n,3,4,1) = [\frac{1}{2}(3n-5)]$, and the graphs of H_1^8, H_j^{10} $(j = 1,2,3,4)$, H_j^{12} $(j = 1,2,3)$, M_j^{2m} $(j = 1,2,\ldots,6)$ and K_1^{2m+1} are the elements of $\operatorname{Min} G_E(n,3,4,1)$ for $n = 8$, 10, 12, $2m$ and $2m+1$ respectively.*

If G has a cut vertex it must be x_1 or x_4 since $D(G) \leq 3$. It is easily seen that the basic graph M_1^6 is the only graph which can have a cut vertex. This together with Lemma 6 gives

THEOREM 3. *For every $n \geq 4$, $M_E(n,3,4,1) = [\frac{1}{2}(3n-4)]$ and the graphs of H_1^8, H_j^{10} $(j = 1,2,3,4)$, H_j^{12} $(j = 1,2,3)$, M_j^{2m} $(j = 1,2,\ldots,7)$ and M_1^{2m+1} are the elements of $\operatorname{Min} G_E(n,3,4,1)$ for $n = 8,10,12,2m$ and $2m+1$ respectively.*

6. The structures of $\operatorname{Min} G_E(n,3,\lambda,1)$, $\lambda \geq 5$.

Bollobás (1968b) proved that if $\lambda \geq 6$ and $n \geq 6$, then $M_E(n,3,\lambda,1) = [\frac{1}{2}(3n-6)]$. In the following we show that the structures of $\operatorname{Min} G_E(n,3,\lambda,1)$, $\lambda \geq 6$, coincide with the structures of $\operatorname{Min} G_E(n,3,\lambda',1)$, $\lambda' \geq 5$.

As in the previous section we can restrict our structural study of extremal graphs to the case when G has no cut vertex. Then obviously the graphs of L_j^{2m+1} $(j = 1,2)$ and L_j^{2m} $(j = 1,2,3,4)$ defined in Section 4 are the only extremal graphs. From these graphs it is clear that if G has a cut vertex then it should be either x_1 or x_4. In either case it is easily seen that $D(G)>3$ if x_1 or x_4 is a cut vertex. Consequently,

THEOREM 4. *If $\lambda \geq 6$ and $n \geq 6$, then $M_E(n,3,\lambda,1) = [\frac{1}{2}(3n-6)]$, and the elements of* $\operatorname{Min} G_E(n,3,\lambda,1)$ *and* $\operatorname{Min} G_V(n,3,\lambda,1)$ *coincide.*

We now consider the case $\lambda = 5$. The following constructions show that $M_E(n,3,5,1) \leq [\frac{1}{2}(3n-5)]$. Denote by P_j^{2m} $(j=1,2)$ the class obtained from the class L_j^{2m-1} by adding a vertex x_{2m} and connecting it to x_1 and one of x_2 or x_3 (see Figure 3). The class K_4^{2m+1} (K_5^{2m+1}) is obtained from the class P_1^{2m} (P_2^{2m}) by adding a vertex x_{2m+1}, joined to each vertex in the neighbour set of x_{2m}. The class K_6^{2m+1} is obtained from the class $K_1^{2m'+1}$, $m'<m$, by adding $m-m'$ triangles with x_1 being a common vertex in all triangles and the other two being new vertices. Put $P_3^{2m} = K^{2m}$. Clearly the graphs of P_j^{2m} $(j=1,2,3)$ and K_j^{2m+1} $(j=1,2,\ldots,6)$ belong to the class $G_E(n,3,5,1)$, for $n=2m$ and $n=2m+1$, respectively, and they have $[\frac{1}{2}(3n-5)]$ edges.

It follows from observation (1) of Section 2 and Theorem 4 above that

$$[\tfrac{1}{2}(3n-6)] \leq M_E(n,3,5,1) \leq [\tfrac{1}{2}(3n-5)].$$

As in the case $\lambda = 4$ we can restrict our structural study of extremal graphs to the case when G has no cut vertex. Therefore G belongs to the class $G_V(n,3,\lambda',1)$ for some λ'. If $\lambda'=4$ then, by Theorem 1, $M \geq [\frac{1}{2}(3n-5)]$. If $\lambda' \geq 5$ then, since for n odd the graphs of $\operatorname{Min} G_V(n,3,\lambda',1)$ do not belong to the class $G_E(n,3,5,1)$ (see Figure 3), $M \geq [\frac{1}{2}(3n-5)]$. Hence $M_E(n,3,5,1) = [\frac{1}{2}(3n-5)]$.

With only minor modification the proof of Lemma 3 can be used to prove

LEMMA 7. *Let $G \in \operatorname{Min} G_E(n,3,5,1)$. Then G possesses two adjacent vertices of degree 2 for every $n \geq 5$ except possibly $n=9$ for which case the only other possible structure is the graph H^9 of Figure 2.*

We can now easily establish the following theorem.

THEOREM 5. *If $n \geq 5$, then $M_E(n,3,5,1) = [\frac{1}{2}(3n-5)]$, and the graphs of H^9, P_j^{2m} $(j=1,2,3)$ and K_j^{2m+1} $(j=1,2,\ldots,6)$ are the elements of* $\operatorname{Min} G_E(n,3,5,1)$ *when $n=9$, $n=2m$ and $n=2m+1$, respectively.*

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Department of Mathematics
University of Waterloo
Ontario, N2L 3G1
Canada