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# Embedding Theorem for Inhomogeneous Besov and Triebel–Lizorkin Spaces on RD-spaces

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*Abstract.* In this article we prove an embedding theorem for inhomogeneous Besov and Triebel–Lizorkin spaces on RD-spaces. The crucial idea is to use the geometric density condition on the measure.

## 1 Introduction and Statement of Main Results

Spaces of homogeneous type were introduced by Coifman and Weiss [CW1] in the early 1970s. A *quasi-metric* $d$ on a set $X$ is a function $d\colon X\times X\to[0,\infty)$ satisfying (i) $d(x,y)=d(y,x)\geq 0$ for all $x,y\in X$; (ii) $d(x,y)=0$ if and only if $x=y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0\in[1,\infty)$ such that for all $x$, $y$, $z\in X$,

$$d(x,y)\leq A_0\big[\,d(x,z)+d(z,y)\big].$$

We define the quasi-metric ball by $B(x,r):=\{y\in X: d(x,y)<r\}$ for $x\in X$ and $r>0$. Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open. We say that $(X,d,\mu)$ is a space of homogeneous type in the sense of Coifman and Weiss if $d$ is a quasi-metric and $\mu$ is a nonnegative Borel regular measure on $X$ satisfying the *doubling condition*: if $x\in X, r>0$, then $0<\mu(B(x,r))<\infty$ and

$$\mu\big(\,B(x,2r)\big)\leq C\mu\big(\,B(x,r)\big), \tag{1.1}$$

where $\mu$ is assumed to be defined on a $\sigma$−algebra which contains all Borel sets and all balls $B(x,r)$ and the constant $0<C<\infty$ is independent of $x\in X$ and $r>0$.

We point out that the doubling condition (1.1) implies that there exists a positive constant $\omega:=\log_2 C$ (the *upper dimension* of $\mu$) such that for all $x\in X$, $\lambda\geq 1$ and $r>0$,

$$\mu\big(\,B(x,\lambda r)\big)\leq C\lambda^{\omega}\mu\big(\,B(x,r)\big).$$

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Spaces of homogeneous type play a prominent role in many fields of mathematics. In particular, they constitute natural generalizations of manifolds admitting all kinds of singularities and still providing rich geometric structure; see [S1, S2]. Analysis on metric measure spaces has been studied quite intensively in [CW2, DJDS, HS, DH, H1, H2]. For example, Coifman and Weiss [CW2] introduced atomic Hardy space $H^p_{at}$ for $p \in (0,1]$ and proved that if $T$ is a Calderón–Zygmund singular integral operator and is bounded on $L^2$, then $T$ extends a bounded operator from $H^p$ to $L^p$ for suitable $p \le 1$. In many applications, however, the additional assumptions on the measure $\mu$ are required. For instance, in order to provide the maximal function characterization of the Hardy spaces $H^p_{at}$ on spaces of homogeneous type, Macías and Segovia [MS] showed that the quasi-metric $d$ can be replaced by another quasi-metric $\widetilde{d}$ such that the topologies induced on $X$ by $d$ and $\widetilde{d}$ coincide. Moreover, $\widetilde{d}$ has the following regularity property. There exist constants $C > 0$ and $0 < \theta < 1$ such that for all $0 < r < \infty$ and all $x, x', y \in X$,

$$|\widetilde{d}(x,y) - \widetilde{d}(x',y)| \le C\widetilde{d}(x,x')^{\theta}\big[\widetilde{d}(x,y) + \widetilde{d}(x',y)\big]^{1-\theta} \tag{1.2}$$

and the measure $\mu$ satisfies the following property. For

$$B(x,r) = \big\{y \in X : \widetilde{d}(x,y) < r\big\},$$
$$\mu\big(B(x,r)\big) \sim r. \tag{1.3}$$

Note that property (1.3) is much stronger than the doubling condition. Macías and Segovia provided the maximal function characterization for Hardy spaces $H^p(X)$ with $(1+\theta)^{-1} < p \le 1$, on spaces of homogeneous type $(X, d, \mu)$ that satisfy the regularity condition (1.2) on the quasi-metric $d$ and property (1.3) on the measure $\mu$.

In [NS], Nagel and Stein developed the product $L^p$ $(1 < p < \infty)$ theory in the setting of the Carnot–Carathéodory spaces formed by vector fields satisfying Hörmander's finite rank condition. The particular Carnot–Carathéodory spaces studied in [NS] are metric spaces with a measure $\mu$ satisfying the conditions $\mu(B(x,sr)) \sim s^{m+2}\mu(B(x,r))$ for $s \ge 1$ and $\mu(B(x,sr)) \sim s^4\mu(B(x,r))$ for $s \le 1$. These conditions on the measure are weaker than property (1.3) but are still stronger than the original doubling condition (1.1). In [HMY1], motivated by the work of Nagel and Stein, Besov and Triebel–Lizorkin spaces were developed on spaces of homogeneous type with the regularity condition (1.2) on the quasi-metric $d$ and the measure $\mu$ satisfying the following reversed doubling condition: there are constants $\kappa \in (0, \omega]$ and $c \in (0,1]$ such that

$$c\lambda^{\kappa}\mu\big(B(x,r)\big) \le \mu\big(B(x,\lambda r)\big) \tag{1.4}$$

for all

$$x \in X, \quad 0 < r < \sup_{x,y\in X} d(x,y)/2, \quad \text{and} \quad 1 \le \lambda < \sup_{x,y\in X} d(x,y)/2r.$$

An RD-space $(X, d, \mu)$ is a space of homogeneous type in sense of Coifman and Weiss where the quasi-metric $d$ satisfies (1.2) and the measure $\mu$ satisfies the "reverse" doubling property (1.4). For further developments, including analogous theories of function spaces on RD-spaces, we refer the reader to [NS, HMY1, HMY2].

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The main purpose in this paper is to establish an embedding theorem for inhomogeneous Besov and Triebel–Lizorkin spaces on RD-spaces. Note that embedding theorems are essential tools in many fields, including function spaces and especially partial differential equations. See [FJ, J, T] for embedding theorems on $\mathbb{R}^n$. Han [H3], Han and Lin [HL], and Yang [Y] have proved embedding theorems for Besov and Triebel–Lizorkin spaces on spaces of homogeneous type $(X, d, \mu)$, where the quasi-metric $d$ satisfies (1.2) and however, measure $\mu$ satisfies (1.3), respectively. Therefore, the result in this paper generalizes the embedding results to a more general setting.

Let us now recall some notation. Let $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. Throughout this paper, we use $C$ to denote positive constants whose value may vary from line to line. Constants with subscripts, such as $C_1$, do not change in different occurrences. By $V_r(x)$ we denote the measure of $B(x, r)$, the ball centered at $x$ with radius $r > 0$, and by $V(x, y)$, we denote the measure of $B(x, y)$, the ball centered at $x$ with radius $d(x, y) > 0$. In addition, we use the notation $a \lesssim b$ to mean that there is a constant $C > 0$ such that $a \le Cb$, and the notation $a \sim b$ to mean that $a \lesssim b \lesssim a$. The implicit constants, $C$, are meant to be independent of other relevant quantities. Also, for two topological spaces, $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \hookrightarrow \mathfrak{B}$ means a linear and continuous embedding. For $p > 1$, let $p'$ be its conjugate index.

We now recall test functions and distributions on RD-spaces $(X, d, \mu)$.

***Definition 1.1*** (Test functions, [HMY1]) Fix $x_0 \in X$, $r > 0$, $\gamma > 0$, and $\beta \in (0, \theta)$. A function $f$ defined on $X$ is said to be a *test function of type* $(x_0, r, \beta, \gamma)$ *centered at* $x_0 \in X$ if $f$ satisfies the following two conditions.

(i) (Size condition) For all $x \in X$,

$$|f(x)| \le C \frac{1}{V_r(x_0) + V(x, x_0)} \Big( \frac{r}{r + d(x, x_0)} \Big)^{\gamma}.$$

(ii) (Hölder regularity condition) For all $x, y \in X$ satisfying $d(x, y) < (2A_0)^{-1} (r + d(x, x_0))$,

$$|f(x) - f(y)| \le C \Big( \frac{d(x, y)}{r + d(x, x_0)} \Big)^{\beta} \frac{1}{V_r(x_0) + V(x, x_0)} \Big( \frac{r}{r + d(x, x_0)} \Big)^{\gamma}.$$

We denote by $G(x_0, r, \beta, \gamma)$ the set of all test functions of type $(x_0, r, \beta, \gamma)$. The norm of $f$ in $G(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{G(x_0, r, \beta, \gamma)} := \inf\{C > 0 : \text{ (i) and (ii) hold}\}.$$

For each fixed $x_0$, let $G(\beta, \gamma) := G(x_0, 1, \beta, \gamma)$. It is easy to check that for each fixed $x_1 \in X$ and $r > 0$, we have $G(x_1, r, \beta, \gamma) = G(\beta, \gamma)$ with equivalent norms. Furthermore, it is also easy to see that $G(\beta, \gamma)$ is a Banach space with respect to the norm on $G(\beta, \gamma)$.

For $0 < \beta < \theta$ and $\gamma > 0$, let $\mathring{G}(\beta, \gamma)$ be the completion of the space $G(\theta, \gamma)$ in the norm of $G(\beta, \gamma)$. For $f \in G(\beta, \gamma)$, define $\|f\|_{\mathring{G}(\beta, \gamma)} := \|f\|_{G(\beta, \gamma)}$.

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***Definition 1.2*** (Distributions) The *distribution space* $(\mathring{G}(\beta,\gamma))'$ is defined to be the set of all linear functionals $\mathcal{L}$ from $\mathring{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists $C > 0$ such that for all $f \in \mathring{G}(\beta,\gamma)$, $|\mathcal{L}(f)| \le C\|f\|_{\mathring{G}(\beta,\gamma)}$.

We begin with recalling the definition of an approximate to the identity, which plays the same role as the heat kernel $H(s,x,y)$ does in [NS].

***Definition 1.3*** [HMY1] Let $\theta$ be the regularity exponent of $X$. A sequence $\{S_k\}_{k\in\mathbb{N}}$ of linear operators is said to be an approximation to the identity (ATI) if there exist constants $C, C_1 > 0$ such that for all $k \in \mathbb{N}$ and all $x, x', y, y' \in X$, $S_k(x,y)$, the kernel of $S_k$, is a function from $X \times X$ into $\mathbb{C}$ satisfies

(i) $S_k(x,y) = 0$ if $d(x,y) \ge C_1 2^{-k}$ and $|S_k(x,y)| \le C\dfrac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C2^{k\theta} d(x,x')^{\theta} \dfrac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$
for $\rho(x,x') \le \max\{2C_1, 2\}2^{-k}$;

(iii) $|S_k(x,y) - S_k(x,y')| \le C2^{k\theta} d(y,y')^{\theta} \dfrac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$
for $\rho(y,y') \le \max\{2C_1, 2\}2^{-k}$;

(iv) $$|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \le C2^{2k\theta} d(x,x')^{\theta} d(y,y')^{\theta} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$$
for $\rho(x,x') \le \max\{2C_1, 2\}2^{-k}$ and $\rho(y,y') \le \max\{2C_1, 2\}2^{-k}$;

(v) $\int_X S_k(x,y)\,d\mu(y) = 1$;

(vi) $\int_X S_k(x,y)\,d\mu(x) = 1$.

The following constructions, which provide an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type, were given by Christ [Chr].

***Lemma 1.4*** *Let X be a RD-space. Then there exist a collection*

$$\{Q_\alpha^k \subset X : k \in \mathbb{N}, \alpha \in I_k\}$$

*of open subsets, where $I_k$ is some (possible finite) index set, and constants $\delta \in (0,1)$ and $C_2, C_3 > 0$ such that*

(i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$ *for each fixed k and* $Q_\alpha^k \cap Q_\beta^k = \varnothing$ *if* $\alpha \ne \beta$;

(ii) *for any* $\alpha, \beta, k, l$ *with* $l \ge k$, *either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \varnothing$;

(iii) *for each* $(k,\alpha)$ *and each* $l < k$ *there is a unique* $\beta$ *such that* $Q_\alpha^k \subset Q_\beta^l$;

(iv) $\operatorname{diam}(Q_\alpha^k) \le C_2\delta^k$;

(v) *each* $Q_\alpha^k$ *contains some ball* $B(z_\alpha^k, C_3\delta^k)$, *where* $z_\alpha^k \in X$.

In fact, we can think of $Q_\alpha^k$ as being a dyadic cube with a diameter roughly $\delta^k$ and centered at $z_\alpha^k$. In what follows, we always suppose $\delta = 1/2$ (see [HS] for how to remove this restriction). Also, in the following, for $k \in \mathbb{N}, \tau \in I_k$, we will denote by

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$Q_\tau^{k,\nu}$, $\nu = 1, \dots, N(k,\tau)$, the set of all cubes $Q_{\tau'}^{k+M} \subset Q_\tau^k$, where $M$ is a fixed large positive integer.

The inhomogeneous Besov and Triebel–Lizorkin spaces on RD-spaces are defined as follows.

***Definition 1.5*** [HMY1] Suppose that $|s| < \theta$, $\omega$ is the upper dimension of $(X, d, \mu)$. Suppose that $\{S_k\}_{k\in\mathbb{N}}$ is an ATI and let $D_0 = S_0$ $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}_+$. Let $M$ be a fixed large positive integer, and let $Q_{\tau_0}^{0,\nu}$ be as above. The inhomogeneous Besov space $B_p^{s,q}(X)$ is the collection of all $f \in (\mathring{G}(\beta,\gamma))'$ with $\beta, \gamma \in (0,\theta)$ and

$$\max\Big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\Big) < p \le \infty \quad \text{and} \quad 0 < q \le \infty$$

such that

$$\|f\|_{B_p^{s,q}(X)} = \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\Big\}^{\frac{1}{p}} + \Big\{\sum_{k\in\mathbb{Z}_+} 2^{ksq}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|^p\Big]^{q/p}\Big\}^{1/q} < \infty,$$

where $m_{Q_\tau^{0,\nu}}(D_0(f))$ are averages of $D_0(f)$ over $Q_\tau^{0,\nu}$.

The inhomogeneous Triebel–Lizorkin space $F_p^{s,q}(X)$ is the collection of all $f \in (\mathring{G}(\beta,\gamma))'$ with $\beta, \gamma \in (0,\theta)$ and

$$\max\Big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\Big) < p < \infty, \quad \max\Big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\Big) < q \le \infty$$

such that

$$\|f\|_{F_p^{s,q}(X)} = \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\Big\}^{\frac{1}{p}} + \Big\|\Big\{\sum_{k\in\mathbb{Z}_+}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}|D_k(f)(y_\tau^{k,\nu})|^q \chi_{Q_\tau^{k,\nu}}\Big\}^{1/q}\Big\|_{L^p(X)} < \infty.$$

***Remark 1.6*** Let $\lambda_\tau^{0,\nu}(f) := m_{Q_\tau^{0,\nu}}(D_0(f))$ for $\tau \in I_0$ and $\nu = 1, \dots, N(0,\tau)$, and let $\lambda_\tau^{k,\nu}(f) := D_k(f)(y_\tau^{k,\nu})$ for $k \in \mathbb{Z}_+$, $\tau \in I_k$ and $\nu = 1, \dots, N(k,\tau)$, where $y_\tau^{k,\nu}$ is any fixed point in $Q_\tau^{k,\nu}$ as in Definition 1.5. Then we can rewrite the norms of $B_p^{s,q}(X)$ and $F_p^{s,q}(X)$ as following unified versions. If $\max\big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\big) < p \le \infty$ and $0 < q \le \infty$,

$$\|f\|_{B_p^{s,q}(X)} \sim \Big\{\sum_{k\in\mathbb{N}} 2^{ksq}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})|\lambda_\tau^{k,\nu}(f)|^p\Big]^{q/p}\Big\}^{1/q}.$$

If $\max\big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\big) < p < \infty$, $\max\big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\big) < q \le \infty$, then

$$\|f\|_{F_p^{s,q}(X)} \sim \Big\|\Big\{\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}|\lambda_\tau^{k,\nu}(f)|^q \chi_{Q_\tau^{k,\nu}}\Big\}^{1/q}\Big\|_{L^p(X)}.$$

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We would like to point out that the reverse doubling property on the measure played an important role. More precisely, this reverse doubling property ensures that

$$\sum_{k\in\mathbb{Z}:2^{-k}\geq r}\frac{1}{\mu(B(x,2^{-k}))}\leq\frac{C}{\mu(B(x,r))},$$

which is the key to develop the theory of inhomogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type (see [HMY1] for more details). But the reverse doubling property on the measure does not play any role in the proof of the embedding theorem in this paper. However, to provide the embedding theorem for inhomogeneous Besov and Triebel–Lizorkin spaces on RD-spaces, the crucial idea is to use the density condition on the measure, namely

$$\mu(B(x,r)\geq Cr^{\omega} \tag{1.5}$$

for any $x\in X$ and $0<r\leq 1$. We also remark that if $X=\mathbb{R}^n$ or $X$ is a quasi-metric space with nonnegative Borel measure $\mu$ satisfying (1.3), then the density condition (1.5) automatically holds.

The main result of this paper is the following theorem.

***Theorem 1.7*** *Suppose that $-\theta<s_1<s_2<\theta$ and $\mu(B(x,r)\geq Cr^{\omega}$ for any $x\in X$ and $0<r\leq 1$.*

(i) *Let $\max\left\{\frac{\omega}{\omega+\theta},\frac{\omega}{\omega+\theta+s_i}\right\}<p_i\leq\infty, 0<q\leq\infty$, $i=1,2$ and $-\theta<s_1-\omega/p_1=s_2-\omega/p_2<\theta$. Then*

$$B^{s_2,q}_{p_2}\hookrightarrow B^{s_1,q}_{p_1}. \tag{1.6}$$

(ii) *Let $\max\left\{\frac{\omega}{\omega+\theta},\frac{\omega}{\omega+\theta+s_i}\right\}<p_i<\infty$ and $\max\left\{\frac{\omega}{\omega+\theta},\frac{\omega}{\omega+\theta+s_i}\right\}<q_i\leq\infty$ for $i=1,2$, and $-\theta<s_1-\omega/p_1=s_2-\omega/p_2<\theta$. Then*

$$F^{s_2,q_2}_{p_2}\hookrightarrow F^{s_1,q_1}_{p_1}. \tag{1.7}$$

## 2 The Proof of Theorem 1.7

The density property (1.5) on the measure $\mu$ plays a crucial role in the proof of Theorem 1.7. We first recall the following lemma given in [HMY1].

***Lemma 2.1*** *Suppose that $\{S_k\}_{k\in\mathbb{N}}$ is an approximation to the identity as in Definition 1.3. Set $D_k=S_k-S_{k-1}$ for $k\in\mathbb{Z}_+$ and $D_0=S_0$. Then there exists a family of functions $\{\widetilde{D}_k(x,y)\}_{k\in\mathbb{N}}$ such that for any fixed $y^{k,\nu}_\tau\in Q^{k,\nu}_\tau$, $k\in\mathbb{N},\tau\in I_k$ and $\nu\in\{1,\dots,N(k,\tau)\}$ and all $f\in(\mathring{G}(\beta,\gamma))'$ with $0<\beta,\gamma<\theta$ and $x\in X$,*

$$f(x)=\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\int_{Q^{0,\nu}_\tau}\widetilde{D}_0(x,y)d\mu(y)m_{Q^{0,\nu}_\tau}\big(D_0(f)\big)+\sum_{k\in\mathbb{Z}_+}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau)D_k(f)(y^{k,\nu}_\tau), \tag{2.1}$$

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*where the series converges in the norm of* $(\mathring{G}(\beta',\gamma'))'$ *with* $\theta > \beta' > \beta$ *and* $\theta > \gamma' > \gamma$. *Moreover,* $\widetilde{D}_k(x,y)$ *with* $k \in \mathbb{N}$, *the kernel of* $\widetilde{D}_k$, *satisfies that for* $0 < \epsilon < \theta$,

$$|\widetilde{D}_k(x,y)| \le C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x,y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x,y))^{\epsilon}};$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C\Big(\frac{d(x,x')}{2^{-k} + d(x,y)}\Big)^{\epsilon} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x,y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x,y))^{\epsilon}},$$

*for* $d(x,x') \le (2^{-k} + d(x,y))/2$;

$$\int_X \widetilde{D}_k(x,y)d\mu(y) = \int_X \widetilde{D}_k(x,y)d\mu(x) = 0, \quad \textit{when} \quad k \in \mathbb{Z}_+;$$
$$\int_X \widetilde{D}_0(x,y)d\mu(y) = \int_X \widetilde{D}_0(x,y)d\mu(x) = 1.$$

***Remark 2.2*** Set

$$\Xi_\tau^{0,\nu}(x) := \frac{1}{\mu(Q_\tau^{0,\nu})} \int_{Q_\tau^{0,\nu}} \widetilde{D}_0(x,y)d\mu(y)$$

for $\tau \in I_0$ and $\nu = 1,\dots,N(0,\tau)$, and $\Xi_\tau^{k,\nu}(x) := \widetilde{D}_k(x, y_\tau^{k,\nu})$ for $k \in \mathbb{Z}_+, \tau \in I_k$ and $\nu = 1,\cdots,N(k,\tau)$, where $y_\tau^{k,\nu}$ are any fixed points in $Q_\tau^{k,\nu}$. Formula (2.1) can be rewritten as

$$f(x) = \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \Xi_\tau^{k,\nu}(x) \lambda_\tau^{k,\nu}(f). \tag{2.2}$$

**Proof of Thorem 1.7** To prove (1.6), let $f \in B_{p_2}^{s_2,q}(X)$ with

$$|s_2| < \theta, \max\Big(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}\Big) < p_2 \le \infty \quad \text{and} \quad 0 < q \le \infty.$$

Since $s_1 < s_2$ and $s_1 - \omega/p_1 = s_2 - \omega/p_2$, it follows that $p_2 < p_1$. First, we recall the following known estimates of [HMY1, Lemma 3.19]. For $k, k' \in \mathbb{N}, \tau \in I_k, \tau' \in I_{k'}$ and $\nu = 1,\dots,N(k,\tau)$, $\nu' = 1,\dots,N(k',\tau')$

$$|D_k\widetilde{D}_{k'}(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})| \le C2^{-|k-k'|\epsilon} \frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu}) + V(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})} \frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')} + d(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{\epsilon}}. \tag{2.3}$$

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It easy to see that

$$\begin{aligned}(2.4)\quad &\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^\epsilon}\\ &\sim \frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y)}\frac{2^{-(k\wedge k')}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y))^\epsilon}\\ &\sim \frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')}}{(2^{-(k\wedge k')}+d(x,y_{\tau'}^{k',\nu'}))^\epsilon}\end{aligned}$$

for any $x\in Q_\tau^{k,\nu}$, $y\in Q_{\tau'}^{k',\nu'}$. Hence,

$$\begin{aligned}D_k(\Xi_{\tau'}^{0,\nu'})(y_\tau^{k,\nu}) &\lesssim 2^{-k\epsilon}\frac{1}{V_1(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{0,\nu'})}\frac{1}{(1+d(y_\tau^{k,\nu},y_{\tau'}^{0,\nu'}))^\epsilon},\\ \lambda_\tau^{0,\nu}(\Xi_{\tau'}^{k',\nu'}) &\lesssim 2^{-k'\epsilon}\frac{1}{V_1(y_\tau^{0,\nu})+V(y_\tau^{0,\nu},y_{\tau'}^{k',\nu'})}\frac{1}{(1+d(y_\tau^{0,\nu},y_{\tau'}^{k',\nu'}))^\epsilon}.\end{aligned}$$

From these and (2.3), it follows that

$$\begin{aligned}(2.5)\quad &|\lambda_\tau^{k,\nu}(\Xi_{\tau'}^{k',\nu'})|\le\\ &\qquad C2^{-|k-k'|\epsilon}\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^\epsilon}.\end{aligned}$$

By the inhomogeneous discrete Calderón reproducing formula (2.2) and the norm of $B_p^{s,q}(X)$ given in Remark 1.6, the norm $\|f\|_{B_{p_1}^{s_1,q}(X)}$ is dominated by

$$C\Big\{\sum_{k\in\mathbb{N}}2^{ks_1q}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\Big|\sum_{k'\in\mathbb{N}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\lambda_\tau^{k,\nu}(\Xi_{\tau'}^{k',\nu'})\lambda_{\tau'}^{k',\nu'}(f)\Big|^{p_1}\Big]^{q/p_1}\Big\}^{1/q}.$$

Applying (2.5), we deduce that

$$\begin{aligned}\|f\|_{B_{p_1}^{s_1,q}(X)}\lesssim\Big\{&\sum_{k\in\mathbb{N}}2^{ks_1q}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\Big(\sum_{k'\in\mathbb{N}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|\\ &\times 2^{-|k-k'|\epsilon}\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^\epsilon}\Big)^{p_1}\Big]^{q/p_1}\Big\}^{1/q}.\end{aligned}$$

We need to consider two cases.
*Case 1:* $p_1>1$.

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We choose $\epsilon' > 0$ and $\epsilon'' > 0$ such that $\epsilon = \epsilon' + \epsilon''$, $\epsilon'$ can be taken arbitrarily close to $\epsilon$, and using the Hölder inequality and (2.4), we get

$$\begin{aligned}
&\|f\|_{B^{s_1,q}_{p_1}(X)}\\
&\lesssim \Big\{\sum_{k\in\mathbb{N}} 2^{ks_1q}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\sum_{k'\in\mathbb{N}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|^{p_1}\\
&\times 2^{-|k-k'|p_1\epsilon_1}\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{\epsilon}}\Big]^{q/p_1}\Big\}^{1/q}\\
&\lesssim \Big\{\sum_{k\in\mathbb{N}} 2^{ks_1q}\Big[\sum_{k'\in\mathbb{N}}2^{-|k-k'|p_1\epsilon_1}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|^{p_1}\Big]^{q/p_1}\Big\}^{1/q}.
\end{aligned}$$

Applying the $p_2/p_1$-inequality implies that the last term above is dominated by

$$\Big\{\sum_{k\in\mathbb{N}}\Big[\sum_{k'\in\mathbb{N}}2^{-|k-k'|p_2\epsilon_1}2^{ks_1p_2}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})^{p_2/p_1}|\lambda_{\tau'}^{k',\nu'}(f)|^{p_2}\Big]^{q/p_2}\Big\}^{1/q}.$$

Note that $p_2/p_1 < 1$, and using the density condition (1.5), it immediately follows that

$$\mu(Q_{\tau'}^{k',\nu'})^{p_2/p_1} = \mu(Q_{\tau'}^{k',\nu'})\big(\mu(Q_\tau^{k',\nu'})\big)^{\frac{p_2}{p_1}-1} \le C2^{-k'\omega(\frac{p_2}{p_1}-1)}\mu(Q_{\tau'}^{k',\nu'})$$

for any $k' \in \mathbb{N}$, $\tau' \in I_{k'}$ and thus the term above is controlled by

$$\Big\{\sum_{k\in\mathbb{N}}\Big[\sum_{k'\in\mathbb{N}}2^{(k-k')s_1p_2}2^{-|k-k'|p_2\epsilon_1}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k's_2p_2}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|^{p_2}\Big]^{q/p_2}\Big\}^{1/q}.$$

Now we choose $s_1 \in (-\epsilon_1, \epsilon_1)$; applying the Hölder inequality for $q/p_2 > 1$ and the $q/p_2$-inequality for $q/p_2 \le 1$ implies that the last term above is dominated by $C\|f\|_{B^{s_2,q}_{p_2}(X)}$, which implies (1.6) for the case where $p_1 > 1$.

*Case 2:* $p_1 \le 1$.

Using the $p_1$-inequality and the $p_2/p_1$-inequality, we have

$$\begin{aligned}
&\|f\|_{B^{s_1,q}_{p_1}(X)}\\
&\lesssim \Big\{\sum_{k\in\mathbb{N}} 2^{ks_1q}\Big[\sum_{k'\in\mathbb{N}}2^{-|k-k'|\epsilon p_1}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})^{p_1}|\lambda_{\tau'}^{k',\nu'}(f)|^{p_1}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\\
&\mu(Q_\tau^{k,\nu})\Big(\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{\epsilon}}\Big)^{p_1}\Big]^{q/p_1}\Big\}^{1/q}
\end{aligned}$$

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$$\lesssim \Big\{ \sum_{k\in\mathbb{N}} 2^{ks_1 q} \Big[ \sum_{k'\in\mathbb{N}} 2^{-|k-k'|\epsilon p_1} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})^{1-p_1} \mu(Q_{\tau'}^{k',\nu'})^{p_1} \times |\lambda_{\tau'}^{k',\nu'}(f)|^{p_1} \Big]^{q/p_1} \Big\}^{1/q}$$

$$\lesssim \Big\{ \sum_{k\in\mathbb{N}} 2^{ks_1 q} \Big[ \sum_{k'\in\mathbb{N}} 2^{-|k-k'|\epsilon p_2} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})^{p_2/p_1-1} \times V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})^{1-p_2} \mu(Q_{\tau'}^{k',\nu'})^{p_2-1} \mu(Q_{\tau'}^{k',\nu'}) |\lambda_{\tau'}^{k',\nu'}(f)|^{p_2} \Big]^{q/p_2} \Big\}^{1/q}.$$

Note that the doubling property (1.1) implies that

$$V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})^{1-p_2} \mu(Q_{\tau'}^{k',\nu'})^{p_2-1} \lesssim 2^{[k'-(k\wedge k')]\omega(1-p_2)}$$

for $p_2 \le 1$, and the density condition (1.5) implies that

$$V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})^{p_2/p_1-1} \lesssim 2^{-(k\wedge k')\omega p_2(\frac{1}{p_1}-\frac{1}{p_2})} \lesssim 2^{-(k\wedge k')p_2(s_1-s_2)}$$

for $p_2/p_1 < 1$. Therefore, we further obtain

$$\|f\|_{B_{p_1}^{s_1,q}(X)} \lesssim \Big\{ \sum_{k\in\mathbb{N}} \Big[ \sum_{k'\in\mathbb{N}} 2^{-|k-k'|\epsilon p_2} 2^{ks_1p_2} 2^{-k's_2p_2} 2^{[k'-(k\wedge k')]\omega(1-p_2)} 2^{-(k\wedge k')p_2(s_1-s_2)} \times \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k's_2p_2} \mu(Q_{\tau'}^{k',\nu'}) |\lambda_{\tau'}^{k',\nu'}(f)|^{p_2} \Big]^{q/p_2} \Big\}^{1/q}.$$

Applying the Hölder inequality for $q/p_2 > 1$ and the $q/p_2$-inequality for $q/p_2 \le 1$ implies that the last term above is dominated by $C\|f\|_{\dot{B}_{p_2}^{s_2,q}(X)}$ when $p_1 \le 1$. This completes the proof of (1.6).

To show (1.7), we may suppose $\|f\|_{F_{p_2}^{s_2,q_2}(X)} = 1$ without loss of generality. By the inhomogeneous discrete Calderón reproducing formula (2.2) in Remark 2.2, the norm of $F_p^{s,q}(X)$ given in Remark 1.6 and (2.5), we see that

$$\Big\{ \sum_{k=0}^{N} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}} \Big\}^{1/q_1}$$

$$\lesssim \Big\{ \sum_{k=0}^{N} 2^{ks_1q_1} \Big( \sum_{k'\in\mathbb{N}} 2^{-|k-k'|\epsilon} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |\lambda_{\tau'}^{k',\nu'}(f)| \times \frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu}) + V(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})} \frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')} + d(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^\epsilon} \chi_{Q_\tau^{k,\nu}} \Big)^{q_1} \Big\}^{1/q_1}.$$

We claim that for $\max\{\frac{\omega}{\omega+\epsilon}, \frac{\omega}{\omega+\epsilon+s_2}\} < r \le 1$,

$$\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |\lambda_{\tau'}^{k',\nu'}(f)| \times \frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu}) + V(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})} \frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')} + d(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^\epsilon} \chi_{Q_\tau^{k,\nu}}(x)$$
$$\le C 2^{-k'\omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r}-1} \inf_{y\in B} \Big\{ M\Big( \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} |\lambda_{\tau'}^{k',\nu'}(f)| \chi_{Q_{\tau'}^{k',\nu'}} \Big)^r(y) \Big\}^{\frac{1}{r}}, \tag{2.6}$$

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where $B = B(x, 2^{-(k\wedge k')})$ and $M$ is the Hardy–Littlewood maximal function. To prove (2.6), it follows that

$$\begin{aligned}
&\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|\\
&\times\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^\epsilon}\chi_{Q_\tau^{k,\nu}}(x)\\
&\lesssim\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y_{\tau'}^{k',\nu'})}\\
&\qquad\times\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y_{\tau'}^{k',\nu'}))^\epsilon}\\
&\lesssim\Big\{\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})^r|\lambda_{\tau'}^{k',\nu'}(f)|^r\\
&\qquad\times\Big[\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y_{\tau'}^{k',\nu'}))^\epsilon}\Big]^r\Big\}^{\frac{1}{r}}\\
&=\Big\{\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})^{r-1}|\lambda_{\tau'}^{k',\nu'}(f)|^r\\
&\quad\times\int_X\Big[\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y_{\tau'}^{k',\nu'}))^\epsilon}\Big]^r\chi_{Q_{\tau'}^{k',\nu'}}(y)\,d\mu(y)\Big\}^{\frac{1}{r}}.
\end{aligned}$$

Note that the density condition (1.5) implies $C2^{-k'\omega}\le\mu(Q_{\tau'}^{k',\nu'})$. The fact $r-1\le 0$ yields that the last term above is dominated by

$$\begin{aligned}
&2^{-k'\omega(1-\frac{1}{r})}\Big\{\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\int_B|\lambda_{\tau'}^{k',\nu'}(f)|^r\\
&\quad\times\Big[\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y)}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y))^\epsilon}\Big]^r\chi_{Q_{\tau'}^{k',\nu'}}(y)\,d\mu(y)\Big\}^{\frac{1}{r}}\\
&+2^{-k'\omega(1-\frac{1}{r})}\Big\{\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|\lambda_{\tau'}^{k',\nu'}(f)|^r\\
&\quad\times\sum_{m=0}^{\infty}\int_{[2^{(m+1)}B]\setminus[2^mB]}\Big[\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y)}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y))^\epsilon}\Big]^r\\
&\qquad\times\chi_{Q_{\tau'}^{k',\nu'}}(y)\,d\mu(y)\Big\}^{\frac{1}{r}}\\
&=:F+G.
\end{aligned}$$

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We first estimate the term $F$. For any $x' \in B$, we have

$$F \lesssim 2^{-k'\omega(1-\frac{1}{r})}\Big\{\mu(B)^{1-r}\frac{1}{\mu(B)}\int_B \sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|\lambda_{\tau'}^{k',\nu'}(f)|^r\chi_{Q_{\tau'}^{k',\nu'}}(y)d\mu(y)\Big\}^{\frac{1}{r}}$$
$$\lesssim 2^{-k'\omega(1-\frac{1}{r})}\mu(B)^{\frac{1}{r}-1}\Big\{M\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|\lambda_{\tau'}^{k',\nu'}(f)|^r\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r(x')\Big\}^{\frac{1}{r}}.$$

Now we estimate the term $G$. For any $x' \in B$, it follows that

$$\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\sum_{m=0}^{\infty}\int_{[2^{(m+1)}B]\setminus[2^m B]}|\lambda_{\tau'}^{k',\nu'}(f)|^r$$
$$\times\Big[\frac{1}{V_{2^{-(k\wedge k')}}(x)+V(x,y)}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y))^{\epsilon}}\Big]^r\chi_{Q_{\tau'}^{k',\nu'}}(y)d\mu(y)$$
$$\lesssim \mu(B)^{1-r}2^{-m[\epsilon r-\omega(1-r)]}M\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|\lambda_{\tau'}^{k',\nu'}(f)|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r(x').$$

Thus, let $r > \frac{\omega}{\omega+\gamma}$; the above estimate implies that

$$G \le C2^{-k'\omega(1-\frac{1}{r})}\mu(B)^{\frac{1}{r}-1}\Big\{M\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|\lambda_{\tau'}^{k',\nu'}(f)|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r(x')\Big\}^{\frac{1}{r}}.$$

Combining estimates of $F$, $G$ and arbitrariness of $x' \in B$ completes the proof of claim (2.6).

The estimates in (2.3) and (2.6) yield

$$\sum_{k'\in\mathbb{N}}2^{-|k-k'|\epsilon}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\lambda_{\tau'}^{k',\nu'}(f)|$$
$$\times\frac{1}{V_{2^{-(k\wedge k')}}(y_\tau^{k,\nu})+V(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{\epsilon}}\chi_{Q_\tau^{k,\nu}}$$
$$\lesssim\sum_{k'\in\mathbb{N}}2^{-k's_2-|k-k'|\epsilon-k'\omega(1-\frac{1}{r})}\mu(B)^{\frac{1}{r}-1}$$
$$\inf_{y\in B}\Big\{M\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k's_2}|\lambda_{\tau'}^{k',\nu'}(f)|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r(y)\Big\}^{\frac{1}{r}}.$$

for $\frac{\omega}{\omega+\gamma} < r \le 1$. Choose $r$ satisfying $\frac{\omega}{\omega+\gamma} < r < \max\{p_2, q_2\}$ and $r \le 1$, and denote

$$F_{k'}(y) = \Big\{M\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k's_2}|\lambda_{\tau'}^{k',\nu'}(f)|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r(y)\Big\}^{\frac{1}{r}}.$$

Applying the Fefferman–Stein vector-valued maximal function inequality [FS] and the arbitrariness of $y_{\tau'}^{k',\nu'}$ further yields

$$\begin{aligned}\inf_{y\in B} F_{k'}(y) &\le \inf_{y\in B}\Big\{\sum_{k'\in\mathbb{N}}(F_{k'}(y))^{q_2}\Big\}^{1/q_2}\\ &\le \Big\{\mu(B)^{-1}\int_B\Big(\sum_{k'\in\mathbb{N}}(F_{k'}(x))^{q_2}\Big)^{p_2/q_2}d\mu(x)\Big\}^{1/p_2}\\ &\le C\mu(B)^{-1/p_2}\Big\|\Big(\sum_{k'\in\mathbb{N}}(F_{k'}(y))^{q_2}\Big)^{1/q_2}\Big\|_{p_2}\\ &\le C\mu(B)^{-1/p_2}\Big\|\Big\{\sum_{k'\in\mathbb{N}}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k's_2}|\lambda_{\tau'}^{k',\nu'}(f)|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^{q_2}\Big\}^{1/q_2}\Big\|_{p_2}\\ &\le C\mu(B)^{-1/p_2},\end{aligned}$$

which implies that

$$\Big\{\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}\Big\}^{1/q_1}\le C\Big\{\sum_{k=1}^{N}\Big(\sum_{k'\in\mathbb{N}}2^{ks_1-k's_2-|k-k'|\epsilon-k'\omega(1-\frac{1}{r})}\mu(B)^{\frac{1}{r}-1}\mu(B)^{-1/p_2}\Big)^{q_1}\Big\}^{1/q_1}.$$

Since $s_2-s_1=\omega/p_2-\omega/p_1>0$, so $p_1>p_2>p_2/(1+p_2)$. The key point here is to choose $r=p_2/(1+p_2)$ so that $\mu(B)^{\frac{1}{r}-1}\mu(B)^{-1/p_2}=1$. Note that $r<p_2$ and $r<1$. We also assume $q_2>r=p_2/(1+p_2)$ for the moment, but this assumption will be removed at the end of the proof. We obtain

$$\begin{aligned}\Big\{\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}\Big\}^{1/q_1} &\lesssim\Big\{\sum_{k=0}^{N}\Big(\sum_{k'\in\mathbb{Z}_+}2^{ks_1-k's_2-|k-k'|\epsilon-k'\omega(1-\frac{1}{r})}\Big)^{q_1}\Big\}^{1/q_1}\\ &\lesssim\Big\{\sum_{k=0}^{N}\Big(\sum_{k'\in\mathbb{Z}_+}2^{-|k-k'|\epsilon}2^{k'\frac{\omega}{p_1}}2^{(k-k')s_1}\Big)^{q_1}\Big\}^{1/q_1}\le C_4 2^{\frac{N\omega}{p_1}+\frac{1}{q_1}},\end{aligned}\tag{2.7}$$

where $\epsilon$ can be taken arbitrarily close to $\theta$ in RD-space and satisfying $-\epsilon<s_1-\omega/p_1<\epsilon$.

If $q_2/q_1\le 1$, since $s_2-s_1=\omega/p_2-\omega/p_1>0$, applying the $q_2/q_1$-inequality, we have

$$\begin{aligned}\Big\{\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_1} &\lesssim\Big\{\sum_{k=N+1}^{\infty}2^{k(s_1-s_2)q_2}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ks_2q_2}|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2}\\ &\lesssim 2^{N(\frac{\omega}{p_1}-\frac{\omega}{p_2})}\Big\{\sum_{k\in\mathbb{Z}_+}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ks_2q_2}|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2}.\end{aligned}\tag{2.8}$$

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If $\delta := \frac{q_2}{q_1} > 1$, applying the Hölder inequality yields

$$(2.9)\quad \Big\{ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_1}$$
$$\lesssim \Big\{ \Big[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \Big]^{1/\delta'}$$
$$\Big[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \Big\{ \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{\delta} \Big]^{1/\delta} \Big\}^{1/q_1}$$
$$\lesssim \Big\{ \Big[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \Big]^{1/\delta'}$$
$$\Big[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2} |\lambda_\tau^{k,\nu}(f)|^{q_2} \chi_{Q_\tau^{k,\nu}}(x) \Big]^{1/\delta} \Big\}^{1/q_1}$$
$$\lesssim 2^{N(\frac{\omega}{p_1}-\frac{\omega}{p_2})} \Big\{ \sum_{k \in \mathbb{Z}_+} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2} |\lambda_\tau^{k,\nu}(f)|^{q_2} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_2}.$$

Combining (2.8) and (2.9), if $q_2 > \frac{p_2}{1+p_2}$, for $N \in \mathbb{N} \cup \{-1\}$, we obtain

$$(2.10)\quad \Big\{ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_1} \le$$
$$C_4 2^{N(\frac{\omega}{p_1}-\frac{\omega}{p_2})} \Big\{ \sum_{k \in \mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2} |\lambda_\tau^{k,\nu}(f)|^{q_2} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_2}.$$

Now we deduce the estimate of the norm $\|f\|_{F^{s_1,q_1}_{p_1}(X)}$. We write

$$\|f\|_{F^{s_1,q_1}_{p_1}(X)}$$
$$= p_1 \int_0^\infty t^{p_1-1} \mu\Big\{ x : \Big\{ \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_1} > t \Big\} dt$$
$$= p_1 \int_0^{2C_4 2^{1/q_1}} t^{p_1-1} \mu\Big\{ x : \Big\{ \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_1} > t \Big\} dt$$
$$+ p_1 \int_{2C_4 2^{1/q_1}}^\infty t^{p_1-1} \mu\Big\{ x : \Big\{ \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_1} > t \Big\} dt$$
$$:= H + I.$$

To estimate $H$, using (2.10) for $N = -1$, if $q_2 > \frac{p_2}{1+p_2}$ and $t \le C$, we also get

$$H \lesssim p_1 \int_0^{2C_4 2^{1/q_1}} t^{p_1-1} \mu\Big\{ x : \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1} |\lambda_\tau^{k,\nu}(f)|^{q_1} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_1} > t \Big\} dt$$
$$\lesssim p_1 \int_0^{2C_4 2^{1/q_1}} t^{p_1-1} \mu\Big\{ x : \Big\{ \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2} |\lambda_\tau^{k,\nu}(f)|^{q_2} \chi_{Q_\tau^{k,\nu}}(x) \Big\}^{1/q_2} > t \Big\} dt$$

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$$\le \frac{p_1}{p_2}(2C_4 2^{1/q_1})^{p_1-p_2} p_2 \int_0^{2C_4 2^{1/q_1}} t^{p_2-1}$$
$$\times \mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2}|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2} > t\Big\}\,dt$$
$$\lesssim p_2\int_0^{2C_4 2^{1/q_1}} t^{p_2-1}\mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2}|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2} > t\Big\}\,dt$$
$$\le C\|f\|^{p_2}_{\dot F^{s_2,q_2}_{p_2}(X)}.$$

To estimate $I$, applying (2.7) and (2.10), if $q_2 > \frac{p_2}{1+p_2}$, we get

$$I = p_1\sum_{N=0}^{\infty}\int_{2C_4 2^{\omega N/p_1+1/q_1}}^{2C_4 2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1}$$
$$\times \mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_1} > t\Big\}\,dt$$
$$\le p_1\sum_{N=0}^{\infty}\int_{2C_4 2^{\omega N/p_1+1/q_1}}^{2C_4 2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1}\mu\Big\{x : \Big\{\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_1}$$
$$+\Big\{\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_1} > 2^{-1/q_1}t\Big\}\,dt$$
$$\lesssim p_1\sum_{N=0}^{\infty}\int_{2C_4 2^{\omega N/p_1+1/q_1}}^{2C_4 2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1}$$
$$\times \mu\Big\{x : \Big\{\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_1q_1}|\lambda_\tau^{k,\nu}(f)|^{q_1}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_1} > 2^{-1/q_1}t/2\Big\}\,dt$$
$$\le p_1\sum_{N=0}^{\infty}\int_{2C_4 2^{\omega N/p_1+1/q_1}}^{2C_4 2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1}\mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2}$$
$$\times|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2} > C_4 2^{-N(\frac{\omega}{p_1}-\frac{\omega}{p_2})}2^{-1/q_1}t/2\Big\}\,dt$$
$$\le p_1\sum_{N=0}^{\infty}\int_{2C_4 2^{\omega N/p_1+1/q_1}}^{2C_4 2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1}$$
$$\times \mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2}|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2} > C_5 t^{p_1/p_2}\Big\}\,dt$$
$$\le p_1\int_0^{\infty} t^{p_1-1}\mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2}\Big(|\lambda_\tau^{k,\nu}(f)|\chi_{Q_\tau^{k,\nu}}(x)\Big)^{q_2}\Big\}^{1/q_2}$$
$$> C_5 t^{p_1/p_2}\Big\}\,dt$$
$$\le p_2\int_0^{\infty} u^{p_2-1}\mu\Big\{x : \Big\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ks_2q_2}|\lambda_\tau^{k,\nu}(f)|^{q_2}\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/q_2} > C_5 u\Big\}\,du$$
$$\lesssim C\|f\|^{p_2}_{\dot F^{s_2,q_2}_{p_2}(X)},$$

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where we used $t \approx C_4 2^{\omega N/p_1+1/q_1}$. This prove (1.7) *i.e.*, $F^{s_2,q_2}_{p_2} \hookrightarrow F^{s_1,q_1}_{p_1}$ with $q_2 > p_2/(1+p_2)$. To remove this assumption for $q_2$, note that for any $q$,

$$\max\Big\{\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s_2}\Big\} < q \le \frac{p_2}{1+p_2},$$

it is easy to see that $F^{s_2,q}_{p_2} \hookrightarrow F^{s_2,q_2}_{p_2}$ for $q_2 > p_2/(1+p_2)$. The proof of Theorem 1.7 is completed. ■

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