

Building a Stationary Stochastic Process From a Finite-Dimensional Marginal

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Abstract. If \mathfrak{A} is a finite alphabet, $\mathcal{U} \subset \mathbb{Z}^D$, and $\mu_{\mathcal{U}}$ is a probability measure on $\mathfrak{A}^{\mathcal{U}}$ that “looks like” the marginal projection of a stationary stochastic process on $\mathfrak{A}^{\mathbb{Z}^D}$, then can we “extend” $\mu_{\mathcal{U}}$ to such a process? Under what conditions can we make this extension ergodic, (quasi)periodic, or (weakly) mixing? After surveying classical work on this problem when $D = 1$, we provide some sufficient conditions and some necessary conditions for $\mu_{\mathcal{U}}$ to be extendible for $D > 1$, and show that, in general, the problem is not formally decidable.

1 Introduction

1.1 The Markov Extension in \mathbb{Z}

Let \mathfrak{A} be a finite alphabet, and let $\mathfrak{A}^{\mathbb{Z}}$ be the space of *bi-infinite sequences* on \mathfrak{A} . A *stationary stochastic process* is a probability measure μ on $\mathfrak{A}^{\mathbb{Z}}$ so that, for any $V \in \mathbb{N}$, $b_0, b_1, \dots, b_V \in \mathfrak{A}$, and any $k \in \mathbb{Z}$

$$\mu\{\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}}; a_0 = b_0, \dots, a_V = b_V\} = \mu\{\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}}; a_k = b_0, \dots, a_{k+V} = b_V\}.$$

Let \mathcal{U} be the interval $[0 \dots U] \subset \mathbb{Z}$. The *projection map* $\text{pr}_{\mathcal{U}}: \mathfrak{A}^{\mathbb{Z}} \rightarrow \mathfrak{A}^{\mathcal{U}}$ is the map sending the sequence $[a_n]_{n \in \mathbb{Z}}$ to the sequence $[a_n]_{n \in \mathcal{U}}$. With this map, we can project μ down to a *marginal* measure, $\mu_{\mathcal{U}} := \text{pr}_{\mathcal{U}}^*[\mu]$, on the space $\mathfrak{A}^{\mathcal{U}}$. This marginal is then *locally stationary*: for any $V < U$, any $b_0, b_1, \dots, b_V \in \mathfrak{A}$, and any $k \in \mathbb{Z}$ so that $V + k \leq U$ also,

$$\mu_{\mathcal{U}}\{\mathbf{a} \in \mathfrak{A}^{\mathcal{U}}; a_0 = b_0, \dots, a_V = b_V\} = \mu_{\mathcal{U}}\{\mathbf{a} \in \mathfrak{A}^{\mathcal{U}}; a_k = b_0, \dots, a_{k+V} = b_V\}.$$

Can we *reverse* this process? Given a locally stationary measure $\mu_{\mathcal{U}}$ upon $\mathfrak{A}^{\mathcal{U}}$, can we *extend* it to a stationary stochastic process μ on $\mathfrak{A}^{\mathbb{Z}}$, so that $\text{pr}_{\mathcal{U}}^*[\mu] = \mu_{\mathcal{U}}$? Yes, and furthermore, we can do so in a canonical fashion, via the so-called *Markov Extension*.

An intuitive description of the Markov Extension is this: We randomly “choose” the coordinates a_0, \dots, a_U according to the probability measure $\mu_{\mathcal{U}}$. We then randomly chose the coordinate a_{U+1} , again according to $\mu_{\mathcal{U}}$ (now treated as a probability measure on $\mathfrak{A}^{\mathcal{U}+1}$), but *conditioned* upon the fact that we have already fixed coordinates a_1, \dots, a_U . Next, we randomly chose the coordinate a_{U+2} , again according to $\mu_{\mathcal{U}}$ (now treated as a probability measure on $\mathfrak{A}^{\mathcal{U}+2}$), but *conditioned* upon the fact that we have already fixed coordinates a_2, \dots, a_{U+1} . Inductively, we get a U -step Markov process on \mathfrak{A} .

To formally construct the Markov Extension, we need a bit of notation:

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- If $\mathbf{a} = [a_n]_{n \in \mathbb{Z}}$ is an element of $\mathfrak{A}^{\mathbb{Z}}$, and $\mathcal{V} \subset \mathbb{Z}$, then let $\mathbf{a}_{\mathcal{V}} := [a_v]_{v \in \mathcal{V}}$
- If μ is a measure upon $\mathfrak{A}^{\mathbb{Z}}$, $\mathcal{V} \subset \mathbb{Z}$, and $\mathbf{b} \in \mathfrak{A}^{\mathcal{V}}$, then let “ $\mu[\mathbf{b}]$ ” denote the measure of the associated cylinder set:

$$\mu[\mathbf{b}] := \mu\{\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}}; \mathbf{a}_{\mathcal{V}} = \mathbf{b}\}.$$

- Suppose $\mathcal{V} \subset \mathbb{Z}$ and $k \in \mathbb{Z}$ are such that $(k + \mathcal{V}) \subset \mathcal{U}$. If $\mathbf{b} := [b_v]_{v \in \mathcal{V}}$ is an element of $\mathfrak{A}^{\mathcal{V}}$, then let \mathbf{b}' be the “shift” of \mathbf{b} by k : that is, $\mathbf{b}' := [b'_v]_{v \in (k + \mathcal{V})}$, where, for all $v \in \mathcal{V}$, $b'_v = b_{v-k}$. Then define:

$$\mu_{\mathcal{U}}[\mathbf{b}] := \mu_{\mathcal{U}}\{\mathbf{a} \in \mathfrak{A}^{\mathcal{U}}; \mathbf{a}_{(k + \mathcal{V})} = \mathbf{b}'\}.$$

(because $\mu_{\mathcal{U}}$ is locally stationary, it doesn’t matter which k we use in this definition, if more than one k is available)

The *Markov Extension* of $\mu_{\mathcal{U}}$ is the probability measure μ_{mrk} , where, for any $N \geq U$, and $\mathbf{b} \in \mathfrak{A}^{\{0 \dots N\}}$,

$$\mu_{\text{mrk}}[\mathbf{b}] := \mu_{\mathcal{U}}[\mathbf{b}_{\mathcal{U}}] \cdot \prod_{k=1}^{N-U} \mu_{\mathcal{U}} \left[\frac{\mathbf{b}_{[k \dots U+k]}}{\mathbf{b}_{[k \dots U+k]}} \right]$$

Here, $[k \dots U + k) := \{k, k + 1, \dots, k + U - 1\}$, while $[k \dots U + k] := \{k, k + 1, \dots, k + U\}$, and $\mu_{\mathcal{U}} \left[\frac{\mathbf{b}_{[k \dots U+k]}}{\mathbf{b}_{[k \dots U+k]}} \right]$ is the *conditional probability*:

$$\mu_{\mathcal{U}} \left[\frac{\mathbf{b}_{[k \dots U+k]}}{\mathbf{b}_{[k \dots U+k]}} \right] := \frac{\mu_{\mathcal{U}}[\mathbf{b}_{[k \dots U+k]}]}{\mu_{\mathcal{U}}[\mathbf{b}_{[k \dots U+k]}]}$$

μ_{mrk} is a stationary probability measure on $\mathfrak{A}^{\mathbb{N}}$. Define the probabilities of cylinder sets indexed by negative coordinates by simply *shifting* them into the positive domain. Thus, μ_{mrk} is defined on all cylinder sets in $\mathfrak{A}^{\mathbb{Z}}$. It is straightforward to check that the probability measure thus defined is stationary, and that its marginal projection upon $\mathfrak{A}^{\mathcal{U}}$ is equal to $\mu_{\mathcal{U}}$.

This construction indicates that a stationary extension of the measure $\mu_{\mathcal{U}}$ always exists. In general, there may be many such extensions. Intuitively, μ_{mrk} is an extension built so as to provide the maximum amount of “random choice” at each successive coordinate. Hence, the following variational principle is not too surprising:

Theorem (Maximal Entropy Property) *Of all the different stationary extensions of $\mu_{\mathcal{U}}$ that exist, μ_{mrk} is the one possessing the largest process entropy, which we define as:*

$$H(\mu_{\text{mrk}}) := \lim_{N \rightarrow \infty} \frac{-1}{N} \sum_{\mathbf{a} \in \mathfrak{A}^{\{1 \dots N\}}} \mu_{\text{mrk}}[\mathbf{a}] \log_2(\mu_{\text{mrk}}[\mathbf{a}]).$$

Proof See, for example, [21]. ■

Under what circumstances do *ergodic* extensions of $\mu_{\mathcal{U}}$ exist? Can we build an extension measure which is supported only on *periodic* words of some fixed periodicity? Also, what happens if \mathcal{U} is *not* just an interval inside \mathbb{Z} ?

1.2 Extension on Lattices

Now, let $D > 0$, and let \mathbb{Z}^D be a D -dimensional lattice. Then $\mathfrak{A}^{\mathbb{Z}^D}$ is the space of D -dimensional configurations on \mathfrak{A} . If $\mathbf{k} \in \mathbb{Z}^D$, then the *shift by \mathbf{k}* is the map $\text{Shift}^{\mathbf{k}}: \mathfrak{A}^{\mathbb{Z}^D} \rightarrow \mathfrak{A}^{\mathbb{Z}^D}$ so that, if $\mathbf{a} := [a_{\mathbf{n}}]_{\mathbf{n} \in \mathbb{Z}^D}$, then $\text{Shift}^{\mathbf{k}} \mathbf{a} := [a'_{\mathbf{n}}]_{\mathbf{n} \in \mathbb{Z}^D}$, where $a'_{\mathbf{n}} = a_{\mathbf{n}-\mathbf{k}}, \forall \mathbf{n} \in \mathbb{Z}^D$.

A *stationary stochastic process* is a probability measure μ on $\mathfrak{A}^{\mathbb{Z}^D}$ that is *invariant* under all shift maps. That is, if $\mathcal{V} \subset \mathbb{Z}^D$ is any finite subset, and $\mathbf{b} \in \mathfrak{A}^{\mathcal{V}}$, then for any $\mathbf{k} \in \mathbb{Z}^D$,

$$\mu[\text{Shift}^{\mathbf{k}}(\mathbf{b})] = \mu[\mathbf{b}].$$

If $\mathcal{U} \subset \mathbb{Z}^D$, and $\mathbf{k} \in \mathbb{Z}^D$, then define $\text{Shift}^{\mathbf{k}} \mathcal{U} = \mathcal{U} + \mathbf{k}$, and define $\text{Shift}^{\mathbf{k}}: \mathfrak{A}^{\mathcal{U}} \rightarrow \mathfrak{A}^{\mathcal{U}+\mathbf{k}}$ so that, if $\mathbf{a} := [a_{\mathbf{n}}]_{\mathbf{n} \in \mathcal{U}}$, then $\text{Shift}^{\mathbf{k}} \mathbf{a} := [a'_{\mathbf{n}}]_{\mathbf{n} \in \mathcal{U}+\mathbf{k}}$, where $a'_{\mathbf{n}} = a_{\mathbf{n}-\mathbf{k}}, \forall \mathbf{n} \in \mathcal{U} + \mathbf{k}$. A probability measure $\mu_{\mathcal{U}}$ on $\mathfrak{A}^{\mathcal{U}}$ is *locally stationary* if for any $\mathcal{V} \subset \mathcal{U}$, any $\mathbf{b} \in \mathfrak{A}^{\mathcal{V}}$, and any $\mathbf{k} \in \mathbb{Z}^D$ so that $\text{Shift}^{\mathbf{k}} \mathcal{V} \subset \mathcal{U}$ also,

$$\mu_{\mathcal{U}}[\text{Shift}^{\mathbf{k}}(\mathbf{b})] = \mu_{\mathcal{U}}[\mathbf{b}].$$

The Extension Problem Given a locally stationary measure $\mu_{\mathcal{U}}$ upon $\mathfrak{A}^{\mathcal{U}}$, can we extend it to a stationary stochastic process μ on $\mathfrak{A}^{\mathbb{Z}^D}$, so that $\text{pr}_{\mathcal{U}}^*[\mu] = \mu_{\mathcal{U}}$?

The Extension Problem does not always have solutions, as examples in Section 3 will show. If we can solve the Extension Problem, can we construct an extension which is ergodic? (quasi) Periodic or (weakly) mixing?

1.3 Extension on Group Modules

Now, let \mathbb{G} be an arbitrary group, and let \mathcal{M} be a \mathbb{G} -module: an arbitrary set equipped with a \mathbb{G} -action. A few examples of this to keep in mind:

- $\mathcal{M} := \mathbb{Z}^D$ and $\mathbb{G} := \mathbb{Z}^D$, also, acting upon \mathcal{M} by *translation*.
- $\mathcal{M} := (\mathbb{Z}/P_1) \oplus (\mathbb{Z}/P_2) \oplus \dots \oplus (\mathbb{Z}/P_D)$, and $\mathbb{G} := \mathbb{Z}^D$ acts upon \mathcal{M} by *translation* with periodic boundary conditions.
- \mathbb{G} is an arbitrary group, \mathbb{H} an arbitrary subgroup, and $\mathcal{M} := \mathbb{G}/\mathbb{H}$ is the set of *right cosets*. \mathbb{G} acts upon \mathcal{M} by multiplication: if $\mathbf{g} \in \mathbb{G}$ and $(\mathbf{k}\mathbb{H}) \in \mathcal{M}$, then $\mathbf{g} \cdot (\mathbf{k}\mathbb{H}) := (\mathbf{g} \cdot \mathbf{k})\mathbb{H}$. (Every *transitive* \mathbb{G} -module is of this type, and every \mathbb{G} -module can be written as a disjoint union of transitive \mathbb{G} -modules.)
- Let $\mathfrak{A}^{\mathcal{M}}$ be the space of \mathcal{M} -indexed configurations on \mathfrak{A} . If $\mathbf{g} \in \mathbb{G}$ then the *shift by \mathbf{g}* is the map $\text{Shift}^{\mathbf{g}}: \mathfrak{A}^{\mathcal{M}} \rightarrow \mathfrak{A}^{\mathcal{M}}$ so that, if $\mathbf{a} := [a_m]_{m \in \mathcal{M}}$, then $\text{Shift}^{\mathbf{g}} \mathbf{a} := [a'_m]_{m \in \mathcal{M}}$, where $a'_m = a_{\mathbf{g}^{-1} \cdot m}, \forall m \in \mathcal{M}$.

A \mathbb{G} -invariant stochastic process is a probability measure μ on $\mathfrak{A}^{\mathcal{M}}$ that is *invariant* under the shift action of \mathbb{G} . That is, if $\mathcal{V} \subset \mathcal{M}$ is any finite subset, and $\mathbf{b} \in \mathfrak{A}^{\mathcal{V}}$, then for any $\mathbf{g} \in \mathbb{G}$,

$$\mu[\text{Shift}^{\mathbf{g}}(\mathbf{b})] = \mu[\mathbf{b}].$$

If $\mathcal{U} \subset \mathcal{M}$ and $\mathbf{g} \in \mathbb{G}$, then define $\text{Shift}^{\mathbf{g}} \mathcal{U} = \mathbf{g} \cdot \mathcal{U} = \{\mathbf{g} \cdot u ; u \in \mathcal{U}\}$, and define $\text{Shift}^{\mathbf{g}}: \mathfrak{A}^{\mathcal{U}} \rightarrow \mathfrak{A}^{\mathbf{g} \cdot \mathcal{U}}$ so that, if $\mathbf{a} := [a_u]_{u \in \mathcal{U}}$, then $\text{Shift}^{\mathbf{g}} \mathbf{a} := [a'_u]_{u \in \mathbf{g} \cdot \mathcal{U}}$, where $a'_u = a_{\mathbf{g}^{-1} \cdot u}$, $\forall u \in \mathbf{g} \cdot \mathcal{U}$. A probability measure $\mu_{\mathcal{U}}$ on $\mathfrak{A}^{\mathcal{U}}$ is *locally stationary* if for any \mathcal{V} subset \mathcal{U} , any $\mathbf{b} \in \mathfrak{A}^{\mathcal{V}}$, and any $\mathbf{g} \in \mathbb{G}$ so that $\text{Shift}^{\mathbf{g}} \mathcal{V} \subset \mathcal{U}$ also,

$$\mu_{\mathcal{U}}[\text{Shift}^{\mathbf{g}}(\mathbf{b})] = \mu_{\mathcal{U}}[\mathbf{b}].$$

Again, we ask:

The (Group Module) Extension Problem Given a locally stationary measure $\mu_{\mathcal{U}}$ upon $\mathfrak{A}^{\mathcal{U}}$, can we *extend* it to a stationary stochastic process μ on $\mathfrak{A}^{\mathcal{M}}$, so that $\text{pr}_{\mathcal{U}}^*[\mu] = \mu_{\mathcal{U}}$?

If $\mathcal{M} = \mathbb{Z}^D = \mathbb{G}$, then this is just the Extension Problem on a D -dimensional lattice. If $\mathcal{M} := (\mathbb{Z}/P_1) \oplus (\mathbb{Z}/P_2) \oplus \dots \oplus (\mathbb{Z}/P_D)$ and $\mathbb{G} := \mathbb{Z}^D$, then a \mathbb{G} -invariant measure on $\mathfrak{A}^{\mathcal{M}}$ is “equivalent” to a stationary stochastic process on $\mathfrak{A}^{\mathbb{Z}^D}$ which is supported only on *periodic* configurations with fundamental domain $[0 \dots P_1) \times [0 \dots P_2) \times \dots \times [0 \dots P_D)$. In Section 6, we will demonstrate that, if $\mathcal{U} \subset [0 \dots P_1) \times [0 \dots P_2) \times \dots \times [0 \dots P_D) \subset \mathbb{Z}^D$ is some “small enough” domain, then any locally stationary measure $\mu_{\mathcal{U}}$ can be identified with a locally invariant measure $\mu_{\mathcal{U}'}$, where $\mathcal{U}' \subset \mathcal{M}$ is the obvious “representation” of \mathcal{U} inside \mathcal{M} .

1.4 Organization of this Paper

In Section 2, we motivate the Extension Problem by discussing applications to the *Invariant Measure Problem* for subshifts of finite type and cellular automata. In Section 3, we show that the Extension Problem is not trivial by providing examples of locally stationary measures which *cannot* be extended. These examples imply two necessary conditions for extendability: the *Entropy Condition* and the *Tiling Condition*.

In Section 4, we review basic harmonic analysis on configuration space, treating it as a compact abelian group, and characterise the Extension Problem in terms of constructing a suitable set of Fourier coefficients. We use this in Section 5, where we consider extension on *finite* \mathbb{G} -modules, and show that, if ν is an extendible measure with full support, and μ is “close enough” to ν , then μ is also extendible. A similar result can be developed for constructing *periodic* extensions, but first we need a tool to “reduce” the Extension Problem on an infinite module to an extension problem on a suitably chosen finite module, which we develop in Section 6, via the concept of “envelopes”.

In Section 7, we show that an extendible, locally stationary measure with full support can be “embedded” in any ergodic \mathbb{Z}^D -dynamical system, in the sense that it is a marginal projection of a stationary \mathbb{Z}^D -process generated by a partition on that system.

In Section 8, we combine the results of Section 5 and Section 6 to investigate when a measure has an almost-surely periodic extension, and provide examples of measures which *never* have periodic extensions, as well as measures which *only* have periodic extensions. Then we use the results of Section 7 to show that “almost all”

extendible measures have extensions which are ergodic, mixing, weakly mixing, or quasiperiodic.

In Section 9, we show that the Extension Problem is, in general, formally undecidable.

1.5 Preliminaries and Notation

If we treat \mathfrak{A} as a *discrete* topological space, and endow $\mathfrak{A}^{\mathcal{M}}$ with the Tychonoff product topology, then $\mathfrak{A}^{\mathcal{M}}$ is a compact, metrizable space. If \mathcal{M} is finite, then $\mathfrak{A}^{\mathcal{M}}$ is finite and discrete. If \mathcal{M} is infinite, then $\mathfrak{A}^{\mathcal{M}}$ is uncountable and *totally disconnected*.

The topology on $\mathfrak{A}^{\mathcal{M}}$ is generated by *cylinder sets*. If $\mathcal{U} \subset \mathcal{M}$ is finite, and $\mathbf{b} \in \mathfrak{A}^{\mathcal{U}}$, then the associated cylinder set is:

$$\{\mathbf{a} \in \mathfrak{A}^{\mathcal{M}} ; \mathbf{a}_{\mathcal{U}} = \mathbf{b}\}.$$

Here, by “ $\mathbf{a}_{\mathcal{U}}$ ” we mean the element $[a_u]_{u \in \mathcal{U}}$, where $\mathbf{a} = [a_m]_{m \in \mathcal{M}}$. Normally, we will use the symbol “ \mathbf{b} ” to denote both the word \mathbf{b} and the cylinder set it induces—the distinction will be clear from context. For example, if μ is some measure, then “ $\mu[\mathbf{b}]$ ” indicates the measure of the cylinder set defined by \mathbf{b} .

Whenever we speak of measures on $\mathfrak{A}^{\mathcal{M}}$, we will mean measures on the *Borel sigma-algebra* generated by the product topology.

If \mathcal{M} is a \mathbb{G} -module, then $\text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{M}}]$ is the space of all \mathbb{G} -invariant probability measures on $\mathfrak{A}^{\mathcal{M}}$. This is a convex subset of $\text{Meas}[\mathfrak{A}^{\mathcal{M}}]$, the space of *all probability measures* on $\mathfrak{A}^{\mathcal{M}}$, which, in turn, is a convex subset of the real vector space $\text{Meas}[\mathfrak{A}^{\mathcal{M}} ; \mathbb{R}]$ of *real-valued measures* on $\mathfrak{A}^{\mathcal{M}}$.

The elements of $\text{Meas}[\mathfrak{A}^{\mathcal{M}} ; \mathbb{C}]$ (*complex-valued measures* on $\mathfrak{A}^{\mathcal{M}}$) act as linear functionals on $\mathbf{C}(\mathfrak{A}^{\mathcal{M}} ; \mathbb{C})$ (the Banach space of *complex-valued, continuous functions*). This induces a *weak-* topology* on $\text{Meas}[\mathfrak{A}^{\mathcal{M}} ; \mathbb{C}]$, making it into a locally convex topological vector space.

$\text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{M}}]$ is a compact subset of $\text{Meas}[\mathfrak{A}^{\mathcal{M}} ; \mathbb{C}]$ under this topology.

When $\mathbb{G} = \mathcal{M} = \mathbb{Z}^D$, we will refer to $\text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{M}}]$ as “ $\text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathbb{Z}^D}]$ ”.

If $\mathcal{U} \subset \mathcal{M}$, then $\text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ is the space of all locally \mathbb{G} -invariant probability measures on $\mathfrak{A}^{\mathcal{U}}$. $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ is the set of all *extendable probability measures*: measures which can be extended to a \mathbb{G} -invariant measure on $\mathfrak{A}^{\mathcal{M}}$. Notice that:

$$\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}] \text{ is a compact, convex subset of } \text{Meas}[\mathfrak{A}^{\mathcal{U}} ; \mathbb{C}].$$

This is because the marginal projection map $\text{pr}_{\mathcal{U}}^* : \text{Meas}[\mathfrak{A}^{\mathcal{M}} ; \mathbb{C}] \rightarrow \text{Meas}[\mathfrak{A}^{\mathcal{U}} ; \mathbb{C}]$ is *linear* and *continuous*, and $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ is simply the image of the compact, convex subset $\text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{M}}]$ under $\text{pr}_{\mathcal{U}}^*$.

2 Applications

2.1 Subshifts of Finite Type

Let $\mathcal{U} \subset \mathbb{Z}^D$ be finite, and suppose that $\mathfrak{B} \subset \mathfrak{A}^{\mathcal{U}}$ is some set of “admissible” \mathcal{U} -words. The *subshift of finite type* defined by \mathfrak{B} is the closed, shift-invariant subset of $\mathfrak{A}^{\mathbb{Z}^D}$:

$$\langle \mathfrak{B} \rangle := \{\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}^D} ; \forall \mathbf{n} \in \mathbb{Z}^D, \mathbf{a}_{\mathcal{U}+\mathbf{n}} \in \mathfrak{B}\}.$$

One-dimensional subshifts of finite type were first studied by Parry [17] and Smale [22]; excellent recent introductions are [11] and [10]. Higher dimensional subshifts are closely related to tilings [13], [14], [18], and involve many additional subtleties; see, for example [16], [15]. Of particular interest is

The Nontriviality Problem For a given set \mathfrak{B} , is the corresponding set $\langle \mathfrak{B} \rangle$ is even nonempty?

The Nontriviality Problem is known to be *formally undecidable*; see [19], [2], or [9].

Theorem 1 Let \mathcal{U} and \mathfrak{B} be as above. $\langle \mathfrak{B} \rangle$ is nontrivial if and only if there is some locally stationary probability measure $\mu_{\mathcal{U}}$ on $\mathfrak{A}^{\mathcal{U}}$, with $\text{supp}[\mu_{\mathcal{U}}] \subset \langle \mathfrak{B} \rangle$, such that $\mu_{\mathcal{U}}$ has a stationary extension.

Proof Suppose that such a $\mu_{\mathcal{U}}$ existed, and let μ be a stationary extension. Clearly, any μ -generic configuration in $\mathfrak{A}^{\mathbb{Z}^D}$ must satisfy the membership criteria of $\langle \mathfrak{B} \rangle$. Hence, $\langle \mathfrak{B} \rangle$ must be nonempty.

Conversely, if $\langle \mathfrak{B} \rangle$ was nonempty, then by the Krylov-Bogoliov theorem [26], there are stationary probability measures whose support is contained in $\langle \mathfrak{B} \rangle$. Let μ be one of these measures, and let $\mu_{\mathcal{U}} := \text{pr}_{\mathcal{U}}^*[\mu]$. Then $\text{supp}[\mu] \subset \mathfrak{B}$. ■

Let $\text{Meas}^{\text{ext}}[\mathfrak{B}]$ be the set of extendible measures supported on \mathfrak{B} .

Corollary 2 It is formally undecidable whether, for a given subset $\mathfrak{B} \subset \mathfrak{A}^{\mathcal{U}}$, the set $\text{Meas}^{\text{ext}}[\mathfrak{B}]$ is nonempty. ■

However, it is easily decidable whether $\text{Meas}^{\text{stat}}[\mathfrak{B}]$ itself is nonempty. The set of all real-valued measures supported on \mathfrak{B} is a finite-dimensional vector space, and the stipulation that an element of this vector space be a locally stationary probability measure takes the form of a finite system of linear equations and inequalities; solving such a system is a decidable problem.

2.2 Cellular Automata

Let $\mathcal{U} \subset \mathbb{Z}^D$ be finite (metaphorically speaking, \mathcal{U} is a “neighbourhood of zero”) and let $\phi: \mathfrak{A}^{\mathcal{U}} \rightarrow \mathfrak{A}$. For every $\mathbf{n} \in \mathbb{Z}^D$, define $\phi_{\mathbf{n}} := \phi \circ \text{Shift}^{-\mathbf{n}}: \mathfrak{A}^{\mathcal{U}+\mathbf{n}} \rightarrow \mathfrak{A}$.

The *cellular automata* determined by ϕ is then the function $\Phi: \mathfrak{A}^{\mathbb{Z}^D} \rightarrow \mathfrak{A}^{\mathbb{Z}^D}$ sending $[a_{\mathbf{n}}]_{\mathbf{n} \in \mathbb{Z}^D} \mapsto [\phi_{\mathbf{n}}(\mathbf{a}_{\mathcal{U}+\mathbf{n}})]_{\mathbf{n} \in \mathbb{Z}^D}$. ϕ is called the *local transformation rule* for Φ . Cellular automata were first investigated by Von Neumann [25] and Ulam [24], and later extensively studied by Hedlund [6], Wolfram [27], and others; more recent surveys are [23], [5], [12], [3].

Any cellular automaton on \mathbb{Z}^D can be represented by a subshift of finite type on $\mathbb{Z}^D \times \mathbb{Z}$. Simply define

$$\tilde{\mathcal{U}} := (\mathcal{U} \times \{0\}) \sqcup \underbrace{\{(0, 0, \dots, 0, 1)\}}_D$$

and then set $\tilde{\mathfrak{B}} := \{\mathbf{a} \in \mathfrak{A}^{\tilde{\mathcal{U}}} ; a_{(0,0,\dots,0,1)} = \phi(\mathbf{a}_{(\mathcal{U} \times \{0\})})\}$

If $\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}^D \times \mathbb{Z}}$, then \mathbf{a} can be seen as a \mathbb{Z} -indexed sequence of configurations in $\mathfrak{A}^{\mathbb{Z}^D}$. Clearly, \mathbf{a} is in $\langle \tilde{\mathfrak{B}} \rangle$ if and only if this sequence describes the Φ -orbit of some point in $\mathfrak{A}^{\mathbb{Z}^D}$.

Of course, unless Φ is surjective, not every element of $\mathfrak{A}^{\mathbb{Z}^D}$ will necessarily have a Φ -preimage, and thus, not every element can appear in such a \mathbb{Z} -indexed sequence of configurations. We can obviate this difficulty by concentrating on the *center* of the dynamical system $(\mathfrak{A}^{\mathbb{Z}^D}, \Phi)$.

If X is any compact space, and $T: X \rightarrow X$ continuous, then the *nonwandering set*, $\Omega(X, T)$ is the set of all points $x \in X$ which are *regionally recurrent*: for any neighbourhood U of x , there is some $n \in \mathbb{N}$ so that $T^n(U) \cap U \neq \emptyset$. $\Omega(X, T)$ is a compact T -invariant subset, so we can look at the restricted dynamical system $(\Omega(X, T), T|_{\Omega(X, T)})$ —however, not all elements of $\Omega(X, T)$ will be regionally recurrent under T , when seen in the subspace topology (see [26] for an example)—hence, $\Omega^2(X, T) := \Omega(\Omega(X, T), T|_{\Omega(X, T)})$ may be a proper subset.

By transfinite induction, for any countable ordinal number α , define $\Omega^{\alpha+1}(X, T) := \Omega(\Omega(X, T), T|_{\Omega^\alpha(X, T)})$, and, if γ is a limit ordinal, define $\Omega^\gamma(X, T) := \bigcap_{\alpha < \gamma} \Omega^\alpha(X, T)$. Since X is compact, this descending sequence of compact subsets must become constant at some countable ordinal α , so that $\Omega^{\alpha+1}(X, T) = \Omega^\alpha(X, T)$. The *center* of (X, T) , defined $\mathbf{Z}(X, T) := \Omega^\alpha(X, T)$, is nonempty, compact, and T -invariant. If μ is any T -invariant Radon measure on X , then $\text{supp}[\mu] \subset \mathbf{Z}(X, T)$.

So, treat $(\mathfrak{A}^{\mathbb{Z}^D}, \Phi)$ as a compact topological dynamical system, and let $\mathbf{Z}(\Phi)$ be its center. The restricted map $\Phi_1: \mathbf{Z}(\Phi) \rightarrow \mathbf{Z}(\Phi)$ is surjective, so every element in $\mathbf{Z}(\Phi)$ appears in some \mathbb{Z} -indexed sequence of $\mathfrak{A}^{\mathbb{Z}^D}$ -configurations admissible to $\tilde{\mathfrak{B}}$.

The Invariant Measure Problem Given a local transformation rule $\phi: \mathfrak{A}^{\mathcal{U}} \rightarrow \mathfrak{A}$, describe the set of Φ -invariant, stationary measures on $\mathfrak{A}^{\mathbb{Z}^D}$.

Suppose that we represent the cellular automata as a subshift of finite type in the aforementioned way, and suppose that $\mu_{\tilde{\mathcal{U}}}$ is a locally stationary probability measure on $\mathfrak{A}^{\tilde{\mathcal{U}}}$. It is easy to verify that a stationary extension of $\mu_{\tilde{\mathcal{U}}}$ to $\mathfrak{A}^{\mathbb{Z}^D \times \mathbb{Z}}$ is equivalent to a Φ -invariant, stationary measure on $\mathfrak{A}^{\mathbb{Z}^D}$.

3 Caveats and Counterexamples

3.1 Nonextendability in \mathbb{Z} ; The Entropy Metric

The following counterexample, which first appeared in [1], shows that, even in \mathbb{Z} , locally stationary measures are not always extendible, when the initial domain is “disconnected”.

Suppose that $\mathcal{U} := \{0, 1, 3\}$. If $\mu_{\mathcal{U}}$ is a probability measure on $\mathfrak{A}^{\mathcal{U}}$, then we can treat the functions pr_0, pr_1 , and pr_3 as *random variables* ranging over the domain \mathfrak{A} . So, let $\mu_{\mathcal{U}}$ be any probability measure on $\mathfrak{A}^{\mathcal{U}}$ such that:

- (A) $\text{pr}_0 = \text{pr}_1, \mu_{\mathcal{U}}$ -almost-surely.
- (B) pr_0 and pr_3 are *independent* as random variables. (thus pr_1 and pr_3 are also independent.)

To ensure $\mu_{\mathcal{U}}$ is locally stationary, it suffices to require only that the random variables $\text{pr}_0, \text{pr}_1,$ and pr_3 are *identically distributed*.

The measure $\mu_{\mathcal{U}}$ cannot be extended even to a locally stationary measure on $\mathfrak{A}^{[0 \dots 3]}$, much less a stationary measure on $\mathfrak{A}^{\mathbb{Z}}$. To see this, suppose that $\mu_{[0 \dots 3]}$ was a locally stationary extension. Then condition (A) defining $\mu_{\mathcal{U}}$ implies that, as random variables on the probability space $(\mathfrak{A}^{[0 \dots 3]}, \mu_{[0 \dots 3]})$, $\text{pr}_0 = \text{pr}_1 = \text{pr}_2 = \text{pr}_3$. But by condition (B), pr_0 and pr_3 are independent—a contradiction.

This example can be understood as part of a more general phenomenon. If S is any set, and μ is any probability measure on \mathfrak{A}^S , then μ induces an *entropy metric*, D_μ , on the set $\text{Fin}[S]$ of all finite subsets of S . If $\mathcal{U}, \mathcal{V} \subset S$ are finite, then define

$$H_\mu[\mathcal{U}|\mathcal{V}] := - \sum_{\mathbf{b} \in \mathfrak{A}^\mathcal{V}} \sum_{\mathbf{a} \in \mathfrak{A}^\mathcal{U}} \mu[\mathbf{a}|\mathbf{b}] \log_2(\mu[\mathbf{a}|\mathbf{b}]),$$

$$\text{where } \mu[\mathbf{a}|\mathbf{b}] := \frac{\mu\{\mathbf{c} \in \mathfrak{A}^S; \mathbf{c}_\mathcal{U} = \mathbf{a} \text{ and } \mathbf{c}_\mathcal{V} = \mathbf{b}\}}{\mu\{\mathbf{c} \in \mathfrak{A}^S; \mathbf{c}_\mathcal{V} = \mathbf{b}\}}.$$

Then define: $D_\mu[\mathcal{U}, \mathcal{V}] := H_\mu[\mathcal{U}|\mathcal{V}] + H_\mu[\mathcal{V}|\mathcal{U}]$.

It is easy to check that D_μ is a *metric* on $\text{Fin}[S]$. Furthermore, if S is a \mathbb{G} -module, and μ is a \mathbb{G} -invariant measure, then D_μ is a \mathbb{G} -action invariant metric. If S is a *subset* of some \mathbb{G} -module, and μ is a *locally* \mathbb{G} -invariant measure, then D_μ is a “locally” \mathbb{G} -invariant metric, in the obvious sense.

Now, suppose \mathcal{M} is a \mathbb{G} -module, $\mathcal{U} \subset \mathcal{M}$, and $\mu_{\mathcal{U}}$ is a locally \mathbb{G} -invariant measure on $\mathfrak{A}^\mathcal{U}$. If μ is to be an invariant extension of $\mu_{\mathcal{U}}$, then it must satisfy the condition:

$$\text{For every } \mathcal{V}, \mathcal{W} \in \text{Fin}[\mathcal{U}], \text{ and every } \mathbf{g} \in \mathbb{G}, D_\mu[\mathbf{g} \cdot \mathcal{V}, \mathbf{g} \cdot \mathcal{W}] = D_{\mu_{\mathcal{U}}}[\mathcal{V}, \mathcal{W}].$$

Hence, D_μ is forced to take certain values on a subset of $\text{Fin}[\mathcal{M}]$. The question is: can we define D_μ in the *rest* of $\text{Fin}[\mathcal{M}]$ so that it is a metric? If we cannot, then it is impossible to extend μ .

In the aforementioned counterexample, $D_{\mu_{\mathcal{U}}}[\{0\}, \{1\}] = 0$. Thus, if μ was an extension of $\mu_{\mathcal{U}}$, we would have:

$$D_\mu[\{0\}, \{1\}] = D_\mu[\{1\}, \{2\}] = D_\mu[\{2\}, \{3\}] = 0$$

and hence, $D_\mu[\{0\}, \{3\}] = 0$. But we know that $D_\mu[\{0\}, \{3\}] > 0$, because pr_0 and pr_3 are independent random variables. Hence, no such extension μ can exist.

3.2 Nonextendability in \mathbb{Z}^D ; The Tiling Condition

In the previous counterexample, it seems the problem was that the domain \mathcal{U} was not “connected”. However, in \mathbb{Z}^2 , extendability can fail even when \mathcal{U} is a 2×2 box.

Suppose $\mathcal{U} \subset \mathbb{Z}^D$, and $\mu_{\mathcal{U}} \in \text{Meas}^{\text{stat}}[\mathfrak{A}^\mathcal{U}]$. The *support* of $\mu_{\mathcal{U}}$ is some subset $\text{supp}[\mu_{\mathcal{U}}] \subset \mathfrak{A}^\mathcal{U}$; let $\langle \text{supp}[\mu_{\mathcal{U}}] \rangle$ be the *subshift of finite type* defined by $\text{supp}[\mu_{\mathcal{U}}]$. If $\mu \in \text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathbb{Z}^D}]$ is a stationary extension of $\mu_{\mathcal{U}}$, then any μ -generic configuration $\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}^D}$ must be an element of $\langle \text{supp}[\mu_{\mathcal{U}}] \rangle$.

Thus, we have:

The Tiling Condition μ_U cannot be extendible unless $\langle \text{supp}[\mu_U] \rangle$ is nontrivial.

Intuitively, the configuration \mathbf{a} determines a *tiling* of \mathbb{Z}^D by elements in $\text{supp}[\mu_U]$: for any $\mathbf{k} \in \mathbb{Z}^D$, $\mathbf{a}_{(\mathbf{k}+U)}$ is an element of $\text{supp}[\mu_U]$.

For example, suppose that $D := 2$, $U := [0 \cdots 1] \times [0 \cdots 1]$, and $\mathfrak{A} := \{0, 1, 2\}$. Elements of \mathfrak{A}^U are thus 2×2 words in \mathfrak{A} .

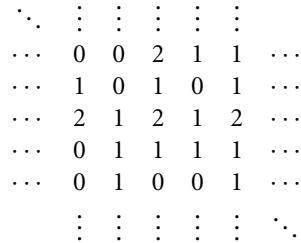


Figure 1: A configuration of letters

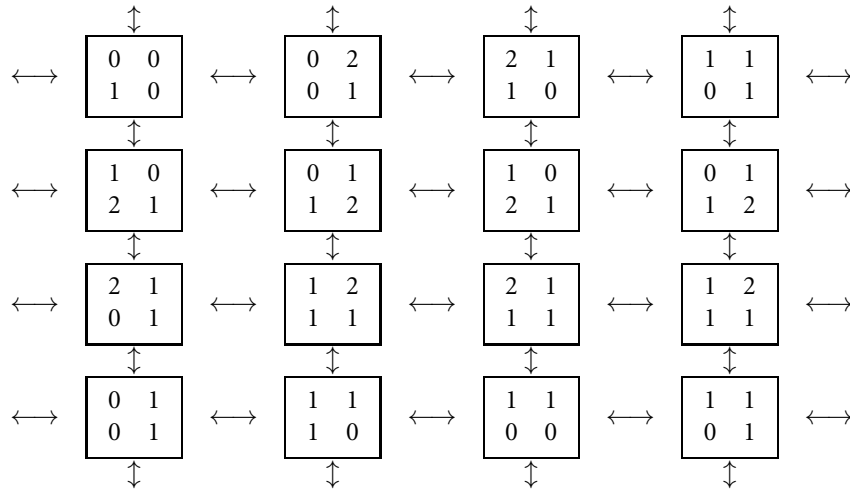


Figure 2: The corresponding assignment of matrices.

Choosing a configuration in $\mathfrak{A}^{\mathbb{Z}^2}$ is equivalent to assigning a 2×2 matrix to each point in the lattice, so that adjacent sides agree. For example, the configuration in Figure 1 is equivalent to the assignment of Figure 2

We will define a locally stationary measure $\mu_{\mathcal{U}}$ so that $\text{supp}[\mu_{\mathcal{U}}]$ cannot tile \mathbb{Z}^2 in this manner. We will do this by explicitly constructing $\text{supp}[\mu_{\mathcal{U}}]$ to tile a *different* space instead, a kind of “pseudolattice” (see Figure 3).

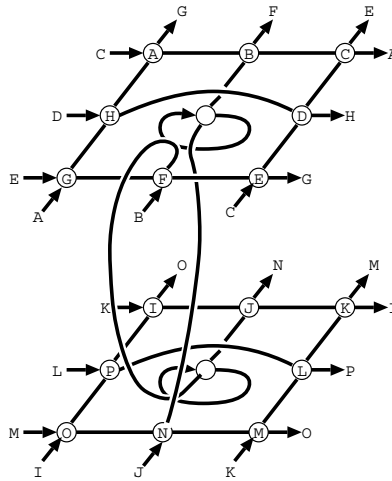


Figure 3: A “pseudolattice”.

Stack two 3×3 grids on top of one another, and then “break” the connection between the central element of each level, and its southern, eastern, and western neighbours. Cross-connect the eastern and western neighbours with each other. Connect the southern neighbour to the central element of the level *above*, and we connect the central element of this level to the southern element of the level *below*. We also maintain the connection between the central element and its northern neighbour,

Now we’ll form a locally stationary measure which tiles *this* space instead. Consider the tiling portrayed in Figure 4. Count every element of $\mathfrak{A}^{2 \times 2}$ as many times as it appears in these two pictures. There are 18 tiles, and each one appears exactly once. Thus, each of the tiles shown gets a probability of $\frac{1}{18}$.

To show that $\mu_{\mathcal{U}}$ is locally stationary, it suffices to check that the *left columns* and *right columns* have the same probability distribution, and that the *top* and *bottom* rows have the same probability distribution. This is easy to confirm.

We claim that one simply *cannot* tile \mathbb{Z} with this collection of blocks. For example, as soon as one lays down a tile of the form $\begin{bmatrix} 0 & 0 \\ 6 & 6 \end{bmatrix}$, one is forced to place a tile $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ immediately above it, since this is the only tile which will “match”. Once one has done this, one *must* place the tile $\begin{bmatrix} 9 & 1 \\ 10 & 0 \end{bmatrix}$ to the left of $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$, and the tile $\begin{bmatrix} 4 & 9 \\ 0 & 10 \end{bmatrix}$ to its right. So far, all the tiles are compatible. However, now, what tile shall we lay down below $\begin{bmatrix} 9 & 1 \\ 10 & 0 \end{bmatrix}$? To be compatible with $\begin{bmatrix} 9 & 1 \\ 10 & 0 \end{bmatrix}$, this tile’s top row should read $\begin{bmatrix} & \end{bmatrix}$. However, to be compatible with the tile $\begin{bmatrix} 0 & 0 \\ 6 & 6 \end{bmatrix}$ to its immediate right, the tile’s right-hand side should read $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$. There is no tile in our collection which meets these two criteria.

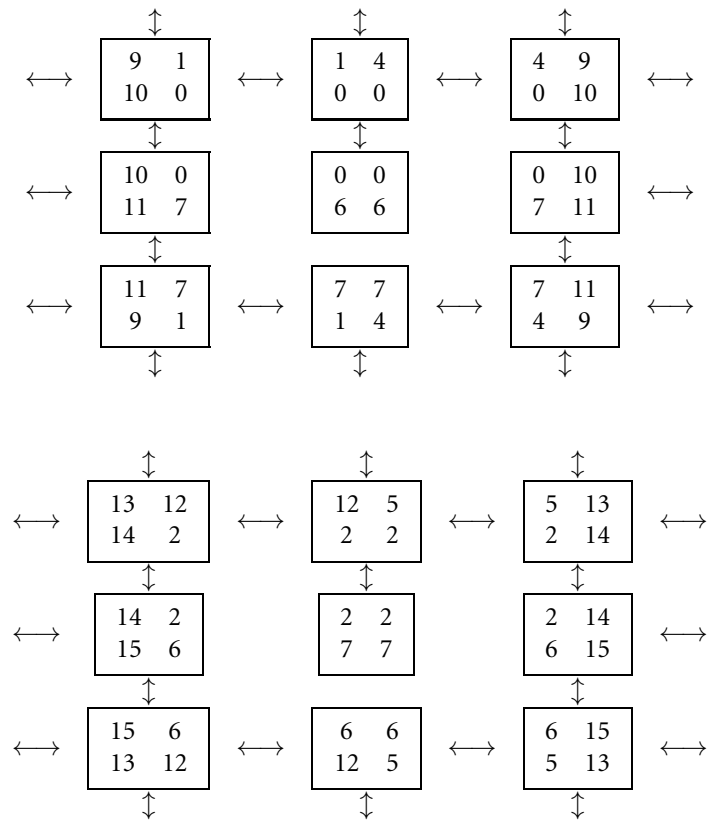


Figure 4: A configuration on the pseudolattice

The Tiling Condition is necessary, but not sufficient. To see this, recall that the set $\text{Meas}^{\text{ext}}[\mathcal{U}]$ is closed as a subset of $\text{Meas}^{\text{stat}}[\mathcal{U}]$. Thus, its complement is open. Hence, every nonextendible measure is surrounded by a neighbourhood of nonextendible measures.

If β is the equidistributed measure (assigning equal probability to every element of $\mathfrak{A}^{\mathcal{U}}$), and $\epsilon > 0$ is small, then consider the measure:

$$\mu_\epsilon := (1 - \epsilon) \cdot \mu_{\mathcal{U}} + \epsilon \cdot \beta$$

μ_ϵ is a convex combination of $\mu_{\mathcal{U}}$ and β . Since $\epsilon > 0$, the support of μ_ϵ is all of $\mathfrak{A}^{\mathcal{U}}$. Thus, μ_ϵ always satisfies the Tiling Condition. However, if ϵ is “sufficiently small”, the measure μ_ϵ will be inside the neighbourhood of nonextendible measures around $\mu_{\mathcal{U}}$.

4 Harmonic Analysis of Extensions

4.1 Configuration Space as a Compact Group

Solving the Extension Problem requires a good way of describing measures, and Harmonic Analysis provides one. To employ this approach, we must reconceive the configuration space as a compact abelian topological group. Hence, from now on, we will operate under the assumption that:

The alphabet \mathfrak{A} is a finite abelian group.

The choice of group structure on \mathfrak{A} is unimportant—if \mathfrak{A} has A elements, then the simplest choice is to let $\mathfrak{A} := \mathbb{Z}/A$.

If we endow $\mathfrak{A}^{\mathcal{M}}$ with the product group structure, it is a compact abelian topological group. What is its dual group?

Let \mathfrak{A} be the dual group of \mathfrak{A} . If $\mathcal{V} \subset \mathcal{M}$ is finite, and, for all $v \in \mathcal{V}$, $\chi_v \in \hat{\mathfrak{A}}$, then $\chi_v \circ \text{pr}_v : \mathfrak{A}^{\mathcal{M}} \rightarrow \mathbb{T}^1$ is the map taking the configuration $[a_m]_{m \in \mathcal{M}}$ to the value $\chi_v(a_v)$. (Here “ \mathbb{T}^1 ” is the unit circle group.)

We will use the notation “ $\bigotimes_{v \in \mathcal{V}} \chi_v$ ” to refer to the map:

$$\left(\prod_{v \in \mathcal{V}} \chi_v \circ \text{pr}_v \right) : \mathfrak{A}^{\mathcal{M}} \rightarrow \mathbb{T}^1$$

$$[a_m]_{m \in \mathcal{M}} \mapsto \prod_{v \in \mathcal{V}} \chi_v(a_v)$$

It is easy to verify the next theorem:

Theorem 3 Let \mathcal{M} be any set. The dual group of $\mathfrak{A}^{\mathcal{M}}$ is the set:

$$\left\{ \bigotimes_{v \in \mathcal{V}} \chi_v ; \mathcal{V} \subset \mathcal{M} \text{ is any finite subset, and, for all } v \in \mathcal{V}, \chi_v \in \hat{\mathfrak{A}}. \right\} \quad \blacksquare$$

4.2 The Fourier Transform

Now, if μ is a measure on $\mathfrak{A}^{\mathcal{M}}$, and $\chi \in \widehat{\mathfrak{A}^{\mathcal{M}}}$, then the *Fourier Coefficient* of μ at χ is defined:

$$\hat{\mu}_\chi = \langle \mu, \chi \rangle := \int_{\mathfrak{A}^{\mathcal{M}}} \bar{\chi} d\mu$$

The *Fourier Transform* of μ is the function: $\hat{\mu}: \widehat{\mathfrak{A}^{\mathcal{M}}} \rightarrow \mathbb{C}$ so that $\hat{\mu}_\chi = \langle \mu, \chi \rangle$.

If $\text{Meas}[\mathfrak{A}^{\mathcal{M}}; \mathbb{C}]$ is endowed with the *total variation norm*, and $\mathbf{C}(\widehat{\mathfrak{A}^{\mathcal{M}}}; \mathbb{C})$ is endowed with the *uniform norm*, then the map

$$\begin{aligned} \text{Four}: \text{Meas}[\mathfrak{A}^{\mathcal{M}}; \mathbb{C}] &\rightarrow \mathbf{C}(\widehat{\mathfrak{A}^{\mathcal{M}}}; \mathbb{C}) \\ \mu &\mapsto \hat{\mu} \end{aligned}$$

is an *injective, bounded linear function* of norm 1 [8]. Thus, the Fourier transform of μ totally characterizes it: if μ and ν are two measures, and $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$.

4.3 Fourier Theory and (local) Stationarity

The shift action of \mathbb{G} upon $\mathfrak{A}^{\mathcal{M}}$ induces a right action of \mathbb{G} upon $\widehat{\mathfrak{A}^{\mathcal{M}}}$. If $\mathbf{g} \in \mathbb{G}$, and $\chi \in \widehat{\mathfrak{A}^{\mathcal{M}}}$, then define:

$$(1) \quad \chi \cdot \mathbf{g} = \chi \circ \text{Shift}^{\mathbf{g}^{-1}}$$

Note that, if $\chi = \prod_{v \in \mathcal{V}} (\chi_v \circ \text{pr}_v)$, then $\chi \cdot \mathbf{g} = \prod_{v \in \mathcal{V}} (\chi_v \circ \text{pr}_{\mathbf{g} \cdot v})$.

If $\mathcal{U} \subset \mathcal{M}$ is not closed under the \mathbb{G} -action, then there is no “shift action” on $\mathfrak{A}^{\mathcal{U}}$. However, we can still treat \mathbb{G} as “acting” upon $\widehat{\mathfrak{A}^{\mathcal{U}}}$ in a certain limited capacity, as follows:

Suppose $\mathcal{V} \subset \mathcal{U}$, and $\chi = \prod_{v \in \mathcal{V}} (\chi_v \circ \text{pr}_v)$. Suppose that $\mathbf{g} \in \mathbb{G}$ is such that $\mathbf{g} \cdot \mathcal{V} \subset \mathcal{U}$ also. Then $\chi \cdot \mathbf{g} = \prod_{v \in \mathcal{V}} (\chi_v \circ \text{pr}_{\mathbf{g} \cdot v})$ is still an element of $\widehat{\mathfrak{A}^{\mathcal{U}}}$.

Theorem 4 1. If $\mu \in \text{Meas}[\mathfrak{A}^{\mathcal{M}}]$, then μ is \mathbb{G} -invariant if and only if, for every $\chi \in \widehat{\mathfrak{A}^{\mathcal{M}}}$ and every $\mathbf{g} \in \mathbb{G}$, $\langle \mu, \chi \rangle = \langle \mu, \chi \cdot \mathbf{g} \rangle$.

2. If $\mathcal{U} \subset \mathcal{M}$, and $\mu \in \text{Meas}[\mathfrak{A}^{\mathcal{U}}]$, then μ is locally \mathbb{G} -invariant if and only if, for every $\chi \in \widehat{\mathfrak{A}^{\mathcal{U}}}$ and every $\mathbf{g} \in \mathbb{G}$ so that $\chi \cdot \mathbf{g}$ is also in $\widehat{\mathfrak{A}^{\mathcal{U}}}$, $\langle \mu, \chi \rangle = \langle \mu, \chi \cdot \mathbf{g} \rangle$.

Proof We will prove Part 2, since Part 1 clearly follows.

Proof of “ \implies ” Let $\chi = \bigotimes_{v \in \mathcal{V}} \chi_v$, for some $\mathcal{V} \subset \mathcal{U}$. Then a simple computation reveals:

$$\langle \mu, \chi \rangle = \sum_{\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}} \mu[\mathbf{a}] \cdot \bar{\chi}(\mathbf{a})$$

Where, by “ $\mu[\mathbf{a}]$ ”, we mean $\mu\{\mathbf{b} \in \mathfrak{A}^{\mathcal{U}}; \mathbf{b}_{\mathcal{V}} = \mathbf{a}\}$. Thus,

$$\begin{aligned} \langle \mu, \chi \cdot \mathbf{g} \rangle &=_{(1)} \sum_{\mathbf{a} \in \mathfrak{A}^{\mathcal{G} \cdot \mathcal{V}}} \mu[\mathbf{a}] \cdot (\tilde{\chi} \circ \text{Shift}^{\mathbf{g}^{-1}}(\mathbf{a})) \\ &=_{(2)} \sum_{\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}} \mu[\text{Shift}^{\mathbf{g}} \mathbf{a}] \cdot (\tilde{\chi} \circ \text{Shift}^{\mathbf{g}^{-1}} \circ \text{Shift}^{\mathbf{g}}(\mathbf{a})) \\ &= \sum_{\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}} \mu[\text{Shift}^{\mathbf{g}} \mathbf{a}] \cdot \tilde{\chi}(\mathbf{a}) \\ &=_{(3)} \sum_{\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}} \mu[\mathbf{a}] \cdot \tilde{\chi}(\mathbf{a}) \\ &= \langle \mu, \chi \rangle \end{aligned}$$

- (1) By definition of $\chi \cdot \mathbf{g}$ (equation (1)).
- (2) Because $\text{Shift}^{\mathbf{g}}: \mathfrak{A}^{\mathcal{U}} \rightarrow \mathfrak{A}^{\mathcal{G} \cdot \mathcal{U}}$ is an isomorphism.
- (3) Because μ is locally \mathbb{G} -invariant.

Proof of “ \Leftarrow ” If $\mathcal{V} \subset \mathcal{U}$ is finite, then for any $\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}$, then it is easy to verify that:

$$\mu[\mathbf{a}] = \text{pr}_{\mathcal{V}}^*[\mu][\mathbf{a}] = \sum_{\chi \in \widehat{\mathfrak{A}^{\mathcal{V}}}} \hat{\mu}_{\chi} \cdot \chi(\mathbf{a})$$

The argument is then very similar to that of “ \Rightarrow ”.

4.4 Fourier Properties of Stationary Extensions

Suppose that $\mathcal{U} \subset \mathcal{M}$, and $\mathcal{V} \subset \mathcal{U}$ is a finite subset, and suppose that $\chi := \bigotimes_{v \in \mathcal{V}} \chi_v$ is some element of $\widehat{\mathfrak{A}^{\mathcal{U}}}$. Then we can also think of χ as an element of $\widehat{\mathfrak{A}^{\mathcal{M}}}$. In other words, $\widehat{\mathfrak{A}^{\mathcal{U}}}$ embeds canonically in $\widehat{\mathfrak{A}^{\mathcal{M}}}$. We will “abuse notation”, and identify elements of $\widehat{\mathfrak{A}^{\mathcal{U}}}$ with their images in $\widehat{\mathfrak{A}^{\mathcal{M}}}$. The following theorem is a straightforward computation:

Theorem 5 Let $\mu_{\mathcal{U}} \in \text{Meas}[\mathfrak{A}^{\mathcal{U}}; \mathbb{C}]$, and let $\mu \in \text{Meas}[\mathfrak{A}^{\mathcal{M}}; \mathbb{C}]$. Then

$$(\text{pr}_{\mathcal{U}}^*[\mu] = \mu_{\mathcal{U}}) \iff (\forall \chi \in \widehat{\mathfrak{A}^{\mathcal{U}}}, \langle \mu, \chi \rangle = \langle \mu_{\mathcal{U}}, \chi \rangle).$$

Thus, we have reduced the Extension Problem to finding a measure μ on $\text{Meas}[\mathfrak{A}^{\mathcal{M}}]$ whose Fourier coefficients agree with those of $\mu_{\mathcal{U}}$ on $\widehat{\mathfrak{A}^{\mathcal{U}}}$. However, we can’t just “fill in” the remaining Fourier coefficients in an arbitrary way. First of all, we must produce something which is \mathbb{G} -invariant. Second of all, we want to end up with a probability measure.

Theorem 6 Let $\mu_{\mathcal{U}} \in \text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$, and let $\mu \in \text{Meas}[\mathfrak{A}^{\mathcal{M}}]$. Then μ is a stationary extension of $\mu_{\mathcal{U}}$ if and only if the following two conditions are satisfied:

- For every $\chi \in \widehat{\mathfrak{A}^{\mathcal{U}}}$, and every $\mathbf{g} \in \mathbb{G}$, $\langle \mu, \chi \cdot \mathbf{g} \rangle = \langle \mu_{\mathcal{U}}, \chi \rangle$.
(This equation must be true even when $\chi \cdot \mathbf{g}$ is no longer in $\widehat{\mathfrak{A}^{\mathcal{U}}}$).
- The Fourier coefficients of μ form a positive definite sequence.

Proof The first condition follows from Part 1 of Theorem 4. Notice that, if more than one \mathbb{G} -translate of χ lies inside $\widehat{\mathfrak{A}^{\mathcal{U}}}$, then all of them will produce the same equation, by Part 2 of Theorem 4 (since $\mu_{\mathcal{U}}$ is locally \mathbb{G} -invariant).

The second condition is just the Bochner-Herglotz theorem to guarantee that the measure μ is nonnegative [8]. This forces μ to be a probability measure, because now $\mu[\mathfrak{A}^{\mathcal{M}}] = \langle \mu, \mathbb{1} \rangle = \langle \mu_{\mathcal{U}}, \mathbb{1} \rangle = \mu_{\mathcal{U}}[\mathfrak{A}^{\mathcal{U}}] = 1$. (Since $\mu_{\mathcal{U}}$ itself is a probability measure). ■

5 Extension on Finite Modules

Suppose that \mathcal{M} is a finite \mathbb{G} -module, $\mathcal{U} \subset \mathcal{M}$, and $\mu_{\mathcal{U}} \in \text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$. We will show that if $\mu_{\mathcal{U}}$ is “sufficiently close” to a product measure, then it is extendible. More generally, we will show:

Theorem 7 Let $\nu_{\mathcal{U}} \in \text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ be an extendible measure, with an invariant extension ν such that $\text{supp}[\nu] = \mathfrak{A}^{\mathcal{M}}$.

There exists an $\epsilon > 0$ so that, if $\mu_{\mathcal{U}} \in \text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ is any measure with $\|\mu_{\mathcal{U}} - \nu_{\mathcal{U}}\|_{\text{var}} < \epsilon$, then $\mu_{\mathcal{U}}$ is also extendible. This ϵ is of the form:

$$\epsilon = \frac{1}{H(\mathcal{M})} \cdot \min_{\mathbf{a} \in \mathfrak{A}^{\mathcal{M}}} \nu[\mathbf{a}]$$

($\min_{\mathbf{a} \in \mathfrak{A}^{\mathcal{M}}} \nu[\mathbf{a}] > 0$ by hypothesis that $\text{supp}[\nu] = \mathfrak{A}^{\mathcal{M}}$), where $H(\mathcal{M})$ is a number determined by the \mathbb{G} -module structure of \mathcal{M} , and which satisfies the following bounds:

- (A) $H(\mathcal{M}) \leq \text{Card}[\widehat{\mathfrak{A}^{\mathcal{M}}}]$.
- (B) $H(\mathcal{M}) \leq \text{Card}[\mathbb{G}/\mathbb{H}] \cdot \text{Card}[\widehat{\mathfrak{A}^{\mathcal{U}}}]$.

where \mathbb{H} is the stabiliser of \mathcal{M} in \mathbb{G} :

$$\mathbb{H} := \{\mathbf{h} \in \mathbb{G} ; \forall m \in \mathcal{M}, \mathbf{h} \cdot m = m\}$$

Proof Define $\delta_{\mathcal{U}} := \mu_{\mathcal{U}} - \nu_{\mathcal{U}}$. Thus, $\delta_{\mathcal{U}}$ is a real-valued measure. Since $\mu_{\mathcal{U}}$ and $\nu_{\mathcal{U}}$ are locally \mathbb{G} -invariant, $\delta_{\mathcal{U}}$ is also¹.

Next we will define δ , a real-valued, \mathbb{G} -invariant measure upon $\mathfrak{A}^{\mathcal{M}}$, in terms of its Fourier coefficients. For every $\chi \in \widehat{\mathfrak{A}^{\mathcal{M}}}$,

- If there is some $\kappa \in \widehat{\mathfrak{A}^{\mathcal{U}}}$ and \mathbf{g} in \mathbb{G} so that $\chi = \kappa \cdot \mathbf{g}$, then let $\hat{\delta}(\chi) := \hat{\delta}_{\mathcal{U}}(\kappa)$.
- Otherwise, let $\hat{\delta}_{\chi} := 0$.

¹Cylinder subsets of $\mathfrak{A}^{\mathcal{U}}$ can have negative $\delta_{\mathcal{U}}$ -measures, but these measures are still preserved under any shift which leaves the cylinder set inside $\mathfrak{A}^{\mathcal{U}}$.

By Part 2 of Theorem 4, the definition of $\hat{\delta}_\chi$ is independent of the choice of κ and \mathbf{g} , if more than one choice is available. By Part 1 of the same theorem, the measure δ is \mathbb{G} -invariant.

Claim 1 δ is a real-valued measure.

Proof Since $\delta_{\mathcal{U}}$ is a real-valued measure, we know that, for every $\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}$, $\widehat{\delta_{\mathcal{U}}}(\bar{\chi}) = \widehat{\delta_{\mathcal{U}}}(\chi)$. It follows that, for every $\chi \in \widehat{\mathfrak{A}}^{\mathcal{M}}$, $\hat{\delta}(\bar{\chi}) = \hat{\delta}(\chi)$, and from this, we conclude that δ is also a real-valued measure. ■

Claim 2 There is a number $H(\mathcal{M})$, determined by the \mathbb{G} -module structure of \mathcal{M} , and satisfying inequalities (A) and (B), so that $\|\delta\|_{\text{var}} \leq H(\mathcal{M}) \cdot \|\delta_{\mathcal{U}}\|_{\text{var}}$.

Proof From elementary harmonic analysis [8], we know that:

- $\|\widehat{\delta_{\mathcal{U}}}\|_{\infty} < \|\delta_{\mathcal{U}}\|_{\text{var}}$.
- $\|\delta\|_{\text{var}} < \|\hat{\delta}\|_1$.

Hence, it suffices to show that $\|\hat{\delta}\|_1 < H(\mathcal{M}) \cdot \|\widehat{\delta_{\mathcal{U}}}\|_{\infty}$, where $H(\mathcal{M})$ is the aforementioned number. To see inequality (A), notice that

$$\|\hat{\delta}\|_1 \leq \text{Card}[\mathcal{M}] \cdot \|\hat{\delta}\|_{\infty} = \text{Card}[\mathcal{M}] \cdot \|\widehat{\delta_{\mathcal{U}}}\|_{\infty}$$

where the second equality follows immediately from the definition of $\hat{\delta}$.

Now for inequality (B). For any $\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}$, let $\mathbb{G} \cdot \chi := \{\mathbf{g} \cdot \chi ; \mathbf{g} \in \mathbb{G}\}$ be the orbit of χ under the action of \mathbb{G} . Then:

$$\begin{aligned} \|\hat{\delta}\|_1 &= \sum_{\chi \in \widehat{\mathfrak{A}}^{\mathcal{M}}} |\hat{\delta}(\chi)| \\ &= \sum_{\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}} \sum_{\xi \in \mathbb{G} \cdot \chi} |\hat{\delta}(\xi)| \\ &= \sum_{\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}} \sum_{\xi \in \mathbb{G} \cdot \chi} |\widehat{\delta_{\mathcal{U}}}(\chi)| \\ &= \sum_{\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}} \text{Card}[\mathbb{G} \cdot \chi] \cdot |\widehat{\delta_{\mathcal{U}}}(\chi)| \end{aligned}$$

But for any $\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}$, $\text{Card}[\mathbb{G} \cdot \chi] < \text{Card}[\mathbb{G}/\mathbb{H}]$. So this expression is less than

$$\begin{aligned} \sum_{\chi \in \widehat{\mathfrak{A}}^{\mathcal{U}}} \text{Card}[\mathbb{G}/\mathbb{H}] \cdot |\widehat{\delta_{\mathcal{U}}}(\chi)| &= \text{Card}[\mathbb{G}/\mathbb{H}] \cdot \|\widehat{\delta_{\mathcal{U}}}\|_1 \\ &\leq \text{Card}[\mathbb{G}/\mathbb{H}] \cdot \text{Card}[\widehat{\mathfrak{A}}^{\mathcal{U}}] \cdot \|\widehat{\delta_{\mathcal{U}}}\|_{\infty}. \quad \blacksquare \end{aligned}$$

Recall that ν is some invariant extension of $\nu_{\mathcal{U}}$. Define:

$$\mu := \nu + \delta$$

Claim 3 μ is a nonnegative, \mathbb{G} -invariant probability measure.

Proof μ is a sum of two real-valued, \mathbb{G} -invariant measures, and thus is also a real-valued, \mathbb{G} -invariant measure.

Also, $\|\nu - \mu\|_{\text{var}} = \|\delta\|_{\text{var}} < H(\mathcal{M}) \|\delta_{\mathcal{U}}\|_{\text{var}} = H(\mathcal{M}) \cdot \|\nu_{\mathcal{U}} - \mu_{\mathcal{U}}\|_{\text{var}}$. Thus,

$$\left(\|\nu_{\mathcal{U}} - \mu_{\mathcal{U}}\|_{\text{var}} < \epsilon := \frac{1}{H(\mathcal{M})} \cdot \min_{\mathbf{a} \in \mathfrak{A}^{\mathcal{M}}} \nu[\mathbf{a}] \right) \implies (\text{For every } \mathbf{a} \in \mathfrak{A}^{\mathcal{M}}, \mu[\mathbf{a}] > 0.)$$

It remains to show that $\mu[\mathfrak{A}^{\mathcal{M}}] = 1$, or, equivalently, that $\langle \mu, \mathbb{1} \rangle = 1$. Since $\langle \nu, \mathbb{1} \rangle = 1$, this is equivalent to showing that $\langle \delta, \mathbb{1} \rangle = 0$. But $\langle \delta, \mathbb{1} \rangle = \langle \delta_{\mathcal{U}}, \mathbb{1} \rangle$, and $\langle \delta_{\mathcal{U}}, \mathbb{1} \rangle = \langle \nu_{\mathcal{U}}, \mathbb{1} \rangle - \langle \mu_{\mathcal{U}}, \mathbb{1} \rangle = 0$. ■

Finally, we want to show that μ is an extension of $\mu_{\mathcal{U}}$. But

$$\text{pr}_{\mathcal{U}}^*[\mu] = \text{pr}_{\mathcal{U}}^*[\nu] + \text{pr}_{\mathcal{U}}^*[\delta] = \nu_{\mathcal{U}} + \delta_{\mathcal{U}} = \mu_{\mathcal{U}}. \quad \blacksquare$$

If ρ is a probability measure on \mathfrak{A} , let $\rho^{\mathcal{U}}$ be the corresponding product measure on $\mathfrak{A}^{\mathcal{U}}$.

Corollary 8 *Let \mathcal{M} and $H(\mathcal{M})$ be as in the previous theorem. Let ρ be a probability measure on \mathfrak{A} with full support, and let*

$$\epsilon := \frac{1}{H(\mathcal{M})} \left(\min_{a \in \mathfrak{A}} \rho(a) \right)^{\text{Card}[\mathcal{M}]}$$

Let $\mathcal{U} \subset \mathcal{M}$. If $\mu \in \text{Meas}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$, and $\|\mu - \rho^{\mathcal{U}}\|_{\text{var}} < \epsilon$, then μ is extendible.

Proof $\rho^{\mathcal{U}}$ extends to the \mathbb{G} -invariant probability measure $\rho^{\mathcal{M}}$ on \mathcal{M} , and

$$\min_{\mathbf{a} \in \mathfrak{A}^{\mathcal{M}}} \rho^{\mathcal{M}}[\mathbf{a}] = \left(\min_{a \in \mathfrak{A}} \rho(a) \right)^{\text{Card}[\mathcal{M}]}. \quad \blacksquare$$

6 Envelopes: Reduction to Smaller Modules

Suppose that \mathcal{M} and $\tilde{\mathcal{M}}$ are \mathbb{G} -modules, and that $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ is a \mathbb{G} -module *homomorphism*, that is, for all $m \in \mathcal{M}$ and $\mathbf{g} \in \mathbb{G}$, $\phi(\mathbf{g} \cdot m) = \mathbf{g} \cdot \phi(m)$.

If $\tilde{\mathbf{a}} := [\tilde{a}_{\tilde{m}}]_{\tilde{m} \in \tilde{\mathcal{M}}} \in \mathfrak{A}^{\tilde{\mathcal{M}}}$, then define the element $\mathbf{a} := [a_m]_{m \in \mathcal{M}} \in \mathfrak{A}^{\mathcal{M}}$, by the formula:

$$(2) \quad \forall m \in \mathcal{M}, \quad a_m := \tilde{a}_{\phi(m)}$$

This determines a function $\mathfrak{A}^{\tilde{\mathcal{M}}} \rightarrow \mathfrak{A}^{\mathcal{M}}$, where $\mathfrak{A}^{\phi}(\tilde{\mathbf{a}}) := \mathbf{a}$.

If $\tilde{\mu}$ is a \mathbb{G} -invariant measure on $\mathfrak{A}^{\tilde{\mathcal{M}}}$, we define the *pullback* of $\tilde{\mu}$ through ϕ to be the measure: $\phi^{\leftarrow} \tilde{\mu} := (\mathfrak{A}^{\phi})^* \tilde{\mu}$. It is easily verified that $\phi^{\leftarrow} \tilde{\mu}$ is a \mathbb{G} -invariant measure on $\mathfrak{A}^{\mathcal{M}}$.

Given a \mathbb{G} -module \mathcal{M} , a subset $\mathcal{U} \subset \mathcal{M}$ and a locally \mathbb{G} -invariant measure $\mu_{\mathcal{U}}$ on $\mathfrak{A}^{\mathcal{U}}$, we want to find a smaller \mathbb{G} -module $\tilde{\mathcal{M}}$, a subset $\tilde{\mathcal{U}} \subset \tilde{\mathcal{M}}$, and a locally \mathbb{G} -invariant measure $\tilde{\mu}_{\tilde{\mathcal{U}}}$ on $\mathfrak{A}^{\tilde{\mathcal{U}}}$, such that, if we can extend $\tilde{\mu}_{\tilde{\mathcal{U}}}$ to a \mathbb{G} -invariant measure $\tilde{\mu}$ on $\mathfrak{A}^{\tilde{\mathcal{M}}}$, then $\mu := \phi^{\leftarrow} \tilde{\mu}$ is an extension of $\mu_{\mathcal{U}}$.

Definition 9 Envelope.

Let \mathcal{M} be a \mathbb{G} -module, and $\mathcal{U} \subset \mathcal{M}$.

An *envelope* for \mathcal{U} is a \mathbb{G} -module $\tilde{\mathcal{M}}$, along with a \mathbb{G} -module homomorphism $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$, such that

- (E1) When restricted to \mathcal{U} , the function ϕ is injective.
- (E2) If $\mathcal{V} \subset \mathcal{U}$, then for any $\mathbf{g} \in \mathbb{G}$ such that $\mathbf{g} \cdot \phi(\mathcal{V}) \subset \phi(\mathcal{U})$, we can find some element $\tilde{\mathbf{g}} \in \mathbb{G}$ so that:
 1. $\tilde{\mathbf{g}} \cdot \mathcal{V} \subset \mathcal{U}$,
 2. For all $v \in \mathcal{V}$, $\phi(\mathbf{g} \cdot v) = \tilde{\mathbf{g}} \cdot \phi(v)$. (Thus, $\phi(\mathbf{g} \cdot \mathcal{V}) = \tilde{\mathbf{g}} \cdot \phi(\mathcal{V})$.)

Example: Envelopes in a Lattice Suppose $\mathbb{G} = \mathcal{M} = \mathbb{Z}^D$, and let $\mathcal{U} \subset \mathbb{Z}^D$ be finite, and small enough that it fits into a box of dimensions $N_1 \times N_2 \times \dots \times N_D$. We will indicate the action of \mathbb{Z}^D on itself with the “+” symbol.

Consider the \mathbb{Z}^D -module:

$$\tilde{\mathcal{M}} := \frac{\mathbb{Z}}{2N_1\mathbb{Z}} \times \frac{\mathbb{Z}}{2N_2\mathbb{Z}} \times \dots \times \frac{\mathbb{Z}}{2N_D\mathbb{Z}}$$

and let $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ be the \mathbb{Z}^D -module homomorphism:

$$\phi(n_1, \dots, n_D) := \left(n_1 + \frac{\mathbb{Z}}{2N_1\mathbb{Z}}, n_2 + \frac{\mathbb{Z}}{2N_2\mathbb{Z}}, \dots, n_D + \frac{\mathbb{Z}}{2N_D\mathbb{Z}} \right)$$

Then $(\tilde{\mathcal{M}}, \phi)$ is an envelope for \mathcal{U} .

Remark In this example, the module

$$\tilde{\mathcal{M}} := \frac{\mathbb{Z}}{N_1\mathbb{Z}} \times \frac{\mathbb{Z}}{N_2\mathbb{Z}} \times \dots \times \frac{\mathbb{Z}}{N_D\mathbb{Z}}$$

with the quotient map $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ would *not* necessarily have worked as an envelope for \mathcal{U} . To see this, suppose that

$$\mathcal{U} := [1 \dots N_1] \times \{1\} \times \{1\} \times \dots \times \{1\}$$

and let $\mathcal{V} := \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 := (1, 1, \dots, 1)$, while $\mathbf{v}_2 := (2, 1, 1, \dots, 1)$. Let $\mathbf{g} := (N_1 - 1, 0, 0, \dots, 0) \in \mathbb{G}$. Then note that

$$\tilde{\mathbf{g}} + \phi(\mathbf{v}_1) = \phi(\mathbf{g} + \mathbf{v}_1) = \phi(N_1, 1, 1, \dots, 1) = \phi(\mathbf{v}_3),$$

where $\mathbf{v}_3 := (N_1, 1, 1, \dots, 1)$, while

$$\tilde{\mathbf{g}} + \phi(\mathbf{v}_2) = \phi(\tilde{\mathbf{g}} + \mathbf{v}_2) = \phi(N_1 + 1, 1, 1, \dots, 1) = \phi(1, 1, 1, \dots, 1) = \phi(\mathbf{v}_1)$$

Now, there is no element $\mathbf{g} \in \mathbb{G}$ so that $\mathbf{g} + \mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_3\}$. Thus, although $\tilde{\mathbf{g}} + \phi(\mathcal{V}) \subset \phi(\mathcal{U})$, we cannot find some $\mathbf{g} \in \mathbb{G}$ so that $\mathbf{g} + \mathcal{V} \subset \mathcal{U}$ and $\phi(\mathbf{g} + \mathcal{V}) = \tilde{\mathbf{g}} + \phi(\mathcal{V})$.

Proposition 10 *Let \mathcal{M} be a \mathbb{G} -module, and $\mathcal{U} \subset \mathcal{M}$. Let $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ be an envelope for \mathcal{U} , and $\tilde{\mathcal{U}} := \phi(\mathcal{U})$.*

1. *For any probability measure $\mu_{\mathcal{U}}$ on $\mathfrak{A}^{\mathcal{U}}$, there is a unique probability measure $\tilde{\mu}_{\tilde{\mathcal{U}}}$ on $\mathfrak{A}^{\tilde{\mathcal{U}}}$ so that $\mu_{\mathcal{U}} = \phi^{\leftarrow} \tilde{\mu}_{\tilde{\mathcal{U}}}$.*
2. *If $\mu_{\mathcal{U}}$ is locally \mathbb{G} -invariant, then so is $\tilde{\mu}_{\tilde{\mathcal{U}}}$.*
3. *If $\tilde{\mu}$ is an extension of $\tilde{\mu}_{\tilde{\mathcal{U}}}$ to a \mathbb{G} -invariant probability measure on $\mathfrak{A}^{\tilde{\mathcal{M}}}$, then $\nu := \phi^{\leftarrow} \tilde{\mu}$ is an extension of $\mu_{\mathcal{U}}$ to a \mathbb{G} -invariant probability measure on $\mathfrak{A}^{\mathcal{M}}$,*

Proof of Part 1 By hypothesis, $\phi|_{\mathcal{U}}: \mathcal{U} \rightarrow \tilde{\mathcal{U}}$ is injective. Let $\psi: \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ be the inverse map, and define $\tilde{\mu}_{\tilde{\mathcal{U}}} := \psi^{\leftarrow} \mu_{\mathcal{U}}$. Thus, $\mu_{\mathcal{U}} = \phi^{\leftarrow} \tilde{\mu}_{\tilde{\mathcal{U}}}$. Since $\phi|_{\mathcal{U}}$ is injective, the measure $\tilde{\mu}_{\tilde{\mathcal{U}}}$ is the unique one satisfying this equation.

Proof of Part 2 Let $\tilde{\mathcal{V}} \subset \tilde{\mathcal{U}}$, and $\tilde{\mathbf{c}} \in \mathfrak{A}^{\tilde{\mathcal{V}}}$. Suppose $\tilde{\mathbf{g}} \in \mathbb{G}$ is such that $\tilde{\mathbf{g}} \cdot \tilde{\mathcal{V}} \subset \tilde{\mathcal{U}}$ as well. We want to show:

$$\tilde{\mu}_{\tilde{\mathcal{U}}}[\text{Shift}^{\tilde{\mathbf{g}}} \tilde{\mathbf{c}}] = \tilde{\mu}_{\tilde{\mathcal{U}}}[\tilde{\mathbf{c}}]$$

Let $\mathcal{V} := \psi(\tilde{\mathcal{V}}) \subset \mathcal{U}$, and let $\mathbf{c} := \mathfrak{A}^{\phi}(\tilde{\mathbf{c}})$, where $\mathfrak{A}^{\phi}: \mathfrak{A}^{\tilde{\mathcal{V}}} \rightarrow \mathfrak{A}^{\mathcal{V}}$ is as defined by equation (2) near the beginning of Section 6. Thus, if $\tilde{\mathbf{c}} = \{\tilde{c}_v\}_{v \in \tilde{\mathcal{V}}}$, then $\mathbf{c} = \{c_v\}_{v \in \mathcal{V}}$, where, for all $v \in \mathcal{V}$, $c_v := \tilde{c}_{\phi(v)}$.

Let \mathbf{C} be the cylinder set in $\mathfrak{A}^{\mathcal{U}}$ associated to \mathbf{c} (and likewise, $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{c}}$). Thus, $\tilde{\mathbf{C}} = \mathfrak{A}^{\psi}(\mathbf{C})$. Since $\tilde{\mathcal{M}}$ is an envelope, there is a $\mathbf{g} \in \mathbb{G}$ satisfying condition (E2). By (E2)(1), $\text{Shift}^{\mathbf{g}} \mathbf{C}$ is also a cylinder set in $\mathfrak{A}^{\mathcal{U}}$, and since $\mu_{\mathcal{U}}$ is locally \mathbb{G} -invariant, $\mu_{\mathcal{U}}[\text{Shift}^{\mathbf{g}} \mathbf{C}] = \mu_{\mathcal{U}}[\mathbf{C}]$.

Claim 1 $\mathfrak{A}^{\psi}(\text{Shift}^{\mathbf{g}} \mathbf{C}) = \text{Shift}^{\tilde{\mathbf{g}}} \tilde{\mathbf{C}}$

Proof Let $\tilde{\mathbf{a}} := [\tilde{a}_{\tilde{u}}]_{\tilde{u} \in \tilde{\mathcal{U}}} \in \mathfrak{A}^{\tilde{\mathcal{U}}}$, and suppose that $\tilde{\mathbf{a}} = \mathfrak{A}^{\psi}(\mathbf{a})$, where $\mathbf{a} := [a_u]_{u \in \mathcal{U}} \in \mathfrak{A}^{\mathcal{U}}$. Then $(\tilde{\mathbf{a}} \in \mathfrak{A}^{\psi}(\text{Shift}^{\mathbf{g}} \mathbf{C})) \iff (\mathbf{a} \in \text{Shift}^{\mathbf{g}} \mathbf{C}) \iff (\forall v \in \mathcal{V}, a_{(\mathbf{g},v)} = c_v) \iff$

$$(1) (\forall v \in \mathcal{V}, \tilde{a}_{(\tilde{\mathbf{g}},\phi(v))} = \tilde{c}_{\phi(v)}) \iff (\tilde{\mathbf{a}} \in \text{Shift}^{\tilde{\mathbf{g}}} \tilde{\mathbf{C}}).$$

(1) Because, for all $v \in \mathcal{V}$, $\tilde{a}_{(\tilde{\mathbf{g}},\phi(v))} = \tilde{a}_{\phi(\mathbf{g},v)} = a_{\mathbf{g},v}$, and $c_v = \tilde{c}_{\phi(v)}$. ■

$$\text{Thus, } \tilde{\mu}_{\tilde{\mathcal{U}}}[\text{Shift}^{\tilde{\mathbf{g}}} \tilde{\mathbf{C}}] = \tilde{\mu}_{\tilde{\mathcal{U}}}[\mathfrak{A}^{\psi}(\text{Shift}^{\mathbf{g}} \mathbf{C})] = \mu_{\mathcal{U}}[\text{Shift}^{\mathbf{g}} \mathbf{C}] = \mu_{\mathcal{U}}[\mathbf{C}] = \tilde{\mu}_{\tilde{\mathcal{U}}}[\tilde{\mathbf{C}}].$$

Proof of Part 3 This is straightforward. ■

7 Embedding of Locally Stationary Measures

Suppose that (X, \mathcal{X}, ν) is a probability space, and T is a ν -preserving action of \mathbb{Z}^D upon X . Let $\mathcal{P}: X \rightarrow \mathfrak{A}$ be a measurable function (i.e. a \mathfrak{A} -labelled, measurable partition of X), and let $\mathcal{P}^{\mathbb{Z}^D}: X \rightarrow \mathfrak{A}^{\mathbb{Z}^D}$ be the map $x \mapsto [\mathcal{P}(T^n(x))]_{\mathbf{n} \in \mathbb{Z}^D}$. The projection of μ through $\mathcal{P}^{\mathbb{Z}^D}$ is then a stationary probability measure on $\mathfrak{A}^{\mathbb{Z}^D}$, called the stochastic process induced by \mathcal{P} and T . Call this measure η .

Suppose that $\mathcal{U} \subset \mathbb{Z}^D$, and $\mu_{\mathcal{U}} \in \text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathcal{U}}]$. The map \mathcal{P} is an embedding of $\mu_{\mathcal{U}}$ in the system $(X, \mathcal{X}, \nu; T)$ if $\text{pr}_{\mathcal{U}}^*[\eta] = \mu_{\mathcal{U}}$. When can $\mu_{\mathcal{U}}$ be thus embedded?

Theorem 11 *Suppose that $\mathcal{U} \subset \mathbb{Z}^D$ is finite, and that $\mu_{\mathcal{U}}$ lies in the interior of $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$. Suppose that $(X, \mathcal{X}, \nu; T)$ is ergodic. Then $\mu_{\mathcal{U}}$ can be embedded in $(X, \mathcal{X}, \nu; T)$.*

Proof We will first show how to construct an “approximate” embedding for $\mu_{\mathcal{U}}$. The approximation method involves a certain degree of error, which can be exactly characterized and then compensated for.

Suppose $U \in \mathbb{N}$, so that $\mathcal{U} \subset \mathcal{B}(U)$. Let $\mu \in \text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathbb{Z}^D}]$ be an extension of $\mu_{\mathcal{U}}$. Then for any $N > 0$, $\mu_{\mathcal{B}(N)} := \text{pr}_{\mathcal{B}(N)}^*[\mu]$ is a locally stationary probability measure on $\mathfrak{A}^{\mathcal{B}(N)}$. Also, if $\mathcal{U}_0 \subset \mathcal{B}(N)$ is any translation of \mathcal{U} , then $\text{pr}_{\mathcal{U}_0}^*[\mu_{\mathcal{B}(N)}] = \mu_{\mathcal{U}_0}$, where $\mu_{\mathcal{U}_0}$ is the obvious “translation” of $\mu_{\mathcal{U}}$ to the domain \mathcal{U}_0 .

The Rokhlin Tower Lemma for \mathbb{Z}^D -actions says that, for any $\epsilon > 0$ and $N \in \mathbb{N}$, there is a subset $R \in \mathcal{X}$ so that the disjoint union:

$$\bigsqcup_{\mathbf{n} \in \mathcal{B}(N+U)} T^{\mathbf{n}}(R)$$

has measure greater than $1 - \epsilon$.

Let $x \in X$ be a generic point for R , and suppose we look at the “name” of x with respect to the partition $\{R, X \setminus R\}$: for all $\mathbf{n} \in \mathbb{Z}^D$, colour the point \mathbf{n} “black” if $T^{\mathbf{n}}x \in R$, and “white” otherwise. Let $\mathcal{R} \subset \mathbb{Z}^D$ be the set of “black” points. The Rokhlin Tower condition is equivalent to saying that the union:

$$\bigsqcup_{\mathbf{r} \in \mathcal{R}} (\mathcal{B}(N + U) + \mathbf{r})$$

is disjoint, and has Cesáro density greater than $1 - \epsilon$ in \mathbb{Z}^D .

To define a measurable function $\mathcal{P}: X \rightarrow \mathfrak{A}$, we will provide a scheme to determine its value at every point in the \mathbb{Z}^D -orbit of x , in terms of the $\{R, X \setminus R\}$ -name of x (this is sometimes called “colouring the name of x ”). The scheme well-defines the values of \mathcal{P} on the orbit of every generic point in X —thus, it defines \mathcal{P} almost everywhere on X .

Defining the value of \mathcal{P} on the \mathbb{Z}^D -orbit of x is equivalent to defining a function $\mathbf{p}: \mathbb{Z}^D \rightarrow \mathfrak{A}$ —in other words, a configuration. Do this as follows: Let $\phi: \mathcal{R} \rightarrow \mathfrak{A}^{\mathcal{B}(N)}$ be some function so that, for each $\mathbf{a} \in \mathfrak{A}^{\mathcal{B}(N)}$, the Cesáro density of the subset $\phi^{-1}(\mathbf{a})$ inside \mathcal{R} is equal to $\mu_{\mathcal{B}(N)}[\mathbf{a}]$ (since the set \mathcal{R} itself has a well-defined Cesáro density,

such a function can always be constructed). For each $\mathbf{u} \in \mathcal{R}$, let $\mathbf{p}_{\mathcal{B}(N)+\mathbf{u}} = \phi(\mathbf{u})$. This immediately defines \mathbf{p} on “most” of \mathbb{Z}^D . Now, fix some $\mathfrak{a} \in \mathfrak{A}$, and label *all* remaining points in \mathbb{Z}^D with the symbol \mathfrak{a} .

The function \mathcal{P} induces a stationary probability measure η on $\mathfrak{A}^{\mathbb{Z}^D}$. $\eta_{\mathfrak{U}} := \text{pr}_{\mathfrak{U}}^*[\eta]$ is “close” to $\mu_{\mathfrak{U}}$, but slightly “enriched” in words that contain big blocks of the “ \mathfrak{a} ” symbol, while impoverished in words that don’t. If we fix $\epsilon > 0$ and $N \in \mathbb{N}$, then $\eta_{\mathfrak{U}} = F_{\epsilon,N}[\mu_{\mathfrak{U}}]$, where $F_{\epsilon,N}: \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}] \rightarrow \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]$ is an *affine function*.

So, if we want to actually produce the measure $\mu_{\mathfrak{U}}$ as an *outcome* of this procedure, we must find some $\nu_{\mathfrak{U}} \in \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]$, so that $\mu_{\mathfrak{U}} = F_{\epsilon,N}[\nu_{\mathfrak{U}}]$. In other words, in order to use this construction to build an embedding of $\mu_{\mathfrak{U}}$ within X , we must find some N and ϵ so that $\mu_{\mathfrak{U}} \in I_{N,\epsilon} := F_{N,\epsilon}(\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}])$.

Claim 1 For any $\delta > 0$, there exist ϵ and N so that $\text{Lbsg}[I_{\epsilon,N}] \geq (1 - \delta) \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]]$, where *Lbsg* is the Lebesgue measure.

Proof $F_{\epsilon,N}$ is affine, and thus, differentiable with a constant derivative, $D_{\epsilon,N}$. For any $\delta_1 > 0$, we can find a small enough ϵ and large enough N that, for every $\mu_{\mathfrak{U}} \in \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]$, $\|F_{\epsilon,N}[\mu_{\mathfrak{U}}] - \mu_{\mathfrak{U}}\|_{\text{var}} < \delta_1$. Thus, for any $\delta_2 > 0$, we can make δ_1 small enough so that $\|D_{\epsilon,N} - \text{Id}\|_{\infty} < \delta_2$ (where $\|\cdot\|_{\infty}$ is the operator norm). Thus, for any δ , we can in turn make δ_2 small enough that the determinant of $D_{\epsilon,N}$ is within δ of 1. Thus, for large enough N and small enough ϵ , $F_{\epsilon,N}: \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}] \rightarrow I_{\epsilon,N}$ is a diffeomorphism, and, if *Lbsg* is the Lebesgue measure, then $\text{Lbsg}[I_{\epsilon,N}] \geq (1 - \delta) \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]]$. ■

Claim 2 For any μ in the interior of $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]$ there exist ϵ and N so that $\mu \in I_{\epsilon,N}$.

Proof Identify $\text{Meas}[\mathfrak{A}^{\mathfrak{U}}; \mathbb{R}]$ with $\mathbb{R}^{\mathfrak{A}^{\mathfrak{U}}}$, endowed with an inner product. $I_{\epsilon,N}$ is convex, so if $\mu \in \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}] \setminus I_{\epsilon,N}$, then there is some unit vector $\mathbf{v} \in \mathbb{R}^{\mathfrak{A}^{\mathfrak{U}}}$, so that $I_{\epsilon,N} \subset \{\mathbf{w} \in \mathbb{R}^{\mathfrak{A}^{\mathfrak{U}}}; \langle \mathbf{w} - \mu, \mathbf{v} \rangle < 0\}$. Fix μ , and regard $m_{\mathbf{v}}$ as a function of \mathbf{v} . The set $\{\mathbf{w} \in \mathbb{R}^{\mathfrak{A}^{\mathfrak{U}}}; \langle \mathbf{w} - \mu, \mathbf{v} \rangle \geq 0\} \cap \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]$ has nontrivial Lebesgue measure $m_{\mathbf{v}} \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]]$, for some $m_{\mathbf{v}} > 0$. Since the unit sphere in $\mathbb{R}^{\mathfrak{A}^{\mathfrak{U}}}$ is compact, there is some $M > 0$ so that $m_{\mathbf{v}} \geq M$ for all \mathbf{v} in the sphere.

Let $\delta < M$, and, by Claim 1, find ϵ and N so that $\text{Lbsg}[I_{\epsilon,N}] \geq (1 - \delta) \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]]$. Then we have $M \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]] > \delta \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]] > \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}] \setminus I_{\epsilon,N}] \geq \text{Lbsg}[\{\mathbf{w} \in \mathbb{R}^{\mathfrak{A}^{\mathfrak{U}}}; \langle \mathbf{w} - \mu, \mathbf{v} \rangle > 0\} \cap \text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]] > M \cdot \text{Lbsg}[\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]]$, a contradiction. ■

We conclude that any point μ in the interior of $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathfrak{U}}]$ is in $I_{\epsilon,N}$ for some ϵ and N , and thus, can be “embedded” in the system $(X, \mathcal{X}, \mu; T)$ via the aforementioned construction. ■

8 (Quasi)Periodic, Ergodic, and Mixing Extensions

8.1 Periodic Probability Measures

If $\mathbb{P} \subset \mathbb{N}^D$, then a configuration $\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}^D}$ is called \mathbb{P} -periodic if, for all $n \in \mathbb{Z}^D$ and $p \in \mathbb{P}$, $a_{n+p} = a_n$. If $\langle \mathbb{P} \rangle$ is the sublattice generated by \mathbb{P} , and $\tilde{\mathcal{M}} := \mathbb{Z}^D / \langle \mathbb{P} \rangle$, with \mathbb{Z}^D acting upon $\tilde{\mathcal{M}}$ by translation, then $\tilde{\mathcal{M}}$ is \mathbb{Z}^D -module. The quotient map $\phi: \mathbb{Z}^D \rightarrow \tilde{\mathcal{M}}$ is a homomorphism of \mathbb{Z}^D -modules. Configuration \mathbf{a} is \mathbb{P} -periodic if and only if $\mathbf{a} = \mathfrak{A}^\phi \tilde{\mathbf{a}}$, for some word $\tilde{\mathbf{a}} \in \mathfrak{A}^{\tilde{\mathcal{M}}}$ (in the notation of Section 6).

In general, if \mathcal{M} is a \mathbb{G} -module, $\tilde{\mathcal{M}}$ is another \mathbb{G} -module, and $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ is a \mathbb{G} -module homomorphism, then we will say that an element $\mathbf{a} \in \mathfrak{A}^{\mathcal{M}}$ is $\tilde{\mathcal{M}}$ -periodic if $\mathbf{a} = \mathfrak{A}^\phi[\tilde{\mathbf{a}}]$, for some $\tilde{\mathbf{a}} \in \mathfrak{A}^{\tilde{\mathcal{M}}}$.

If μ is a \mathbb{G} -invariant measure on $\mathfrak{A}^{\mathcal{M}}$, then μ is $\tilde{\mathcal{M}}$ -periodic if the elements of the space $(\mathfrak{A}^{\mathcal{M}}, \mu)$ are μ -almost surely $\tilde{\mathcal{M}}$ -periodic. This is the case if and only if there is a \mathbb{G} -invariant measure $\tilde{\mu}$ on $\mathfrak{A}^{\tilde{\mathcal{M}}}$, such that $\mu = \phi^\leftarrow[\tilde{\mu}]$.

8.2 Periodic Extensions

Suppose that $\mathcal{U} \subset \mathcal{M}$, and $\mu_{\mathcal{U}}$ is a locally \mathbb{G} -invariant measure upon $\mathfrak{A}^{\mathcal{U}}$. Can we extend $\mu_{\mathcal{U}}$ to a periodic measure on $\mathfrak{A}^{\mathcal{M}}$?

Theorem 12 Suppose that $\tilde{\mathcal{M}}$ is a finite \mathbb{G} -module, a quotient of \mathcal{M} via the map $\phi: \mathcal{M} \rightarrow \tilde{\mathcal{M}}$, and an envelope for \mathcal{U} . Let $H(\tilde{\mathcal{M}})$ be the constant described in Theorem 7.

Let $\tilde{\nu}$ be a \mathbb{G} -invariant measure on $\mathfrak{A}^{\tilde{\mathcal{M}}}$, with full support, and let

$$\epsilon := \frac{1}{H(\tilde{\mathcal{M}})} \cdot \min_{\tilde{\mathbf{a}} \in \mathfrak{A}^{\tilde{\mathcal{U}}}} \tilde{\nu}[\tilde{\mathbf{a}}].$$

Let $\nu = (\mathfrak{A}^\phi)^* \tilde{\nu}$, and let $\nu_{\mathcal{U}} := \text{pr}_{\mathcal{U}}^*[\nu]$. If $\mu_{\mathcal{U}}$ is any locally \mathbb{G} -invariant measure on $\mathfrak{A}^{\mathcal{U}}$ so that $\|\mu_{\mathcal{U}} - \nu_{\mathcal{U}}\|_{\text{var}} < \epsilon$, then $\mu_{\mathcal{U}}$ can be extended to a \mathbb{G} -invariant, $\tilde{\mathcal{M}}$ -periodic probability measure on $\mathfrak{A}^{\mathcal{M}}$.

Proof Let $\tilde{\mathcal{U}} := \phi(\mathcal{U}) \subset \tilde{\mathcal{M}}$. By Part 1 of Theorem 10, the measure $\tilde{\mu}_{\tilde{\mathcal{U}}} := (\phi^{-1})^\leftarrow \mu_{\mathcal{U}}$ is a locally \mathbb{G} -invariant measure on $\mathfrak{A}^{\tilde{\mathcal{U}}}$. Further, if $\tilde{\nu}_{\tilde{\mathcal{U}}} := \text{pr}_{\tilde{\mathcal{U}}}^*[\tilde{\nu}]$, then $\|\tilde{\mu}_{\tilde{\mathcal{U}}} - \tilde{\nu}_{\tilde{\mathcal{U}}}\|_{\text{var}} < \epsilon$. Since $\tilde{\mathcal{M}}$ is finite, we can apply Theorem 7, and extend $\tilde{\mu}_{\tilde{\mathcal{U}}}$ to a \mathbb{G} -invariant measure, $\tilde{\mu}$, on all of $\mathfrak{A}^{\tilde{\mathcal{M}}}$.

Now, define $\mu := \phi^\leftarrow[\tilde{\mu}]$. Then μ is a $\tilde{\mathcal{M}}$ -periodic, \mathbb{G} -invariant measure by construction, and also, $\text{pr}_{\mathcal{U}}^*[\mu] = \mu_{\mathcal{U}}$. ■

Corollary 13 The set $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ has nontrivial interior in the space $\text{Meas}[\mathfrak{A}^{\mathcal{U}}; \mathbb{R}]$, and the set of $\tilde{\mathcal{M}}$ -periodically extendible measures has nontrivial interior within $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$.

Proof Let ρ be any probability measure on \mathfrak{A} with full support, and let $\mu_{\mathcal{U}} := \rho^{\mathcal{U}}$ be the product measure on $\mathfrak{A}^{\mathcal{U}}$. In the notation of Theorem 12, $\rho^{\tilde{\mathcal{M}}}$ is a \mathbb{G} -invariant extension of $\rho^{\tilde{\mathcal{U}}}$, with full support, and induces a $\tilde{\mathcal{M}}$ -periodic extension of $\mu_{\mathcal{U}}$ to

$\mathfrak{A}^{\mathcal{M}}$. By Theorem 12, all measures in an open ball around $\mu_{\mathcal{U}}$ also have $\tilde{\mathcal{M}}$ -periodic extensions. ■

Corollary 14 *Suppose $\mathcal{U} \subset \mathbb{Z}^D$ is finite, and fits inside a box of size $Q_1 \times Q_2 \times \dots \times Q_D$. Suppose that $\mathbf{P} := (P_1, \dots, P_D)$, where $P_1 \geq 2Q_1, P_2 \geq 2Q_2, \dots, P_D \geq 2Q_D$, and let ν be a \mathbf{P} -periodic, stationary probability measure on $\mathfrak{A}^{\mathbb{Z}^D}$. Let $\nu_{\mathcal{U}} := \text{pr}_{\mathcal{U}}^*[\nu]$.*

There is an $\epsilon > 0$ (a function of \mathbf{P} and ν), so that, if $\mu_{\mathcal{U}}$ is any locally stationary probability measure on $\mathfrak{A}^{\mathcal{U}}$ within ϵ of $\nu_{\mathcal{U}}$ in total variation norm, then $\mu_{\mathcal{U}}$ has a \mathbf{P} -periodic extension. ■

For any $\mathbf{P} := (P_1, \dots, P_D)$, let $\text{Meas}^{\mathbf{P}}[\mathfrak{A}^{\mathbb{Z}^D}]$ denote the set of \mathbf{P} -periodic, stationary processes.

If $\mathcal{U} \subset \mathbb{Z}^D$, then let $\text{Meas}^{\mathbf{P}}[\mathfrak{A}^{\mathcal{U}}]$ denote the set of \mathbf{P} -periodically-extendible measures: those elements of $\text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathcal{U}}]$ having an extension that is \mathbf{P} -periodic. The following facts are not difficult to verify:

- $\text{Meas}^{\mathbf{P}}[\mathfrak{A}^{\mathcal{U}}]$ is a closed, convex set.
- If $\mu \in \text{Meas}^{\mathbf{P}}[\mathfrak{A}^{\mathcal{U}}]$ and $\nu \in \text{Meas}^{\mathbf{Q}}[\mathfrak{A}^{\mathcal{U}}]$, then any convex combination of μ and ν is inside $\text{Meas}^{\mathbf{R}}[\mathfrak{A}^{\mathcal{U}}]$, where, for each $d \in [1 \dots D]$, R_d is the lowest common multiple of P_d and Q_d .

Let $\text{Meas}^{\text{per}}[\mathfrak{A}^{\mathcal{U}}]$ be the set of all locally stationary measures possessing a periodic extension of any periodicity. It follows that $\text{Meas}^{\text{per}}[\mathfrak{A}^{\mathcal{U}}]$ is also a convex set.

8.3 Essentially Aperiodic Measures

Not every extendible measure has a periodic extension. This follows from the existence of *essentially aperiodic tile systems*—that is, sets of tiles which can tile the plane, but *only* in an aperiodic fashion. In [19], Raphael Robinson exhibits a collection of six “notched” square tiles, which, along with their 4 rotations, will tile the plane, but *only* in an aperiodic fashion. We can code these six tiles as six 3×3 matrices in the alphabet $\mathfrak{A} := \{0, a, A, b, B, c, C\}$

A C A	a c a	a b a
B 0 d	c 0 c	c 0 c
A B A	a C a	a B a
a C a	a b a	a b a
B 0 C	c 0 c	b 0 b
a B a	a b a	a B a

Each tile has a “0” symbol in its center, surrounded by four “corners” and four “edges”. The tiles must be put together so that these corners and edges “match” according to the following *mapping rules*:

- “b” edges must be matched to “B” edges.
- “c” edges must be matched to “C” edges.
- Where four tiles meet, exactly three corners must be of type “a”, and one of type “A”.

These matching rules can be encoded as a subshift of finite type on the alphabet \mathfrak{A} , defined by some subset $\mathfrak{R} \subset \mathfrak{A}^{\mathcal{U}}$, where $\mathcal{U} := [1 \cdots 3]^2$. Any configuration in $\langle \mathfrak{R} \rangle$ corresponds to some Robinson tiling. Now let μ be a stationary probability measure on $\langle \mathfrak{R} \rangle$, and let $\mu_{\mathcal{U}} := \text{pr}_{\mathcal{U}}^*[\mu]$. Then $\mu_{\mathcal{U}}$ is a locally stationary measure, and $\text{supp}[\mu_{\mathcal{U}}] = \mathfrak{R}$.

We claim that $\mu_{\mathcal{U}}$ is “essentially aperiodic”. To see this, suppose that ν was any extension of $\mu_{\mathcal{U}}$. Then $\text{supp}[\nu] \subset \langle \mathfrak{R} \rangle$, and thus, almost every configuration in the probability space $(\mathfrak{A}^{\mathbb{Z}^2}, \nu)$ is aperiodic.

8.4 Essentially Periodic Measures

At the opposite extreme are *essentially periodic* measures: locally stationary measures which *only* have periodic extensions.

For example, let $\mathfrak{A} := \{0, 1\}$ and $\mathcal{U} := [1 \cdots 9] \times \{0, 1\}$, and let $\mathfrak{B} \subset \mathfrak{A}^{\mathcal{U}}$ be the set:

$$\left(\begin{array}{c} \boxed{\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}} \\ \vdots \\ \boxed{\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}} \\ \boxed{\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}} \\ \boxed{\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}} \\ \boxed{\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}} \end{array} \right)$$

In other words, all blocks are of the form $\boxed{\begin{array}{cc} w_1 & 0 \\ w_0 & 0 \end{array}}$ where w_0 and w_1 are successive 8-bit binary numbers. Let $\mathfrak{B} \subset \mathfrak{A}^{\mathcal{U}}$ be the set containing all elements of \mathfrak{B} and all their horizontal cyclic permutations. \mathfrak{B} defines a subshift of finite type, which contains only the orbit of a single, periodic configuration, having horizontal periodicity 9, and vertical periodicity 256. Call this configuration \mathbf{a}

If $\mu_{\mathcal{U}}$ is the measure on $\mathfrak{A}^{\mathcal{U}}$ assigning equal mass to each of the 2304 elements of \mathfrak{B} , then $\mu_{\mathcal{U}}$ has only one stationary extension: the measure μ which assigns equal mass to each of the 2304 distinct translates of \mathbf{a} . Thus, $\mu_{\mathcal{U}}$ is *essentially periodic*, with period 256×9 .

Note that the periodicity 256×9 is much larger than 2×9 , which was the size of the initial domain \mathcal{U} . Indeed, as this argument makes clear, the periodicity of essentially periodic measure can be made to grow exponentially with the size of the initial domain.

8.5 Ergodic Extensions

A stationary probability measure μ on $\mathfrak{A}^{\mathbb{Z}^D}$ is called *ergodic* if any measurable subset $U \subset \mathfrak{A}^{\mathbb{Z}^D}$ which is invariant under all shifts has μ -measure either zero or one. The set of ergodic measures on $\mathfrak{A}^{\mathbb{Z}^D}$, which we denote by “ $\text{Meas}^{\text{erg}}[\mathfrak{A}^{\mathbb{Z}^D}]$ ”, is exactly the set of *extremal points* of $\text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathbb{Z}^D}]$ (see [4] or [26]). Hence, every stationary measure can be approximated arbitrarily well as a convex combination of ergodic measures.

If $U \subset \mathbb{Z}^D$, and $\mu_U \in \text{Meas}^{\text{stat}}[\mathfrak{A}^U]$, then we say μ is *ergodically extendible* if it can be extended to an ergodic measure on $\mathfrak{A}^{\mathbb{Z}^D}$. The set of ergodically extendible measures will be written as “ $\text{Meas}^{\text{erg}}[\mathfrak{A}^U]$ ”. Since the map $\text{pr}_U^* : \text{Meas}[\mathfrak{A}^{\mathbb{Z}^D}] \rightarrow \text{Meas}[\mathfrak{A}^U]$ is linear, any extremal point of $\text{Meas}^{\text{ext}}[\mathfrak{A}^U]$ has a pr_U^* -preimage which is extremal in $\text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathbb{Z}^D}]$. As a consequence, every extremal point of $\text{Meas}^{\text{ext}}[\mathfrak{A}^U]$ is in $\text{Meas}^{\text{erg}}[\mathfrak{A}^U]$. Hence, every extendible measure on \mathfrak{A}^U can be approximated arbitrarily well as a convex combination of ergodically extendible measures.

We will see in Section 8.6 that, in fact, “almost all” extendible measures are ergodically extendible. However, not every extendible measure is. To see this, suppose that $U \subset \mathbb{Z}^D$ is some finite domain, let \mathfrak{A} and \mathfrak{B} be two *disjoint* alphabets, and suppose that $\mu_U \in \text{Meas}^{\text{stat}}[\mathfrak{A}^U]$ and $\nu_U \in \text{Meas}^{\text{stat}}[\mathfrak{B}^U]$ are two extendible probability measures. Let $\eta_U := \frac{1}{2}\mu_U + \frac{1}{2}\nu_U$. Then η_U is also extendible, and any extension of η_U is of the form $\eta := \frac{1}{2}\mu + \frac{1}{2}\nu$, where μ and ν extend μ_U and ν_U , respectively. η can never be ergodic: $\mathfrak{A}^{\mathbb{Z}^D}$ and $\mathfrak{B}^{\mathbb{Z}^D}$ are disjoint, shift-invariant subsets of $(\mathfrak{A} \sqcup \mathfrak{B})^{\mathbb{Z}^D}$, each having η -measure $\frac{1}{2}$.

Proposition 15 *Let $U \subset \mathbb{Z}^D$ be finite.*

1. *Every ergodically extendible measure on \mathfrak{A}^U is a limit point of periodically extendible measures.*
2. *$\text{Meas}^{\text{per}}[\mathfrak{A}^U]$ is a dense, convex subset of $\text{Meas}^{\text{ext}}[\mathfrak{A}^U]$.*
3. *$\text{Meas}^{\text{per}}[\mathfrak{A}^U]$ contains the entire interior of $\text{Meas}^{\text{ext}}[\mathfrak{A}^U]$.*

Proof Part 2 follows immediately from Part 1, and the fact that $\text{Meas}^{\text{per}}[\mathfrak{A}^U]$ is convex, and the fact that $\text{Meas}^{\text{ext}}[\mathfrak{A}^U]$ is the convex closure of $\text{Meas}^{\text{erg}}[\mathfrak{A}^U]$.

Proof of Part 3 This follows from Part 2, and the fact that, if C a dense, convex subset of a D -dimensional convex set K , then C contains $\text{int}[K]$. To see this, let $x \in \text{int}[K]$, and let B be an open ball around x inside of $\text{int}[K]$. Let S be the boundary of B , and let s_1, \dots, s_D be D equidistant points in S , so that their convex closure, $\text{co}\{s_1, \dots, s_D\}$ is a regular D -simplex containing the centre-point x .

Since C is dense in K , $C \cap B$ is dense in B . Thus, find elements $c_1, \dots, c_D \in C$ so that, for all $d \in [1 \cdots D]$, c_d is “very close” to s_d . Then $\text{co}\{c_1, \dots, c_D\} \subset C$ is a D -simplex “very close” to $\text{co}\{s_1, \dots, s_D\}$, and therefor contains x .

Proof of Part 1 Let $\mu_U \in \text{Meas}^{\text{erg}}[\mathfrak{A}^U]$, and let μ be an ergodic extension of μ_U . Let $\mathbf{a} \in \mathfrak{A}^{\mathbb{Z}^D}$ be a *generic configuration* for μ : in other words, for any finite subset $V \subset \mathbb{Z}^D$ and configuration $\mathbf{b} \in \mathfrak{A}^V$,

$$\mu[\mathbf{b}] = \lim_{N \rightarrow \infty} \text{Freq}[\mathbf{b} \subset \mathbf{a}; \mathfrak{B}(N)]$$

where $\mathcal{B}(N) := [0 \cdots N]^D$ is the D -dimensional cube of side length N , and

$$\begin{aligned} \text{Freq}[\mathbf{b} \subset \mathbf{a}; \mathcal{B}(N)] &:= \frac{\# \text{ of times “}\mathbf{b}\text{” appears inside } \mathbf{a}_{\mathcal{B}(N)}}{N^D} \\ &= \frac{1}{N^D} \sum_{\mathbf{n} \in \mathcal{B}(N)} \mathbb{1}\{\mathbf{a}_{\mathcal{V}+\mathbf{n}} = \mathbf{b}\} \end{aligned}$$

Such generic configurations exist, by the Birkhoff Ergodic Theorem.

In particular, for any $\epsilon > 0$, we can find a large enough N so that, for all $\mathbf{b} \in \mathfrak{A}^{\mathcal{U}}$,

$$|\mu[\mathbf{b}] - \text{Freq}[\mathbf{b} \subset \mathbf{a}; \mathcal{B}(N)]| < \frac{\epsilon}{2}$$

Suppose that all of \mathcal{U} fits inside a cube of side length U . Assume that N is so large that the U -thick boundary of $\mathcal{B}(N)$ is “relatively small”:

$$\frac{\text{Card}[\mathcal{B}(N)] - \text{Card}[\mathcal{B}(N - U)]}{\text{Card}[\mathcal{B}(N)]} < \frac{\epsilon}{2}$$

Now, identify $\mathcal{B}(N)$ with $\mathcal{N} := (\mathbb{Z}/N) \oplus \cdots \oplus (\mathbb{Z}/N)$, and treat $\mathbf{a}_{\mathcal{B}(N)}$ as an element of $\mathfrak{A}^{\mathcal{N}}$. Then this configuration, along with its N^D periodic translations on $\mathfrak{A}^{\mathcal{N}}$, defines a stationary measure on $\mathfrak{A}^{\mathcal{N}}$, which, in turn, defines an \mathcal{N} -periodic, stationary measure on $\mathfrak{A}^{\mathbb{Z}^D}$. Call this measure ν , and then let $\nu_{\mathcal{U}} := \text{pr}_{\mathcal{U}}^*[\nu]$. It is straightforward to verify that

$$\|\nu_{\mathcal{U}} - \mu_{\mathcal{U}}\| < \epsilon$$

and of course, by construction, $\nu_{\mathcal{U}} \in \text{Meas}^{\text{per}}[\mathfrak{A}^{\mathcal{U}}]$. ■

8.6 Mixing, Weak Mixing, and Quasiperiodicity

A stationary probability measure μ on $\mathfrak{A}^{\mathbb{Z}^D}$ is called *weakly mixing* if the stochastic process $(\mathfrak{A}^{\mathbb{Z}^D} \times \mathfrak{A}^{\mathbb{Z}^D}, \mu \otimes \mu)$ is ergodic. μ is called *mixing* if, for any measurable $A, B \subset \mathfrak{A}^{\mathbb{Z}^D}$ of nonzero measure, any any sequence $\{\mathbf{n}_k\}_{k \in \mathbb{N}} \subset \mathbb{Z}^D$ tending to infinity, $\lim_{k \rightarrow \infty} \mu[A \cap \text{Shift}^{\mathbf{n}_k} B] = \mu[A] \cdot \mu[B]$. A function $\phi \in \mathbf{L}^2(\mathfrak{A}^{\mathbb{Z}^D}, \mu)$ is an *eigenfunction* of the system $(\mathfrak{A}^{\mathbb{Z}^D}, \mu)$ if there is a group homomorphism $\chi: \mathbb{Z}^D \rightarrow \mathbb{T}^1$ such that, for all $\mathbf{n} \in \mathbb{Z}^D$, $\text{Shift}^{\mathbf{n}}(\phi) = \chi(\mathbf{n}) \cdot \phi$. The system is called *quasiperiodic* if $\mathbf{L}^2(\mathfrak{A}^{\mathbb{Z}^D}, \mu)$ has an orthonormal basis of eigenfunctions.

All of these concepts can be defined for any measure-preserving \mathbb{Z}^D -action on a probability space (X, \mathcal{X}, μ) . Mixing implies weak mixing implies ergodicity, but weak mixing and quasiperiodicity are mutually exclusive. Furthermore, all of these properties are *inheritable* through morphisms. If $(X, \mathcal{X}, \mu; T)$ and $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}; \hat{T})$ are two measure-preserving \mathbb{Z}^D -actions, then a *morphism* between the systems is a measure-preserving surjection $\Psi: X \rightarrow \hat{X}$ so that, for all $\mathbf{n} \in \mathbb{Z}^D$, $\Psi \circ T^{\mathbf{n}} = \hat{T}^{\mathbf{n}} \circ \Psi$. If Ψ is such a morphism, and $(X, \mathcal{X}, \mu; T)$ is ergodic (respectively: weakly mixing, mixing, or quasiperiodic), then so is $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}; \hat{T})$.

In particular, let $F: X \rightarrow \mathfrak{A}$ be a measurable function, so that F and T together induce a stationary stochastic process on $\mathfrak{A}^{\mathbb{Z}^D}$, having measure $\hat{\mu}$ (see Section 7). If $\hat{X} := \text{supp}[\hat{\mu}] \subset \mathfrak{A}^{\mathbb{Z}^D}$ and $\hat{T} := \text{Shift}$, then the map $F^{\mathbb{Z}^D}: X \rightarrow \mathfrak{A}^{\mathbb{Z}^D}$ is a morphism. Thus, if $(X, \mathcal{X}, \mu; T)$ possesses any of the aforementioned inheritable properties, so does the process $(\mathfrak{A}^{\mathbb{Z}^D}, \hat{\mu})$.

Theorem 16 *Suppose that $\mathcal{U} \subset \mathbb{Z}^D$ is finite, and that $\mu_{\mathcal{U}}$ is in the interior of $\text{Meas}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$. Then $\mu_{\mathcal{U}}$ can be extended to a stationary process μ which is any of: ergodic, mixing, weakly mixing, or quasiperiodic.*

Proof The argument is the same in all four cases. First, find a system $(X, \mathcal{X}, \nu; T)$ which is ergodic, and which also has the property in question (for the first three, this is trivial; for the fourth, it is sufficient to know that ergodic, quasiperiodic systems exist). Next, use Theorem 11 to embed $\mu_{\mathcal{U}}$ within the desired process. Let $\mu \in \text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathbb{Z}^D}]$ be the stochastic process generated by this embedding. Then μ itself has the desired property. ■

The same argument works for any other “inheritable” property of dynamical systems. The interpretation: knowledge of the local marginal $\mu_{\mathcal{U}}$ tells you basically nothing about the asymptotic dynamical properties of the process μ .

9 Decidability Questions

In Section 2.1, we showed:

It is formally undecidable whether, for a given subset $\mathfrak{B} \subset \mathfrak{A}^{\mathcal{U}}$, the set $\text{Meas}^{\text{ext}}[\mathfrak{B}]$ is nonempty.

This raises the question of whether the Extension Problem itself is formally decidable.

Let \mathbb{R}_{\dagger} be the set of all *recursively computable* (r.c.) real numbers: that is, real numbers whose decimal expansion can be generated by some Turing Machine [7]. \mathbb{R}_{\dagger} is a countable field, containing all rational and real-algebraic numbers. Let $\text{Meas}_{\dagger}[\mathfrak{A}^{\mathcal{M}}; \mathbb{R}]$ be the set of r.c., real-valued measures: those such that, if $\mathcal{V} \subset \mathcal{M}$ is finite, and $\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}$, then the measure of \mathbf{a} is an element of \mathbb{R}_{\dagger} . (Of course, some “exotic” measurable subsets of $\mathfrak{A}^{\mathbb{Z}^D}$ may have non r.c. measures). $\text{Meas}_{\dagger}[\mathfrak{A}^{\mathcal{M}}; \mathbb{R}]$ is a vector space over the field \mathbb{R}_{\dagger} .

Let $\text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{M}}]$ be the set of \mathbb{G} -invariant probability measures, etc. Clearly, when we ask about the “formal decidability” of the Extension Problem, what we are *really* referring to is the Extension Problem for r.c. measures:

If $\mathcal{U} \subset \mathcal{M}$, and $\mu_{\mathcal{U}} \in \text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$, is μ extendible to a \mathbb{G} -invariant measure on $\mathfrak{A}^{\mathcal{U}}$?

Note that we do not require the extension itself to be r.c. If a recursive decision procedure explicitly *constructs* an extension, then this extension will be r.c. by nature. However, it is conceivable that some recursive decision procedure might exist

which demonstrates the existence of an extension by “nonconstructive” means. It is conceivable that, although we can recursively decide that $\mu_{\mathcal{U}}$ is extendible, no r.c. extension exists.

A subset $S \subset \text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ is called *recursively decidable* (r.d.) if there is a Turing machine \mathbb{M} , so that, when given any $\mu \in \text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ as input, \mathbb{M} halts after some finite number of steps, and outputs either “yes” or “no”, depending upon whether or not μ is an element of S .

A subset $S \subset \text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ is called *recursively enumerable* (r.e.) if there is a Turing machine \mathbb{M} , so that, when given any integer $n \in \mathbb{N}$ as input, \mathbb{M} halts after a finite number of steps, and produces as output some measure $F_{\mathbb{M}}[n] \in S$, and so that the function $F_{\mathbb{M}}: \mathbb{N} \rightarrow S$ instantiated by \mathbb{M} is *surjective*. In other words, \mathbb{M} provides a mechanism to systematically “list” all elements of S .

Equivalently, $S \subset \text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ is *recursively enumerable* if there is a Turing machine \mathbb{M} , so that, when given any $\mu \in \text{Meas}_{\dagger}^{\mathbb{G}}[\mathfrak{A}^{\mathcal{U}}]$ as input, \mathbb{M} halts after some finite number of steps *unless* μ is *not* in S , in which case \mathbb{M} never halts.

The following facts are easy to verify: Any r.d. set is r.e., but the converse is not true. However, if both S and its complement are r.e., then S is r.d. Finally, although a countable union of r.d. sets is not necessarily itself r.d., it is still r.e. [7].

Theorem 17 *Let $\mathcal{U} \subset \mathbb{Z}^D$ be a finite subset. Then*

1. For any $\mathbf{P} \in \mathbb{N}^D$, $\text{Meas}_{\dagger}^{\mathbf{P}}[\mathfrak{A}^{\mathcal{U}}]$ is r.d.
2. $\text{Meas}_{\dagger}^{\text{per}}[\mathfrak{A}^{\mathcal{U}}]$ is r.e.
3. $\text{Meas}_{\dagger}^{\text{stat}}[\mathfrak{A}^{\mathcal{U}}] \setminus \text{Meas}_{\dagger}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ is r.e. ■

Proof of Part 1 If $\mu_{\mathcal{U}} \in \text{Meas}_{\dagger}^{\text{stat}}[\mathfrak{A}^{\mathcal{U}}]$, we want to know whether the set $S := \{\mu \in \text{Meas}_{\dagger}^{\mathbf{P}}[\mathfrak{A}^{\mathcal{U}}] ; \text{pr}_{\mathcal{U}}^*[\mu] = \mu_{\mathcal{U}}\}$ is nonempty.

Suppose $\mathbf{P} := (P_1, \dots, P_D)$. Let $\tilde{\mathcal{M}} := (\mathbb{Z}/P_1) \oplus \dots \oplus (\mathbb{Z}/P_D)$, and suppose that \mathcal{U} maps bijectively into the subset $\tilde{\mathcal{U}} \subset \tilde{\mathcal{M}}$ via the quotient map from $\mathbb{Z}^D \rightarrow \tilde{\mathcal{M}}$. Let $\tilde{\mu}_{\tilde{\mathcal{U}}} \in \text{Meas}_{\dagger}^{\text{stat}}[\mathfrak{A}^{\tilde{\mathcal{U}}}]$ be the projected image of $\mu_{\mathcal{U}}$.

The vector space of \mathbf{P} -periodic, signed measures on $\mathfrak{A}^{\mathcal{U}}$ is linearly isomorphic to the finite dimensional vector space $\text{Meas}[\mathfrak{A}^{\tilde{\mathcal{M}}} ; \mathbb{R}]$. The image of S under this isomorphism is the affine set

$$\tilde{S} := \{\mu \in \text{Meas}[\mathfrak{A}^{\tilde{\mathcal{M}}} ; \mathbb{R}] ; \mu \text{ a stationary probability measure, and } \text{pr}_{\tilde{\mathcal{U}}}^*[\mu] = \tilde{\mu}_{\tilde{\mathcal{U}}}\}.$$

\tilde{S} is the solution set of a finite system of linear equations and linear inequalities in μ :

- $\mu[\mathfrak{A}^{\tilde{\mathcal{M}}}] = 1$.
- For all $n \in \mathbb{Z}^D$, $\text{Shift}_*^n \mu = \mu$.
- $\text{pr}_{\tilde{\mathcal{U}}}^*[\mu] = \tilde{\mu}_{\tilde{\mathcal{U}}}$.
- For all $\mathbf{a} \in \mathfrak{A}^{\tilde{\mathcal{M}}}$, $\mu[\mathbf{a}] \geq 0$.

Thus, it is r.d. whether \tilde{S} is nonempty, and thus, whether $\mu_{\mathcal{U}}$ has a \mathbf{P} -periodic extension.

Proof of Part 2 $\text{Meas}_\dagger^{\text{per}}[\mathfrak{A}^{\mathcal{U}}]$ is a countable union of recursively decidable sets, and thus, r.e.

Proof of Part 3 Suppose $\mu \in \text{Meas}[\mathfrak{A}^{\mathbb{Z}^D}; \mathbb{C}]$ has Fourier transform $[\hat{\mu}_\chi]_{\chi \in \widehat{\mathfrak{A}^{\mathbb{Z}^D}}}$, let $\mathcal{V} \subset \mathbb{Z}^D$ be a finite subset, and let $\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}$. It is easy to verify:

$$\mu[\mathbf{a}] = \sum_{\chi \in \widehat{\mathfrak{A}^{\mathcal{V}}}} \hat{\mu}_\chi \cdot \overline{\chi(\mathbf{a})}$$

Thus, if $\mu_{\mathcal{U}} \in \text{Meas}^{\text{stat}}[\mathfrak{A}^{\mathcal{U}}]$, then by Theorems 4 and 5, μ is an extension of $\mu_{\mathcal{U}}$ if and only if:

- For all $\chi \in \widehat{\mathfrak{A}^{\mathcal{U}}}$, $\hat{\mu}_\chi = \langle \mu_{\mathcal{U}}, \chi \rangle$
- For all $\mathbf{n} \in \mathbb{Z}^D$, and all $\chi \in \widehat{\mathfrak{A}^{\mathbb{Z}^D}}$, if $\xi := \chi \circ \text{Shift}^{\mathbf{n}}$ then $\hat{\mu}_\chi = \hat{\mu}_\xi$.
- For all finite $\mathcal{V} \subset \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}^{\mathcal{V}}$, $\sum_{\chi \in \widehat{\mathfrak{A}^{\mathcal{V}}}} \hat{\mu}_\chi \cdot \overline{\chi(\mathbf{a})} > 0$.

Thus, an extension for $\mu_{\mathcal{U}}$ is equivalent to a set of Fourier coefficients satisfying a countable collection of linear equations and inequalities.

For all $N \in \mathbb{N}$, let $\mathcal{B}(N) := [0 \cdots N]^D$, and let $\Xi_N := \widehat{\mathfrak{A}^{\mathcal{B}(N)}}$. If N is large enough that $\mathcal{U} \subset \mathcal{B}(N)$, then we can start by trying to define all the Fourier Coefficients in the set $\{\mu_\chi; \chi \in \Xi_N\}$. The three sets of linear constraints listed above now become a finite system of linear equations and inequalities—if the solution set is nonempty, call it S_N .

Claim 1 *Suppose that, for all $N \in \mathbb{N}$, the set S_N is nonempty. Then $\mu_{\mathcal{U}}$ is extendible.*

Proof S_N is a compact subset of the finite dimensional vector space \mathbb{C}^{Ξ_N} . Furthermore, if S_{N+1} is also nonempty, then any vector in S_{N+1} , when projected to \mathbb{C}^{Ξ_N} , determines an element in S_N . Call this projection map pr_N .

Fix N , and, for all $M > N$, let $\tilde{S}_N^M := \text{pr}_N \circ \text{pr}_{N+1} \circ \cdots \circ \text{pr}_{M-1}(S_M)$, a nonempty compact subset of S_N . Also, $\tilde{S}_N^{M+1} \supset \tilde{S}_N^{M+2} \supset \tilde{S}_N^{M+3} \supset \cdots$. Thus, $\tilde{S}_N := \bigcap_{M > N} \tilde{S}_N^M$ is a nonempty compact subset. Further, $\text{pr}_N(\tilde{S}_{N+1}) = \tilde{S}_N$. Thus, any element of \tilde{S}_N can be “extended” to an element of \tilde{S}_{N+1} , which can then be “extended” to \tilde{S}_{N+1} , etc.

Pick any element $\hat{\mu}_N \in \tilde{S}_N$, and inductively extend it in this fashion, producing $\hat{\mu}_M \in \tilde{S}_M$, for every $M > N$. Once this is done, the collection of vectors $\{\hat{\mu}_M\}_{M > N}$ defines a single element $\hat{\mu} \in \mathbb{C}^{\widehat{\mathfrak{A}^{\mathbb{Z}^D}}}$. $\hat{\mu}$ is the Fourier transform of some measure μ , and by construction, μ is a stationary probability measure, and an extension of $\mu_{\mathcal{U}}$. ■

Hence, if $\mu_{\mathcal{U}}$ is not extendible, then, by contradiction, there must be some $N \in \mathbb{N}$ so that S_N is empty. Since S_N is the solution set of a finite system of linear equations and inequalities, it is r.d. whether S_N is empty.

Hence, by successively checking the nonemptiness of S_N for each $N \in \mathbb{N}$, we have a recursive procedure which will halt if $\mu_{\mathcal{U}}$ is not extendible, and tell us so. (If $\mu_{\mathcal{U}}$ is extendible, however, the procedure will never halt). Thus we can recursively enumerate the elements of $\text{Meas}_\dagger^{\text{stat}}[\mathfrak{A}^{\mathcal{U}}] \setminus \text{Meas}_\dagger^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$. ■

Theorem 18 *Let $\mathcal{U} \subset \mathbb{Z}^D$ be finite. The set $\text{Meas}_\dagger^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ is not r.e.*

Proof Recall that, if $\mathfrak{T} \subset \mathfrak{A}^{\mathcal{U}}$, then $\langle \mathfrak{T} \rangle$ is the associated *subshift of finite type* (see Section 2.1). Let $\mathbf{N} := \{\mathfrak{T} ; \langle \mathfrak{T} \rangle \text{ is not trivial}\}$, and let $\mathbf{T} := \{\mathfrak{T} ; \langle \mathfrak{T} \rangle \text{ is trivial}\}$. Recall that \mathbf{N} is not r.d. (see [19], [2], or [9]).

Claim 1 *Suppose $\mathfrak{T} \subset \mathfrak{A}^{\mathcal{U}}$. If $\mathfrak{T} \in \mathbf{T}$, then there is some $N \in \mathbb{N}$ so that no configuration in $\mathfrak{A}^{\mathcal{B}(N)}$ is \mathfrak{T} -admissible.*

Proof Suppose that, for every $N \in \mathbb{N}$, there was a configuration $\mathbf{a}^{[N]} \in \mathfrak{A}^{\mathcal{B}(N)}$ that was \mathfrak{T} -admissible, that is: for all $n \in \mathcal{B}(N)$, if $n + \mathcal{U} \subset \mathcal{B}(N)$, then $\mathbf{a}^{[N]}_{n+\mathcal{U}} \in \mathfrak{T}$. Extend $\mathbf{a}^{[N]}$ to an element of $\mathfrak{A}^{\mathcal{Z}^D}$ by filling all the remaining entries in some arbitrary fashion—call the extended configuration $\mathbf{b}^{[N]}$.

Since $\mathfrak{A}^{\mathcal{Z}^D}$ is compact, the sequence $[\mathbf{b}^{[N]}]_{N \in \mathbb{N}}$ has a convergent subsequence—call it $[\mathbf{b}^{[N_k]}]_{k \in \mathbb{N}}$ —which converges to some limit $\mathbf{b} \in \mathfrak{A}^{\mathcal{Z}^D}$.

For any $M \in \mathbb{N}$, there is some $K \in \mathbb{N}$ so that, for all $k > K$, $\mathbf{b}^{[N_k]}_{\mathcal{B}(M)} = \mathbf{b}_{\mathcal{B}(M)}$. Hence, the central “ $\mathcal{B}(M)$ -block” of \mathbf{b} is \mathfrak{T} -admissible. This is true for every M ; we conclude that \mathbf{b} is \mathfrak{T} -admissible. Thus, the set $\langle \mathfrak{T} \rangle$ is nonempty, since it contains \mathbf{b} . ■

Claim 2 *The set \mathbf{T} is r.e.*

Proof Fix $\mathfrak{T} \subset \mathfrak{A}^{\mathcal{U}}$. For any finite N , it is r.d. whether or not $\mathfrak{A}^{\mathcal{B}(N)}$ contains a \mathfrak{T} -admissible configuration (there are only a finite number of cases to check). Suppose we perform this procedure for every $N \in \mathbb{N}$. By Claim 1, if $\mathfrak{T} \in \mathbf{T}$, then we will eventually find an N where no \mathfrak{T} -admissible configuration exists. Thus, we have a procedure which will halt if $\mathfrak{T} \in \mathbf{T}$, and tell us so. ■

As a consequence, since \mathbf{N} is *not* r.d., we conclude that \mathbf{N} is not even r.e.

Claim 3 *Suppose that $\text{Meas}_{\dagger}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ was r.e. Then \mathbf{N} is also r.e.*

Proof Clearly, $\mathbf{N} = \{\mathfrak{T} \subset \mathfrak{A}^{\mathcal{U}} ; \text{for some } \mu \in \text{Meas}_{\dagger}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}], \text{supp}[\mu] = \mathfrak{T}\}$. Hence, any recursive procedure for enumerating the elements of $\text{Meas}_{\dagger}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ would also provide a means for enumerating the elements of \mathbf{N} . ■

By contradiction, $\text{Meas}_{\dagger}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ cannot be r.e. ■

10 Conclusion

Although $\text{Meas}_{\dagger}^{\text{ext}}[\mathfrak{A}^{\mathcal{U}}]$ itself is not recursively denumerable, both its complement and topological interior are (Section 9). As yet, however, no efficient procedure exists for determining when a locally stationary measure is extendible. So far the only substantive result in this direction is Theorem 12, which says, loosely, that if $\mu_{\mathcal{U}}$ is “sufficiently close” to a periodically extendible measure with full support, then $\mu_{\mathcal{U}}$ itself is periodically extendible.

The existence of mixing, ergodic, *etc.* extensions is well-characterized in Section 8.6. However, as yet, no useful work has been done characterizing the *entropy*

of these extensions. In particular, we might ask: given that μ_U is extendible, what do the *maximal-entropy* extensions of μ_U look like? Is the maximal-entropy extension *unique*? Does it possess some kind of “Markov” property, analogous to the Markov Extension in \mathbb{Z} ? Perhaps it is some kind of *Markov Random Field* [20]. Indeed, in general, what would a “Markov extension” of a locally stationary measure look like, if anything? In the nonprobabilistic, purely symbolic setting, the construction analogous to a Markov extension is a \mathbb{Z}^D -subshift of finite type, but these are still poorly understood. Even topological Markov shifts—the simplest subshifts of finite type—do not generalize easily to higher dimensions [16]. The maximal entropy measures for such subshifts have been studied in [15]; perhaps similar techniques can be applied to maximal-entropy extensions of probability measures.

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