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STABILITY OF HALF-LINEAR NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAYS

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Abstract

In this paper, we study the mean square asymptotic stability of a generalized half-linear neutral stochastic differential equation with variable delays applying fixed point theory. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some results due to Burton, Zhang and Luo. Two examples are given to illustrate our results.

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1. Introduction

It is well known that stochastic delay differential equations, also known as stochastic functional differential equations, are a natural generalization of stochastic ordinary differential equations by allowing the coefficients to depend on values in the past. As a simple example let us mention the differential equation

$$dN(t) = aN(t)(K - N(t - \tau))\,dt + \sigma\,dW(t), \quad t \geq 0,$$

which is frequently used to model the dynamics of a population size taking into account time to maturity τ and random fluctuations. Recently, the theory and applications of stochastic delay differential equations have been studied by many authors (see, for example, [5, 8, 10] and the references therein).

On the other hand, Lyapunov's direct method has been successfully used to investigate stability properties of a wide variety of differential equations. However, there are many difficulties encountered in the study of stability by means of Lyapunov's direct method. Recently, Burton [1–4], Luo [9] and Zhang [11, 12] studied stability using fixed point theory which overcame the difficulties encountered in the study of stability by means of Lyapunov's direct method.

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Hitherto, fixed point theory has been used almost exclusively to deal with the stability of deterministic differential equations, not for stochastic differential equations. Very recently, Luo [9] studied the mean square asymptotic stability of a class of linear scalar neutral stochastic differential equations. For more details of stability with regard to stochastic differential equations, we refer to [7, 8] and the references therein.

Motivated by previous work, in this paper we study the mean square asymptotic stability of a half-linear neutral stochastic differential equation with variable delays applying fixed point theory. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved. Two examples are given to illustrate our results. The results presented in this paper improve and generalize the main results in [2, 9, 11, 12].

2. Main results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space and $W(t)$ denote a one-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(t)$. Let $a(t), \bar{a}(t), b(t), \bar{b}(t), c(t), e(t), q(t) \in C(R^+, R)$ and $\tau(t), \delta(t) \in C(R^+, R^+)$ with $t - \tau(t) \to \infty$ and $t - \delta(t) \to \infty$ as $t \to \infty$. Here $C(S_1, S_2)$ denotes the set of all continuous function $\phi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$.

Burton in [2] and Zhang in [12] studied the equation

$$x'(t) = -\bar{b}(t)x(t - \tau(t)) \tag{2.1}$$

and proved the following theorems.

THEOREM A (Burton [2]). *Suppose that $\tau(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |\bar{b}(s + r)|\, ds + \int_0^t |\bar{b}(s + r)| e^{-\int_s^t \bar{b}(u+r)\, du} \int_{s-r}^{s} |\bar{b}(u + r)|\, du\, ds \leq \alpha$$

for all $t \geq 0$ and $\int_0^\infty \bar{b}(s)\, ds = \infty$. Then for every continuous initial function $\phi : [-r, 0] \to R$, the solution $x(t) = x(t, 0, \phi)$ of (2.1) is bounded and tends to zero as $t \to \infty$.

THEOREM B (Zhang [12]). *Suppose that τ is differentiable, the inverse function $g(t)$ of $t - \tau(t)$ exists, and there exists a constant $\alpha \in (0, 1)$ such that for $t \geq 0$, $\liminf_{t\to\infty} \int_0^t \bar{b}(g(s))\, ds > -\infty$ and*

$$\int_{t-\tau(t)}^{t} |\bar{b}(g(s))|\, ds + \int_0^t e^{-\int_s^t \bar{b}(g(u))\, du} |\bar{b}(s)||\tau'(s)|\, ds + \int_0^t e^{-\int_s^t \bar{b}(g(u))\, du} |\bar{b}(g(s))| \int_{s-\tau(s)}^{s} |\bar{b}(g(v))|\, dv\, ds \leq \alpha < 1. \tag{2.2}$$

Then the zero solution of (2.1) is asymptotically stable if and only if $\int_0^t \bar{b}(g(s))\,ds \to \infty$, *as* $t \to \infty$.

Obviously, Theorem B improves Theorem A. Recently, Zhang [11] studied the half-linear equation

$$x'(t) = -\bar{a}(t)x(t) + b(t)g(x(t - \tau(t))) \tag{2.3}$$

where $g : R \to R$ is continuous and obtained the following theorem.

THEOREM C (Zhang [11]). *Suppose that* $\tau(t) \geq 0$ *such that for* $t \geq 0$, $t - \tau(t) \to \infty$ *as* $t \to \infty$, *and there exists a constant* $L > 0$, *for* $|x|, |y| \leq L$, $|g(x) - g(y)| \leq |x - y|$ *and* $q(0) = 0$. *For* $t > 0$, $\liminf_{t\to\infty} \int_0^t \bar{a}(s)\,ds > -\infty$ *and*

$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t \bar{a}(u)\,du} |b(s)|\,ds < 1. \tag{2.4}$$

Then the zero solution of (2.3) is asymptotically stable if and only if $\int_0^t \bar{a}(s)\,ds \to \infty$, *as* $t \to \infty$.

Very recently, Luo [9] considered the linear neutral stochastic differential equation

$$\begin{aligned} d[x(t) - q(t)x(t - \tau(t))] = {} & [a(t)x(t) + b(t)x(t - \tau(t))]\,dt \\ & [c(t)x(t) + e(t)x(t - \delta(t))]\,dW(t) \end{aligned} \tag{2.5}$$

and obtained the following theorem.

THEOREM D (Luo [9]). *Let* $\tau(t)$ *be differentiable. Assume that there exist a constant* $\alpha \in (0, 1)$ *and a continuous function* $h(t) : [0, \infty) \to R$ *such that for* $t \geq 0$, $\liminf_{t\to\infty} \int_0^t h(s)\,ds > -\infty$ *and*

$$\begin{aligned} &|q(t)| + \int_{t-\tau(t)}^t |a(s) + h(s)|\,ds \\ &\quad + \int_0^t e^{-\int_s^t h(u)\,du} |(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s)) + b(s) - q(s)h(s)|\,ds \\ &\quad + \int_0^t e^{-\int_s^t h(u)\,du} |h(s)| \int_{s-\tau(s)}^s |a(u) + h(u)|\,du\,ds \\ &\quad + \left(\int_0^t e^{-2\int_s^t h(u)\,du} (|c(s)| + |e(s)|)^2\,ds \right)^{1/2} \leq \alpha < 1. \end{aligned}$$

Then the zero solution of (2.5) is mean square asymptotically stable if and only if $\int_0^t h(s)\,ds \to \infty$, *as* $t \to \infty$.

In this paper, we consider the following half-linear neutral stochastic differential equation

$$\begin{aligned} d[x(t) - q(t)f(x(t - \tau(t)))] &= [a(t)x(t) + b(t)f(x(t - \tau(t)))]\,dt \\ &\quad + [c(t)x(t) + e(t)g(x(t - \delta(t)))]\,dW(t), \end{aligned} \tag{2.6}$$

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with the initial condition

$$x(s) = \phi(s) \quad \text{for } s \in [m(0), 0],$$

where the functions $f : R \to R$, $g : R \to R$ are continuous, $\phi \in C([m(0), 0], R)$, $x : [m(0), \infty) \times \Omega \to R$, and

$$m(0) = \min \big\{ \inf\{s - \tau(s), s \geq 0\}, \inf\{s - \delta(s), s \geq 0\} \big\} \leq 0.$$

Note that (2.6) becomes (2.5) for $f(x) \equiv g(x) \equiv x$. Thus we know that (2.6) includes (2.1), (2.3) and (2.5) as special cases. Our aim here is to generalize Theorems B, C and D to apply to (2.6).

For any $\phi \in C([m(0), 0], R)$, we define $\|\phi\| = \sup_{s \in [m(0),0]} |\phi(s)|$. For each $\lambda > 0$, we define $C(\lambda) := \{\phi \in C([m(0), 0], R) : \|\phi\| \leq \lambda\}$. Denote by F the Banach space of all $\mathcal{F}$-adapted processes $\psi(t, \omega) : [m(0), \infty) \times \Omega \to R$ which are almost surely continuous in t with norm

$$\|\psi\|_F = \left\{ E\left(\sup_{s \geq m(0)} |\psi(s, \omega)|^2 \right) \right\}^{1/2}.$$

Moreover, we define $F(\lambda) = \{\psi \in F : \|\psi\|_F \leq \lambda\}$ for each $\lambda > 0$ and let

$$\|\psi\|_F^{[r,t]} = \left\{ E\left(\sup_{s \in [r,t]} |\psi(s, \omega)|^2 \right) \right\}^{1/2}, \quad \text{for } r < t,$$

and

$$\|\psi\|_F^{[0,\infty)} = \left\{ E\left(\sup_{s \geq 0} |\psi(s, \omega)|^2 \right) \right\}^{1/2}.$$

THEOREM 2.1. *Suppose that τ is differentiable, and there exist continuous functions $h(t) : [0, \infty) \to R$ and constants $L > 0$, $\alpha \in (0, 1)$, $\beta \in (0, 1]$ such that for $t \geq 0$:*

(i) $\liminf_{t \to \infty} \int_0^t h(s)\, ds > -\infty$;

(ii)
$$\begin{aligned} &\beta|q(t)| + \int_{t-\tau(t)}^{t} |a(s) + h(s)|\, ds \\ &+ \int_0^t e^{-\int_s^t h(u)\, du} |h(s)| \int_{s-\tau(s)}^{s} |a(u) + h(u)|\, du\, ds \\ &+ \int_0^t e^{-\int_s^t h(u)\, du} (|(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s))| \\ &+ \beta|b(s) - q(s)h(s)|)\, ds \\ &+ 2\left(\int_0^t e^{-2\int_s^t h(u)\, du} (|c(s)| + \beta|e(s)|)^2\, ds \right)^{1/2} \leq \alpha < 1; \end{aligned}$$

(iii) $|f(x) - f(y)| \leq \beta|x - y|$ *and* $|g(x) - g(y)| \leq \beta|x - y|$ *for all* $x, y \in F(L)$ *with* $f(0) = g(0) = 0$.

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Then the zero solution of (2.6) is mean square asymptotically stable if and only if

$$\int_0^t h(s)\,ds \to \infty, \qquad \text{as } t \to \infty. \tag{2.7}$$

PROOF. We suppose that (2.7) holds. Choose $\delta > 0$, $\delta < L$ such that $2\delta K + \alpha L \leq L$, where $K = \sup_{t\geq 0}\{e^{-\int_0^t h(s)\,ds}\}$. Let $\phi \in C(\delta)$ and set

$$S = \big\{x : [m(0), \infty) \times \Omega \to R \mid x(t, \omega) = \phi(t) \text{ for } t \in [m(0), 0],\\ x(t, \omega) \in F(L) \text{ for } t \geq 0,\ E|x(t, \omega)|^2 \to 0 \text{ as } t \to \infty\big\}.$$

Then it is easy to check that S is a closed subset of F. From the definitions of $\|\cdot\|$, for any $x \in S$ and $t > 0$,

$$\|x\|_F = \max\{\|\phi\|, \|x\|_F^{[0,\infty)}\} \leq L. \tag{2.8}$$

Define an operator $P : S \to S$ by $(Px)(t) = \phi(t)$ for $t \in [m(0), 0]$ and for $t \geq 0$,

$$\begin{aligned}(Px)(t) = {} & \left(\phi(0) - q(0)f(\phi(-\tau(0))) - \int_{-\tau(0)}^0 (a(s) + h(s))\phi(s)\,ds\right)e^{-\int_0^t h(s)\,ds}\\ & + q(t)f(x(t-\tau(t))) + \int_{t-\tau(t)}^t (a(s) + h(s))x(s)\,ds\\ & + \int_0^t e^{-\int_s^t h(u)\,du}((a(s-\tau(s)) + h(s-\tau(s)))\\ & \quad \times (1-\tau'(s))x(s-\tau(s)) + (b(s) - q(s)h(s))f(x(s-\tau(s))))\,ds\\ & - \int_0^t e^{-\int_s^t h(u)\,du}h(s)\left(\int_{s-\tau(s)}^s (a(u) + h(u))x(u)\,du\right)ds\\ & + \int_0^t e^{-\int_s^t h(u)\,du}(c(s)x(s) + e(s)g(x(s-\delta(s))))\,dW(s)\\ := {} & \sum_{i=1}^5 I_i(t). \end{aligned} \tag{2.9}$$

We now show the mean square continuity of P on $[0, \infty)$. Let $x \in S$, $T_1 > 0$ and $|r|$ be sufficiently small. Then

$$E|(Px)(T_1 + r) - (Px)(T_1)|^2 \leq 5\sum_{i=1}^5 E|I_i(T_1 + r) - I_i(T_1)|^2.$$

It is easy to verify that

$$E|I_i(T_1 + r) - I_i(T_1)|^2 \to 0, \qquad \text{as } r \to 0,\ i = 1, 2, 3, 4.$$

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From the last term I_5 in (2.9), we have

$$
\begin{aligned}
E&|I_5(T_1+r)-I_5(T_1)|^2\\
&= E\bigg|\int_0^{T_1} e^{-\int_s^{T_1} h(u)\,du}(e^{-\int_{T_1}^{T_1+r} h(u)\,du}-1)\\
&\qquad\times(c(s)x(s)+e(s)g(x(s-\delta(s))))\,dW(s)\\
&\qquad+\int_{T_1}^{T_1+r} e^{-\int_s^{T_1+r} h(u)\,du}(c(s)x(s)+e(s)g(x(s-\delta(s))))\,dW(s)\bigg|^2\\
&\le 2E\int_0^{T_1} e^{-2\int_s^{T_1} h(u)\,du}|e^{-\int_{T_1}^{T_1+r} h(u)\,du}-1|^2\\
&\qquad\times|c(s)x(s)+e(s)g(x(s-\delta(s)))|^2\,ds\\
&\qquad+2E\int_{T_1}^{T_1+r} e^{-2\int_s^{T_1+r} h(u)\,du}|c(s)x(s)+e(s)g(x(s-\delta(s)))|^2\,ds\\
&\to 0 \quad \text{as } r\to 0.
\end{aligned}
$$

Therefore P is mean square continuous on $[0,\infty)$.

Next, we verify that $\|Px\|_F\le L$. As $\phi\in C(\delta)$ and $x\in S$,

$$
\begin{aligned}
\|Px\|_F^{[0,\infty)} &= \bigg\{E\Big(\sup_{s\ge 0}|Px(s)|^2\Big)\bigg\}^{1/2} = \bigg\{E\Big(\sup_{s\ge 0}\Big|\sum_{i=1}^{5} I_i(s)\Big|^2\Big)\bigg\}^{1/2}\\
&\le \sum_{i=1}^{5}\bigg\{E\Big(\sup_{s\ge 0}\big|I_i(s)\big|^2\Big)\bigg\}^{1/2}.
\end{aligned}
\tag{2.10}
$$

It follows from (2.9), (2.10), condition (ii), (iii) and Doob's L^p-inequality (see [6]) that

$$
\begin{aligned}
\|Px\|_F^{[0,\infty)} \le \bigg(&|\phi(0)|+|q(0)|\cdot\beta|\phi(-\tau(0))|\\
&+\int_{-\tau(0)}^{0}|a(v)+h(v)|\cdot|\phi(v)|\,dv\bigg)\cdot\sup_{s\ge 0}\{e^{-\int_0^s h(v)\,dv}\}\\
&+\bigg(E\sup_{s\ge 0}\bigg(|q(s)|\cdot\beta|x(s-\tau(s))|\\
&\qquad+\int_{s-\tau(s)}^{s}|a(v)+h(v)|\cdot|x(v)|\,dv\bigg)^2\bigg)^{1/2}\\
&+\bigg(E\sup_{s\ge 0}\bigg(\int_0^s e^{-\int_v^s h(u)\,du}(|(a(v-\tau(v))\\
&\qquad+h(v-\tau(v)))(1-\tau'(v))|\cdot|x(v-\tau(v))|\\
&\qquad+|b(v)-q(v)h(v)|\cdot\beta|x(v-\tau(v))|)\,dv\bigg)^2\bigg)^{1/2}
\end{aligned}
$$

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$$
\begin{aligned}
&+\Big(E\sup_{s\ge 0}\Big(\int_0^s e^{-\int_v^s h(u)\,du}|h(v)|\Big(\int_{v-\tau(v)}^v |a(u)\\
&\qquad +h(u)|\cdot|x(u)|\,du\Big)\,dv\Big)^2\Big)^{1/2}\\
&+2\sup_{s\ge 0}\Big(E\int_0^s e^{-2\int_v^s h(u)\,du}(|c(v)|\cdot|x(v)|\\
&\qquad\qquad +|e(v)|\cdot\beta|x(v-\delta(v))|)^2\,dv\Big)^{1/2}\\
\le{}& \delta K\Big(1+\beta|q(0)|+\int_{-\tau(0)}^0 |a(v)+h(v)|\,dv\Big)\\
&+\|x\|_F\cdot\sup_{s\ge 0}\Big\{\beta|q(s)|+\int_{s-\tau(s)}^s |a(v)+h(v)|\,dv\\
&\qquad +\int_0^s e^{-\int_v^s h(u)\,du}\big(|(a(v-\tau(v))+h(v-\tau(v)))(1-\tau'(v))|\\
&\qquad +\beta|b(v)-q(v)h(v)|\big)\,dv\\
&\qquad +\int_0^s e^{-\int_v^s h(u)\,du}|h(v)|\Big(\int_{v-\tau(v)}^v |a(u)+h(u)|\,du\Big)\,dv\\
&\qquad +2\Big(\int_0^s e^{-2\int_v^s h(u)\,du}(|c(v)|+\beta|e(v)|)^2\,dv\Big)^{1/2}\Big\}\\
\le{}& 2\delta K+\alpha L\le L.
\end{aligned}
$$

It follows from (2.8) that

$$\|Px\|_F=\max\{\|\phi\|,\ \|Px\|_F^{[0,\infty)}\}\le L.$$

Thirdly, we verify that $E|(Px)(t)|^2\to 0$ as $t\to\infty$. Since $E|x(t)|^2\to 0$, $t-\delta(t)\to\infty$ as $t\to\infty$, for each $\epsilon>0$, there exists a $T_1>0$ such that $s\ge T_1$ implies $E|x(s)|^2<\epsilon$ and $E|x(s-\delta(s))|^2<\epsilon$. By condition (ii), for $t\ge T_1$, the last term I_5 in (2.9) satisfies

$$
\begin{aligned}
E\big|I_5(t)\big|^2\le{}& E\int_0^{T_1} e^{-2\int_s^t h(u)\,du}(|c(s)|\cdot|x(s)|+|e(s)|\cdot|g(x(s-\delta(s)))|)^2\,ds\\
&+E\int_{T_1}^t e^{-2\int_s^t h(u)\,du}(|c(s)|\cdot|x(s)|+|e(s)|\cdot|g(x(s-\delta(s)))|)^2\,ds\\
\le{}& (\|x\|_F^{[m(0),T_1]})^2\int_0^{T_1} e^{-2\int_s^t h(u)\,du}(|c(s)|+\beta|e(s)|)^2\,ds+\alpha\epsilon\\
\le{}& \|x\|_F^2\cdot e^{-2\int_{T_1}^t h(u)\,du}\int_0^{T_1} e^{-2\int_s^{T_1} h(u)\,du}(|c(s)|+\beta|e(s)|)^2\,ds+\alpha\epsilon\\
\le{}& L^2\alpha^2 e^{-2\int_{T_1}^t h(u)\,du}+\alpha\epsilon.
\end{aligned}
$$

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By (2.7), there exists $T_2 > T_1$ such that $L^2\alpha^2 e^{-2\int_{T_1}^t h(u)\,du} < \epsilon$ for $t \geq T_2$. Thus for $t \geq T_2$,

$$E\big|I_5(t)\big|^2 < \epsilon + \alpha\epsilon.$$

This proves that $E\big|I_5(t)\big|^2 \to 0$, as $t \to \infty$. Similarly, we can show that $E\big|I_i(t)\big|^2 \to 0$, $i = 1, 2, 3, 4$, as $t \to \infty$. Thus $E|(Px)(t)|^2 \to 0$ as $t \to \infty$. Hence $Px \in S$.

Now we show that $P : S \to S$ is a contraction mapping. For any $x, y \in S$,

$$\begin{aligned}
\|Px - Py\|_F &= \Big(E \sup_{s \geq m(0)} \big|(Px)(s) - (Py)(s)\big|^2\Big)^{1/2} \\
&= \Big(E \sup_{s \geq 0} \Big| q(s)(f(x(s-\tau(s))) - f(y(s-\tau(s)))) \\
&\qquad + \int_{s-\tau(s)}^{s} (a(v) + h(v))(x(v) - y(v))\,dv \\
&\qquad + \int_0^s e^{-\int_v^s h(u)\,du}((a(v-\tau(v)) + h(v-\tau(v))) \\
&\qquad\quad \times (1-\tau'(v))(x(v-\tau(v)) - y(v-\tau(v))) \\
&\qquad + (b(v) - q(v)h(v))(f(x(v-\tau(v))) - f(y(v-\tau(v))))) \,dv \\
&\qquad - \int_0^s e^{-\int_v^s h(u)\,du} h(v) \Big(\int_{v-\tau(v)}^{v} (a(u) + h(u))(x(u) - y(u))\,du\Big)\,dv \\
&\qquad + \int_0^s e^{-\int_v^s h(u)\,du}(c(v)(x(v) - y(v)) + e(v)(g(x(v-\delta(v))) \\
&\qquad - g(y(v-\delta(v))))) \,dW(v)\Big|^2\Big)^{1/2} \\
&\leq \|x-y\|_F \cdot \sup_{s \geq 0} \Big\{\beta|q(s)| + \int_{s-\tau(s)}^{s} |a(v) + h(v)|\,dv \\
&\qquad + \int_0^s e^{-\int_v^s h(u)\,du}(\beta|b(v) - q(v)h(v)| + |(a(v-\tau(v)) \\
&\qquad + h(v-\tau(v)))(1-\tau'(v))|)\,dv \\
&\qquad + \int_0^s e^{-\int_v^s h(u)\,du}|h(v)| \int_{v-\tau(v)}^{v} |a(u) + h(u)|\,du\,dv \\
&\qquad + 2\Big(\int_0^s e^{-2\int_v^s h(u)\,du}(|c(v)| + \beta|e(v)|)^2\,dv\Big)^{1/2}\Big\} \\
&\leq \alpha\|x - y\|_F.
\end{aligned}$$

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Therefore, $P : S \to S$ is contraction mapping and so P has a fixed point $x \in S$, which is a solution of (2.6) with $x(s) = \phi(s)$ on $[m(0), 0]$ and $E|x(t)|^2 \to 0$ as $t \to \infty$.

To obtain the mean square asymptotic stability, we need to show that the zero solution of (2.6) is mean square stable. From (ii) we can choose $\varepsilon > 0$ such that $\alpha^2 + \varepsilon < 1$. Thus we can find a constant $N > 0$ such that

$$\begin{aligned}\left(1+\frac{1}{N}\right)&\bigg(\beta|q(t)| + \int_{t-\tau(t)}^{t} |a(s)+h(s)|\,ds \\ &+ \int_0^t e^{-\int_s^t h(u)\,du}(|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s))| \\ &\qquad + \beta|b(s)-q(s)h(s)|)\,ds \\ &+ \int_0^t e^{-\int_s^t h(u)\,du}|h(s)| \int_{s-\tau(s)}^{s} |a(u)+h(u)|\,du\,ds\bigg)^2 \\ &+ 4(1+N)\int_0^t e^{-2\int_s^t h(u)\,du}(|c(s)|+\beta|e(s)|)^2\,ds \le \alpha^2+\varepsilon < 1. \quad (2.11)\end{aligned}$$

Let $\epsilon > 0$ and $\epsilon < L$ be given and choose $\delta_0 > 0$ and $\delta_0 < \epsilon$ satisfying the condition

$$4(1+N)\delta_0^2 K^2 + (\alpha^2+\varepsilon)\epsilon < \epsilon,$$

where N is defined in (2.11). If $x(t) = x(t, 0, \phi)$ is a solution of (2.6) with $\|\phi\| < \delta_0$, then $x(t) = (Px)(t)$ defined in (2.9). We claim that $E|x(t)|^2 < \epsilon$ for all $t \ge 0$. Notice that $E|x(t)|^2 = \|\phi(t)\|^2 < \epsilon$ for $t \in [m(0), 0]$. If there exists $t^* > 0$ such that $E|x(t^*)|^2 = \epsilon$ and $E|x(t)|^2 < \epsilon$ for $t \in [m(0), t^*)$, then (2.9) and (2.11) imply that

$$\begin{aligned}E|x(t^*)|^2 &\le (1+N)\|\phi\|^2\left(1+\beta|q(0)| + \int_{-\tau(0)}^{0}|a(s)+h(s)|\,ds\right)^2 e^{-2\int_0^{t^*} h(u)\,du} \\ &\quad + \epsilon\left(1+\frac{1}{N}\right)\bigg(\beta|q(t^*)| + \int_{t^*-\tau(t^*)}^{t^*}|a(s)+h(s)|\,ds \\ &\quad + \int_0^{t^*} e^{-\int_s^{t^*} h(u)\,du}\left(\int_{s-\tau(s)}^{s}|a(u)+h(u)|\,du\right)|h(s)|\,ds \\ &\quad + \int_0^{t^*} e^{-\int_s^{t^*} h(u)\,du}(|(a(s-\tau(s))+h(s-\tau(s))) \\ &\qquad \times (1-\tau'(s))| + \beta|b(s)-q(s)h(s)|)\,ds\bigg)^2 \\ &\quad + \epsilon\int_0^{t^*} e^{-2\int_s^{t^*} h(u)\,du}(|c(s)|+\beta|e(s)|)^2\,ds \\ &\le (1+N)\delta_0^2\left(1+\beta|q(0)| + \int_{-\tau(0)}^{0}|a(s)+h(s)|\,ds\right)^2 e^{-2\int_0^{t^*} h(u)\,du} \\ &\quad + (\alpha^2+\varepsilon)\epsilon < \epsilon, \quad (2.12)\end{aligned}$$

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which contradicts the definition of t^*. Thus the zero solution of (2.6) is mean square asymptotically stable if (2.7) holds.

Conversely, we suppose that (2.7) fails. From (i) there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} h(u)\,du = \zeta$ for some $\zeta \in R$. Then, we can choose a constant $J > 0$ satisfying $\int_0^{t_n} h(u)\,du \in [-J, J]$ for all $n \geq 1$. Define

$$\omega(s) := |(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s))| + \beta|b(s) - q(s)h(s)| + |h(s)| \int_{s-\tau(s)}^{s} |a(u) + h(u)|\,du$$

for all $s \geq 0$. From (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n} h(u)\,du} \omega(s)\,ds \leq \alpha,$$

which implies that

$$\int_0^{t_n} e^{\int_0^s h(u)\,du} \omega(s)\,ds \leq \alpha e^{\int_0^{t_n} h(u)\,du} \leq e^J.$$

Therefore, the sequence $\{\int_0^{t_n} e^{\int_0^s h(u)\,du}\omega(s)\,ds\}$ has a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s h(u)\,du} \omega(s)\,ds = \gamma$$

for some $\gamma > 0$. Let k be an integer such that

$$\int_{t_k}^{t_n} e^{\int_0^s h(u)\,du} \omega(s)\,ds < \frac{\delta_1}{8K} \tag{2.13}$$

for all $n \geq k$, where $0 < \delta_1 < 1$ satisfies $8\delta_1^2 K^2 e^{2J} + (\alpha^2 + \varepsilon) < 1$.

We now consider the solution $x(t) = x(t, t_k, \phi)$ of (2.6) with $\|\phi(t_k)\| = \delta_1$ and $\|\phi(t)\| < \delta_1$ for $t < t_k$. By the similar method in (2.12), we have $E|x(t)|^2 < 1$ for $t \geq t_k$. We may choose ϕ so that

$$G(t_k) := \phi(t_k) - q(t_k) f(\phi(t_k - \tau(t_k))) - \int_{t_k-\tau(t_k)}^{t_k} (a(s) + h(s))\phi(s)\,ds \geq \frac{\delta_1}{2}. \tag{2.14}$$

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It follows from (2.9), (2.13) and (2.14) with $x(t) = (Px)(t)$ that for $n \geq k$,

$$\begin{aligned}
E\Big|x(t_n) - q(t_n)f(x(t_n - \tau(t_n))) - \int_{t_n-\tau(t_n)}^{t_n} (a(s) + h(s))x(s)\,ds\Big|^2 \\
&\geq G^2(t_k)e^{-2\int_{t_k}^{t_n} h(u)\,du} - 2G(t_k)e^{-\int_{t_k}^{t_n} h(u)\,du} \int_{t_k}^{t_n} e^{-\int_s^{t_n} h(u)\,du}\omega(s)\,ds \\
&\geq G(t_k)e^{-\int_{t_k}^{t_n} h(u)\,du} \\
&\quad \times \left(G(t_k)e^{-\int_{t_k}^{t_n} h(u)\,du} - 2e^{-\int_0^{t_n} h(u)\,du} \int_{t_k}^{t_n} e^{\int_0^s h(u)\,du}\omega(s)\,ds\right) \\
&\geq G(t_k)e^{-2\int_{t_k}^{t_n} h(u)\,du}\left(G(t_k) - 2e^{-\int_0^{t_k} h(u)\,du} \int_{t_k}^{t_n} e^{\int_0^s h(u)\,du}\omega(s)\,ds\right) \\
&\geq \frac{\delta_1}{2}e^{-2\int_{t_k}^{t_n} h(u)\,du}\left(\frac{\delta_1}{2} - 2K\int_{t_k}^{t_n} e^{\int_0^s h(u)\,du}\omega(s)\,ds\right) \\
&\geq \frac{\delta_1^2}{8}e^{-2J} > 0.
\end{aligned} \tag{2.15}$$

If the zero solution of (2.6) is mean square asymptotically stable, then $E|x(t)|^2 = E|x(t, t_k, \phi)|^2 \to 0$ as $t \to \infty$. Since $t_n - \tau(t_n) \to \infty$, $t_n - \delta(t_n) \to \infty$ as $n \to \infty$ and conditions (ii) and (iii) hold,

$$E\Big|x(t_n) - q(t_n)f(x(t_n - \tau(t_n))) - \int_{t_n-\tau(t_n)}^{t_n} (a(s) + h(s))x(s)\,ds\Big|^2 \to 0, \quad \text{as } n \to \infty,$$

which contradicts (2.15). Thus (2.7) is necessary for Theorem 2.1. This completes the proof. □

REMARK 2.2. Theorem 2.1 is still true if condition (ii) is satisfied for $t \geq t_a$ with some $t_a \in R^+$.

REMARK 2.3. Theorem 2.1 improves Theorem D under different conditions.

Choosing $h(t) \equiv -a(t)$ in Theorem 2.1, we have the following corollary.

COROLLARY 2.4. *Suppose that τ is differentiable, and there exist constants $L > 0$, $\alpha \in (0, 1)$, $\beta \in (0, 1]$ such that for $t \geq 0$:*

(i) $\liminf_{t\to\infty} \int_0^t -a(s)\,ds > -\infty$;

(ii) $\beta|q(t)| + \beta\int_0^t e^{\int_s^t a(u)\,du}|b(s) + q(s)a(s)|\,ds$
$+\ 2(\int_0^t e^{2\int_s^t a(u)\,du}(|c(s)| + \beta|e(s)|)^2\,ds)^{1/2} \leq \alpha < 1$;

(iii) $|f(x) - f(y)| \leq \beta|x - y|$ *and* $|g(x) - g(y)| \leq \beta|x - y|$ *for all* $x, y \in F(L)$ *with* $f(0) = g(0) = 0$.

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Then the zero solution of (2.6) is mean square asymptotically stable if and only if $\int_0^t a(s)\,ds \to \infty$ *as* $t \to \infty$.

Let $h(t) \equiv -b(p(t))$ in Theorem 2.1. Then we have the following corollary.

COROLLARY 2.5. *Suppose that* τ *is differentiable, the inverse function* $p(t)$ *of* $t - \tau(t)$ *exists, and there exist constants* $L > 0$, $\alpha \in (0, 1)$ *and* $\beta \in (0, 1]$ *such that for* $t \geq 0$*:*

(i) $\liminf_{t\to\infty} \int_0^t -b(p(s))\,ds > -\infty$;

(ii)

$$\begin{aligned}\beta|q(t)| &+ \int_{t-\tau(t)}^{t} |a(s) - b(p(s))|\,ds \\ &+ \int_0^t e^{\int_s^t b(p(u))\,du}|b(p(s))| \int_{s-\tau(s)}^{s} |a(u) - b(p(u))|\,du\,ds \\ &+ \int_0^t e^{\int_s^t b(p(u))\,du}(|(a(s-\tau(s)) - b(s))(1-\tau'(s))| \\ &\qquad\qquad + \beta|b(s) + q(s)b(p(s))|)\,ds \\ &+ 2\Big(\int_0^t e^{2\int_s^t b(p(u))\,du}(|c(s)| + \beta|e(s)|)^2\,ds\Big)^{1/2} \leq \alpha < 1;\end{aligned}$$

(iii) $|f(x) - f(y)| \leq \beta|x-y|$ *and* $|g(x) - g(y)| \leq \beta|x-y|$ *for all* $x, y \in F(L)$ *with* $f(0) = g(0) = 0$.

Then the zero solution of (2.6) is mean square asymptotically stable if and only if $\int_0^t b(p(s))\,ds \to \infty$ *as* $t \to \infty$.

REMARK 2.6. When $q(t) \equiv a(t) \equiv c(t) \equiv e(t) \equiv 0$, $f(x) \equiv g(x) \equiv x$ with $\beta \equiv 1$, and $b(t) \equiv -\bar{b}(t)$, from the proof process of Theorem 2.1, we know that the conclusion of Corollary 2.5 still holds if condition (ii) is replaced by (2.2). Therefore, Corollary 2.5 is a generalization of Theorem B.

When $q(t) \equiv c(t) \equiv e(t) \equiv 0$ and $a(t) \equiv -\bar{a}(t)$ in Theorem 2.1, we obtain the following corollary.

COROLLARY 2.7. *Suppose that* τ *is differentiable, and there exist continuous functions* $h(t) : [0, \infty) \to R$ *and constants* $L > 0$, $\alpha \in (0, 1)$, $\beta \in (0, 1]$ *such that for* $t \geq 0$*:*

(i) $\liminf_{t\to\infty} \int_0^t h(s)\,ds > -\infty$;

(ii) $\int_{t-\tau(t)}^t |h(s) - \bar{a}(s)|\,ds + \int_0^t e^{-\int_s^t h(u)\,du}|h(s)| \int_{s-\tau(s)}^s |h(u) - \bar{a}(u)|\,du\,ds + \int_0^t e^{-\int_s^t h(u)\,du}(|(h(s-\tau(s)) - \bar{a}(s-\tau(s)))(1-\tau'(s))| + \beta|b(s)|)\,ds \leq \alpha < 1$;

(iii) *if* $|x| \leq L$, $|y| \leq L$ *and* $|g(x) - g(y)| \leq \beta|x-y|$ *for all* $x, y \in R$ *with* $g(0) = 0$.

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Then the zero solution of (2.3) is asymptotically stable if and only if $\int_0^t h(s)\,ds \to \infty$ *as* $t \to \infty$.

REMARK 2.8. Choosing $h(t) \equiv \bar{a}(t)$ and $\beta \equiv 1$, Corollary 2.7 reduces to Theorem C.

3. Two Examples

In this section, we give two examples to illustrate the applications of our main results.

EXAMPLE 3.1. Consider the half-linear neutral stochastic delay differential equation

$$d\Big(x(t) - \frac{1}{3}x^2\Big(t - \frac{t}{4}\Big)\Big) = \Big(-2x(t) + \frac{2}{3}x^2\Big(t - \frac{t}{4}\Big)\Big)\,dt + \Big(\frac{1}{4}x(t) + \frac{3}{8}x^3\Big(t - \frac{t}{2}\Big)\Big)\,dW(t). \tag{3.1}$$

Then the zero solution of (3.1) is mean square asymptotically stable.

PROOF. It is easy to verify that $t - \tau(t) = t - t/4 \to \infty$ and $t - \delta(t) = t - t/2 \to \infty$ as $t \to \infty$. Since $|x^2| \le |x|$ and $|x^3| \le |x|$ when $|x| \le 1$ for $x \in R$, we can choose $L = 1/2$ and $\beta = 1$ such that condition (iii) of Theorem 2.1 holds. Choosing $h(t) = 2$ in Theorem 2.1, we have $|q(t)| = 1/3$,

$$\int_{t-\tau(t)}^{t} |a(s) + h(s)|\,ds = \int_0^t e^{-\int_s^t h(u)\,du}|h(s)| \int_{s-\tau(s)}^{s} |a(u) + h(u)|\,du\,ds = 0,$$

$$\int_0^t e^{-\int_s^t h(u)\,du}(|(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s))| + |b(s) - q(s)h(s)|)\,ds = 0,$$

and

$$2\Big(\int_0^t e^{-2\int_s^t h(u)\,du}(|c(s)| + |e(s)|)^2\,ds\Big)^{1/2} = \frac{5}{4}\Big(\int_0^t e^{-4(t-s)}\,ds\Big)^{1/2} = \frac{5}{8}(1 - e^{-4t})^{1/2} \le \frac{5}{8}.$$

It is easy to check that $\int_0^t h(s)\,ds \to \infty$ as $t \to \infty$. Let $\alpha = 1/3 + 5/8$. Then $\alpha = 23/24 < 1$ and the zero solution of (3.1) is mean square asymptotically stable by Theorem 2.1. □

EXAMPLE 3.2. Consider the delay differential equation

$$x'(t) = -\frac{19}{100}x(t) + \frac{1}{5}\sin\Big(\frac{1}{10}x(t - e^{-t})\Big). \tag{3.2}$$

Then the zero solution of (3.2) is asymptotically stable.

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PROOF. It is easy to verify that $t-\tau(t)=t-e^{-t}\to\infty$ as $t\to\infty$. Because $|\sin\frac{1}{10}x|\le\frac{1}{10}|x|$ for $x\in R$, we can choose $\beta=\frac{1}{10}$ and L for any positive constant. Then condition (iii) of Corollary 2.7 is satisfied. Choosing $h(t)\equiv 0.29$ in Corollary 2.7, we have

$$\int_{t-\tau(t)}^{t}|h(s)-\bar{a}(s)|\,ds=\int_{t-e^{-t}}^{t}(0.29-0.19)\,ds\to 0.1e^{-t}\le 0.1,$$

$$\int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^{s}|h(u)-\bar{a}(u)|\,du\,ds$$
$$\le\int_0^t e^{-0.29(t-s)}(0.29\times 0.1)\,ds\le 0.1$$

and

$$\int_0^t e^{-\int_s^t h(u)\,du}(|(h(s-\tau(s))-\bar{a}(s-\tau(s)))(1-\tau'(s))|+\beta|b(s)|)\,ds$$
$$=\int_0^t e^{-0.29(t-s)}\left[(0.29-0.19)(1+e^{-s})+\frac{1}{10}\times\frac{1}{5}\right]ds\le 0.7587.$$

It is easy to see that all the conditions of Corollary 2.7 hold for $\alpha=0.1+0.1+0.7587=0.9587<1$. Thus Corollary 2.7 implies that the zero solution of (3.2) is asymptotically stable. □

However, Theorem C cannot be used to verify that the zero solution of (3.2) is asymptotically stable. In fact, noticing that $|\sin\frac{1}{10}x-\sin\frac{1}{10}y|\le|x-y|$ for all $x, y\in R$, $b(t)\equiv 0.2$, $\bar{a}(t)\equiv 0.19$ and

$$\int_0^t e^{-\int_s^t \bar{a}(u)\,du}|b(s)|\,ds=0.2\int_0^t e^{-0.19(t-s)}\,ds<1.0527,$$

we can see that the condition (2.4) of Theorem C does not hold with $\alpha=1.05327>1$.

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