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THE MODULUS OF WEAKLY COMPACT MULTIPLIERS ON THE BANACH ALGEBRA $L^1(\mathcal{G})^{**}$ OF A LOCALLY COMPACT GROUP

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Abstract

In this paper we give a necessary and sufficient condition under which the answer to the open problem raised by Ghahramani and Lau ('Multipliers and modulus on Banach algebras related to locally compact groups', *J. Funct. Anal.* **150** (1997), 478–497) is positive.

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1. Introduction

Let $\mathcal{G}$ be a locally compact group with a left Haar measure λ and let $L^1(\mathcal{G})$ be the Banach space of complex-valued integrable functions with respect to left Haar measure λ. With the norm $\|\cdot\|_1$ and with convolution

$$\vartheta * \omega(x) = \int_{\mathcal{G}} \vartheta(y)\omega(y^{-1}x)\, d\lambda(y) \quad (x \in \mathcal{G})$$

as product, $L^1(\mathcal{G})$ becomes a Banach algebra [9]. Let $M(\mathcal{G})$ be the Banach algebra of all bounded complex-valued regular Borel measures on $\mathcal{G}$ equipped with the convolution product $*$ and the total variation norm. The other Banach algebras we shall consider are defined as follows. Let $C(\mathcal{G})$ be the space of all bounded continuous complex-valued functions on $\mathcal{G}$ with the sup-norm, and let $\mathrm{RUC}(\mathcal{G})$ be the Banach space of all $f \in C(\mathcal{G})$ such that the mapping $x \mapsto f_x$ from $\mathcal{G}$ into $C(\mathcal{G})$ is continuous, where $f_x(y) = f(yx)$ for all $x, y \in \mathcal{G}$ [4].

Let $\mathfrak{A}$ be a Banach algebra. We denote the dual of $\mathfrak{A}$ by $\mathfrak{A}^*$. If $\phi \in \mathfrak{A}^*$, then the value of ϕ at an element $\vartheta \in \mathfrak{A}$ will be written as $\langle \phi, \vartheta\rangle$. For every $n \in \mathfrak{A}^{**}$ and $\phi \in \mathfrak{A}^*$, we define the element $n\phi$ of $\mathfrak{A}^*$ by the formula

$$\langle n\phi, \vartheta\rangle = \langle n, \phi\vartheta\rangle \quad (\vartheta \in \mathfrak{A}),$$

where

$$\langle \phi\vartheta, \omega\rangle = \langle \phi, \vartheta\omega\rangle \quad (\omega \in \mathfrak{A}).$$

In this paper we always endow the second dual $\mathfrak{A}^{**}$ with the first Arens product '$\cdot$', defined by

$$\langle m \cdot n, \phi\rangle = \langle m, n\phi\rangle$$

for all $m, n \in \mathfrak{A}^{**}$ and $\phi \in \mathfrak{A}^{*}$. This product turns $\mathfrak{A}^{**}$ into a Banach algebra [10].

Let $\mathfrak{A}$ be a Banach algebra; a bounded operator $T : \mathfrak{A} \to \mathfrak{A}$ is called a *left multiplier* if $T(\vartheta\omega) = T(\vartheta)\omega$ for all $\vartheta, \omega \in \mathfrak{A}$. An element $\vartheta \in \mathfrak{A}$ is said to be a *left weakly completely continuous element* of $\mathfrak{A}$ if the left multiplier ${}_{\vartheta}\Gamma : \omega \mapsto \vartheta\omega$ is a weakly compact operator on $\mathfrak{A}$.

A Banach lattice $\mathfrak{A}$ is called *complete* if every nonempty subset of $\mathfrak{A}$ bounded from above has a supremum in $\mathfrak{A}$; see [3]. Hence for each bounded operator T from a complete Banach lattice $\mathfrak{A}$ into a complete Banach lattice $\mathfrak{B}$, we may define the modulus of T by

$$|T|(p) = \sup\{|T(\vartheta)| : \vartheta \in \mathfrak{A}, |\vartheta| \le p\}$$

for all positive elements $p \in \mathfrak{A}$. Then $|T|$ is bounded; indeed,

$$|T(\vartheta)| \le \|T\| \, \|\vartheta\| \le \|T\| \, \|p\|$$

for all $\vartheta \in \mathfrak{A}$ and for all positive elements $p \in \mathfrak{A}$ with $|\vartheta| \le p$. It is natural to ask whether the modulus of a left multiplier on a complete Banach lattice is also a left multiplier?

The modulus of left multipliers on group algebras has been studied by Ghahramani and Lau [7]. They proved that if $T = {}_{\mu}\Gamma$ on $L^1(\mathcal{G})$ (respectively on $M(\mathcal{G})$) for some $\mu \in M(\mathcal{G})$, then $|T| = {}_{|\mu|}\Gamma$ on $L^1(\mathcal{G})$ (respectively on $M(\mathcal{G})$). To put it another way, they showed that the modulus of a left multiplier on $L^1(\mathcal{G})$ (respectively on $M(\mathcal{G})$) is also a left multiplier on $L^1(\mathcal{G})$ (respectively on $M(\mathcal{G})$). However, they proved that the modulus of a left multiplier on $L^1(\mathcal{G})^{**}$ is not necessarily a left multiplier. They also proved that for any $m \in L^1(\mathcal{G})^{**}$, $|{}_{m}\Gamma|$ is a left multiplier on $L^1(\mathcal{G})^{**}$ if and only if $|{}_{m}\Gamma| = {}_{|m|}\Gamma$. The question comes to mind immediately: under what conditions on a left multiplier $T : L^1(\mathcal{G})^{**} \to L^1(\mathcal{G})^{**}$ can we assert that $|T|$ is a left multiplier on $L^1(\mathcal{G})^{**}$? Ghahramani and Lau in [7] proposed the following open problem.

Problem 1.1. Let T be a weakly compact left multiplier on $L^1(\mathcal{G})^{**}$. Is the modulus operator $|T|$ also a multiplier?

Weakly compact multipliers on a group algebra $L^1(\mathcal{G})$ have been studied by Sakai and Akemann. Sakai [13] showed that if $\mathcal{G}$ is a locally compact noncompact group, then zero is the only left weakly compact multiplier on $L^1(\mathcal{G})$. Akemann [2] proved that if $\mathcal{G}$ is compact, then every left multiplier on $L^1(\mathcal{G})$ is compact. Akemann also characterised weakly compact elements of $L^1(\mathcal{G})$. In fact, he showed that any weakly compact left multiplier on $L^1(\mathcal{G})$ is of the form ${}_{\vartheta}\Gamma$ for some $\vartheta \in L^1(\mathcal{G})$.

Ghahramani and Lau [5–7] obtained some results on the question of the existence of nonzero weakly compact left multipliers on $L^1(\mathcal{G})^{**}$. Losert [11], among other things, proved that the existence of a nonzero weakly compact left multiplier on $L^1(\mathcal{G})^{**}$ is equivalent to compactness of $\mathcal{G}$; see also [1].

Our main purpose in this paper is to give a necessary and sufficient condition under which the answer to Problem 1.1 is positive. In fact, we show that the modulus of a weakly compact left multiplier on $L^1(\mathcal{G})^{**}$ is a left multiplier if and only if $|T| = {}_{\vartheta}\Gamma$ for some $\vartheta \in L^1(\mathcal{G})$.

2. The results

We commence this section with the main result of the paper. First, we recall the main definitions and results that we require later. Let $\mathfrak{A}$ be a C^*-algebra. A linear functional $p \in \mathfrak{A}^*$ is called *positive* if $\langle p, \phi\rangle \geq 0$ for all positive elements $\phi \in \mathfrak{A}$. It is well known from [12] that if p is a bounded linear functional on a unital C^*-algebra, then p is positive if and only if $\|p\| = \langle p, 1\rangle$.

THEOREM 2.1. *Let $\mathcal{G}$ be a locally compact group and let T be a weakly compact left multiplier on $L^1(\mathcal{G})^{**}$. Then $|T|$ is a left multiplier on $L^1(\mathcal{G})^{**}$ if and only if there exists $\vartheta \in L^1(\mathcal{G})$ such that $|T| = {}_{\vartheta}\Gamma$ on $L^1(\mathcal{G})^{**}$.*

PROOF. Let $|T|$ be a nonzero left multiplier on $L^1(\mathcal{G})^{**}$. Since T is weakly compact, $|T|$ is a nonzero weakly compact left multiplier on $L^1(\mathcal{G})^{**}$; see [3]. It follows that $\mathcal{G}$ is compact. So $L^1(\mathcal{G})$ is an ideal in $L^1(\mathcal{G})^{**}$. Hence for every $\omega_1, \omega_2 \in L^1(\mathcal{G})$,

$$|T|(\omega_1 \cdot \omega_2) = |T|(\omega_1) \cdot \omega_2 \in L^1(\mathcal{G}).$$

This shows that $|T|$ restricted to $L^1(\mathcal{G})$ is a weakly compact left multiplier on $L^1(\mathcal{G})$. By [2], there exists $\vartheta \in L^1(\mathcal{G})$ such that $|T| = {}_{\vartheta}\Gamma$ on $L^1(\mathcal{G})$. Define the function $S : L^1(\mathcal{G})^{**} \to L^1(\mathcal{G})^{**}$ by

$$S(m) = |T|(m) - {}_{\vartheta}\Gamma(m)$$

for all $m \in L^1(\mathcal{G})^{**}$. Note that if $\imath \in C(\mathcal{G})$, then $\imath = \omega\jmath$ for some $\omega \in L^1(\mathcal{G})$ and $\jmath \in C(\mathcal{G})$. Since $m \cdot \omega \in L^1(\mathcal{G})$, we have

$$\langle |T|(m) - {}_{\vartheta}\Gamma(m), \omega\jmath\rangle = \langle |T|(m \cdot \omega) - {}_{\vartheta}\Gamma(m \cdot \omega), \jmath\rangle = 0.$$

Hence $\langle S(m), \imath\rangle = 0$. This implies that

$$\langle |T|(m), \imath\rangle = \langle {}_{\vartheta}\Gamma(m), \imath\rangle$$

for all $\imath \in C(\mathcal{G})$. We show that $S = 0$ on $L^1(\mathcal{G})^{**}$. Suppose that $S(m) \neq 0$ for some $m \in L^1(\mathcal{G})^{**}$. Without loss of generality, we may assume that m is hermitian. Let $m = m_+ - m_-$ be the Jordan decomposition of m. Hence there exists a positive functional p on $L^1(\mathcal{G})^*$ such that $S(p) \neq 0$. Choose a continuous function ψ with compact support K such that $\|\psi\|_\infty \leq 1$ and

$$|\langle {}_{\vartheta}\Gamma(p), \psi\rangle| \geq \|{}_{\vartheta}\Gamma(p)\|_1 - (5/12)\|S(p)\|.$$

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Let V be an open set with compact closure for which $K \subseteq V$. Let θ be a continuous function on $\mathcal{G}$ such that

$$0 \leq \theta(\mathcal{G}) \leq 1, \quad \theta(K) = 1, \quad \text{and } \theta(\mathcal{G} \setminus V) = 0.$$

There exists an element ϕ in the unit ball of $L^1(\mathcal{G})^*$ such that

$$|\langle S(p), \phi\rangle| \geq \tfrac{23}{24}\|S(p)\|.$$

Define the function $\kappa : \mathcal{G} \to \mathbb{C}$ by

$$\kappa(x) := \phi(x) - \theta(x)\psi(x) + \psi(x)$$

for all $x \in \mathcal{G}$. Then κ is an element in the unit ball of $L^1(\mathcal{G})^*$ and $\langle S(p), \kappa\rangle = \langle S(p), \phi\rangle$. Choose a constant $\eta \in \mathbb{C}$ such that $\|\eta(\phi - \theta\psi) + \psi\|_\infty \leq 1$ and

$$|\langle {}_\vartheta\Gamma(p), \eta(\phi - \theta\psi) + \psi\rangle| = |\langle {}_\vartheta\Gamma(p), \phi - \theta\psi\rangle| + |\langle {}_\vartheta\Gamma(p), \psi\rangle|.$$

Then

$$\begin{aligned}\|{}_\vartheta\Gamma(p)\|_1 &\geq |\langle {}_\vartheta\Gamma(p), \phi - \theta\psi\rangle| + |\langle {}_\vartheta\Gamma(p), \psi\rangle| \\ &\geq |\langle {}_\vartheta\Gamma(p), \phi - \theta\psi\rangle| + \|{}_\vartheta\Gamma(p)\|_1 - \tfrac{5}{12}\|S(p)\|.\end{aligned}$$

This implies that

$$|\langle {}_\vartheta\Gamma(p), \phi - \theta\psi\rangle| \leq \tfrac{5}{12}\|S(p)\|.$$

Hence

$$\begin{aligned}|\langle |T|(p), \kappa\rangle| &\geq |\langle S(p), \kappa\rangle| + |\langle {}_\vartheta\Gamma(p), \psi\rangle| - |\langle {}_\vartheta\Gamma(p), \phi - \theta\psi\rangle| \\ &\geq \tfrac{1}{8}\|S(p)\| + \|{}_\vartheta\Gamma(p)\|_1.\end{aligned}$$

It follows that

$$\||T|(p)\| \geq \tfrac{1}{8}\|S(p)\| + \|{}_\vartheta\Gamma(p)\|_1.$$

On the other hand,

$$\|\,|T|(p)\| = \langle |T|(p), 1\rangle = \langle {}_\vartheta\Gamma(p), 1\rangle \leq \|{}_\vartheta\Gamma(p)\|_1.$$

Hence that $\|S(p)\| = 0$ and so $S(p) = 0$. This contradiction shows

$$|T|(m) = {}_\vartheta\Gamma(m)$$

for all $m \in L^1(\mathcal{G})^{**}$. The converse is clear. □

As a corollary of this theorem we have the following result.

Corollary 2.2. *Let $\mathcal{G}$ be a locally compact group and let m be a left weakly completely continuous element of $L^1(\mathcal{G})^{**}$. If $|{}_m\Gamma|$ is a left multiplier on $L^1(\mathcal{G})^{**}$, then $|m| \in L^1(\mathcal{G})$.*

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PROOF. Let $|_m\Gamma|$ be a left multiplier on $L^1(\mathcal{G})^{**}$. Then $|_m\Gamma| = {}_{|m|}\Gamma$ on $L^1(\mathcal{G})^{**}$; see [7, Lemma 3.6]. Invoke Theorem 2.1 to conclude that there exists $\vartheta \in L^1(\mathcal{G})$ such that ${}_{|m|}\Gamma = {}_{\vartheta}\Gamma$ on $L^1(\mathcal{G})^{**}$. Now, choose a right identity E of $L^1(\mathcal{G})^{**}$; see [8]. Then

$$ {}_{|m|}\Gamma(E) = {}_{\vartheta}\Gamma(E). $$

Thus $|m| = \vartheta \in L^1(\mathcal{G})$, and the proof is complete. □

Another consequence of Theorem 2.1 is the following result due to Ghahramani and Lau [7].

COROLLARY 2.3. *Let $\mathcal{G}$ be a locally compact group. If $\vartheta \in L^1(\mathcal{G})$ is a weakly completely continuous element of $L^1(\mathcal{G})^{**}$, then $|\vartheta|$ is also a weakly completely continuous element of $L^1(\mathcal{G})^{**}$. Furthermore, $|_{\vartheta}\Gamma| = {}_{|\vartheta|}\Gamma$ on $L^1(\mathcal{G})^{**}$.*

PROOF. Let $\vartheta \in L^1(\mathcal{G})$ be a weakly completely continuous element of $L^1(\mathcal{G})^{**}$. Let E be a positive right identity of $L^1(\mathcal{G})^{**}$. Then

$$ \begin{aligned} |_{\vartheta}\Gamma|(E) &= \sup\{|\vartheta \cdot m| : m \in L^1(\mathcal{G})^{**}, |m| \le E\} \\ &\le \sup\{|\vartheta| \cdot |m| : m \in L^1(\mathcal{G})^{**}, |m| \le E\} \\ &\le \sup\{|\vartheta| \cdot E : m \in L^1(\mathcal{G})^{**}, |m| \le E\} \\ &= |\vartheta|. \end{aligned} $$

Since $|E| = E \le E$, we have $|_{\vartheta}\Gamma|(E) = |\vartheta|$. By Theorem 2.1, there exists $\omega \in L^1(\mathcal{G})$ such that $|_{\vartheta}\Gamma| = {}_{\omega}\Gamma$. Thus

$$ |_{\vartheta}\Gamma|(E) = {}_{\omega}\Gamma(E) = \omega. $$

So $\omega = |\vartheta|$, and therefore $|_{\vartheta}\Gamma| = {}_{|\vartheta|}\Gamma$ on $L^1(\mathcal{G})^{**}$. This means that $|\vartheta|$ is a weakly completely continuous element of $L^1(\mathcal{G})^{**}$. □

We give a corollary of this result.

COROLLARY 2.4. *Let $\mathcal{G}$ be a locally compact group and let T be a weakly compact left multiplier on $L^1(\mathcal{G})^{**}$. If the range of T is contained in $L^1(\mathcal{G})$, then $|T|$ is a left multiplier.*

PROOF. Let T be a weakly compact left multiplier on $L^1(\mathcal{G})^{**}$ with $T(L^1(\mathcal{G})^{**}) \subseteq L^1(\mathcal{G})$. Let E be a right identity of $L^1(\mathcal{G})^{**}$. If $m \in L^1(\mathcal{G})^{**}$, then

$$ \begin{aligned} \langle T(m - E \cdot m), \omega\jmath\rangle &= \langle T(m - E \cdot m) \cdot \omega, \jmath\rangle \\ &= \langle T(m \cdot \omega - E \cdot (m \cdot \omega)), \jmath\rangle \\ &= 0 \end{aligned} $$

for all $\omega \in L^1(\mathcal{G})$ and $\jmath \in C(\mathcal{G})$. Hence $\langle T(m - E \cdot m), \imath\rangle = 0$ for all $\imath \in C(\mathcal{G})$. Since $T(m - E \cdot m) \in L^1(\mathcal{G})$, it follows that $T(m - E \cdot m) = 0$. Hence, for any $m \in L^1(\mathcal{G})^{**}$,

$$ T(m) = T(E \cdot m) = T(E) \cdot m. $$

Therefore, there exists $\vartheta \in L^1(\mathcal{G})$ such that $T = {}_{\vartheta}\Gamma$ on $L^1(\mathcal{G})^{**}$. In view of Corollary 2.3, we have $|T| = {}_{|\vartheta|}\Gamma$ on $L^1(\mathcal{G})^{**}$, proving the corollary. □

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By Corollary 2.3, if $\vartheta \in L^1(\mathcal{G})$ is a nonzero weakly completely continuous element of $L^1(\mathcal{G})^{**}$, then $|_{\vartheta}\Gamma|$ is a nonzero weakly compact multiplier on $L^1(\mathcal{G})^{**}$. In the next result, we show that this is not true for all elements $m \in L^1(\mathcal{G})^{**}$ instead of $\vartheta \in L^1(\mathcal{G})$.

In the following, let

$$\mathrm{RUC}(\mathcal{G})^{\perp} = \{m \in L^1(\mathcal{G})^{**} : \langle m, \phi\rangle = 0 \text{ for all } \phi \in \mathrm{RUC}(\mathcal{G})\}.$$

Theorem 2.5. *Let $\mathcal{G}$ be a locally compact group and let ℓ be any element of* $\mathrm{RUC}(\mathcal{G})^{\perp}$. *Then $|_{\ell}\Gamma|$ is a left multiplier on $L^1(\mathcal{G})^{**}$ if and only if $\ell = 0$.*

Proof. Suppose that ℓ is a nonzero element of $\mathrm{RUC}(\mathcal{G})^{\perp}$. Choose $\phi \in L^1(\mathcal{G})^*$ with $\langle \ell, \phi\rangle \neq 0$. Then, for every positive function ρ in $L^1(\mathcal{G})$,

$$\begin{aligned}
\langle |\phi|, \rho\rangle &= \sup\{|\langle \phi, \vartheta\rangle| : \vartheta \in L^1(\mathcal{G}), |\vartheta| \le \rho\} \\
&= \sup\Big\{\Big|\int_{\mathcal{G}} \phi(x)\vartheta(x)\, d\lambda(x)\Big| : \vartheta \in L^1(\mathcal{G}), |\vartheta| \le \rho\Big\} \\
&\le \sup\Big\{\int_{\mathcal{G}} |\phi(x)|\rho(x)\, d\lambda(x) : \vartheta \in L^1(\mathcal{G}), |\vartheta| \le \rho\Big\} \\
&= \|\phi\|_\infty \int_{\mathcal{G}} \rho(x)\, d\lambda(x) = \|\phi\|_\infty \langle 1, \rho\rangle.
\end{aligned}$$

Hence $|\phi| \le \|\phi\|_\infty 1$ and so

$$\begin{aligned}
\rho(\mathcal{G})|\langle \ell, \phi\rangle| &\le \rho(\mathcal{G})\sup\{|\langle \ell, \psi\rangle| : \psi \in L^1(\mathcal{G})^*, |\psi| \le \|\phi\|_\infty 1\} \\
&\le \rho(\mathcal{G})\sup\{\langle |\ell|, |\psi|\rangle : \psi \in L^1(\mathcal{G})^*, |\psi| \le \|\phi\|_\infty 1\} \\
&\le \rho(\mathcal{G})\sup\{\langle |\ell|, \|\phi\|_\infty 1\rangle| : \psi \in L^1(\mathcal{G})^*, |\psi| \le \|\phi\|_\infty 1\} \\
&= \rho(\mathcal{G})\|\phi\|_\infty \langle |\ell|, 1\rangle = \|\phi\|_\infty \langle |\ell| 1, \rho\rangle.
\end{aligned}$$

This shows that $|\ell| 1 \neq 0$. Fix $\omega \in L^1(\mathcal{G})$ with $\langle |\ell| 1, \omega\rangle \neq 0$. Since

$$\langle |\ell| 1, \omega\rangle = \langle |\ell|, 1\omega\rangle = \langle |\ell| \cdot \omega, 1\rangle,$$

it follows that $_{|\ell|}\Gamma(\omega) \neq 0$. On the other hand we have, for any $\vartheta \in L^1(\mathcal{G})$ and $\phi \in L^1(\mathcal{G})^*$, $\vartheta\phi \in \mathrm{RUC}(\mathcal{G})$. Thus

$$\langle \ell \cdot \vartheta, \phi\rangle = \langle \ell, \vartheta\phi\rangle = 0,$$

and hence $\ell \cdot \vartheta = 0$, which implies that $\ell \cdot m = 0$ for all $m \in L^1(\mathcal{G})^{**}$ with $|m| \le \omega$; see [3, p. 234]. Therefore,

$$|_{\ell}\Gamma|(\omega) = \sup\{|_{\ell}\Gamma(m)| : m \in L^1(\mathcal{G})^{**}, |m| \le \omega\} = 0.$$

Now if $|_{\ell}\Gamma|$ is a left multiplier, then from [7] we would have $|_{\ell}\Gamma|(\omega) = {}_{|\ell|}\Gamma(\omega)$, a contradiction. Therefore, $\ell = 0$. The converse is clear. □

As suggested by a referee, it would be worth working on the following open problem: give an example of a weakly compact left multiplier T on $L^1(\mathcal{G})^{**}$ such that $|T|$ is not a left multiplier.

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