25 Compactification of string theory I. Tori and orbifolds

We do not live in a ten-dimensional world, and certainly not in a 26-dimensional world without fermions. But if we don't insist on Lorentz invariance in all directions then there are other possible ways to construct consistent string theories. In this chapter we will uncover many consistent string theories in four dimensions (and in others). If anything, our problem will shortly be an "embarrassment of riches:" we will see that there are vast numbers of possible string constructions. The connection of these various constructions to one another is not always clear. Many of them can be obtained from others by varying the expectation values of the light fields (i.e. the moduli). One might imagine that others could be obtained by exciting massive fields as well. In general, though, this is not known and in any case the meaning of such connections in a theory of gravity is obscure. But, before exploring these deep and difficult questions, we need to acquire some experience with constructing strings in different dimensions.

25.1 Compactification in field theory: the Kaluza–Klein program

The idea that space–time might be more than four-dimensional was first put forward by Kaluza and Klein shortly after Einstein published his general theory of relativity. They argued that *five-dimensional* general coordinate invariance would give rise to both four-dimensional general coordinate invariance and a $U(1)$ gauge invariance, unifying electromagnetism and gravity. In modern language they considered the possibility that space–time is five-dimensional, with the structure $M^4 \times S^1$. This is, on first exposure, a bizarre concept but its implications are readily understood by considering a toy model. Take a single scalar field Φ in five dimensions. Denote the coordinates of M^4 by x^μ as usual and that of the fifth dimension by y,

$$0 \leq y < 2\pi R. \tag{25.1}$$

Because y is a periodic variable, we can expand the field Φ in Fourier modes:

$$\Phi(x, y) = \sum_n \frac{1}{\sqrt{2\pi R}} \phi_n(x) e^{i p_n y}, \quad p_n = \frac{n}{R}. \tag{25.2}$$

Taking a simple free-field Lagrangian for Φ in five dimensions, the Lagrangian, written in terms of the Fourier modes, takes the form

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$$\int d^4xdy\,\mathcal{L} = -\int d^4xdy\,\frac{1}{2}\left[(\partial\phi)^2 + M^2\phi^2)\right]$$
$$= -\int d^4x\sum_n \frac{1}{2}\left[\partial_\mu\phi^2 + \left(M^2 + p_n^2\right)\phi^2\right]. \tag{25.3}$$

So, from a four-dimensional perspective, this theory describes an infinite number of fields, with ever increasing mass. In the gravitational case, symmetry considerations will force $M = 0$. If we set $M = 0$ in our scalar model, we obtain one massless state in four dimensions ($n = 0$) and an infinite tower – the Kaluza–Klein tower – of massive states. If R is very small, say $R \approx M_{\rm p}^{-1}$, the massive states are all extremely heavy. For the physics of the everyday world we can integrate out these massive fields and obtain an effective Lagrangian for the massless field. The effects of the infinite set of massive fields – the signature of extra dimensions – will show up only in tiny, higher-dimensional operators. So, in the end, finding evidence for these extra dimensions is likely to be extremely difficult.

Having understood this simple model, we can turn to Kaluza and Klein's theory of gravitation and electromagnetism. The five-dimensional theory has the Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{g}R. \tag{25.4}$$

Now there is an infinite tower of massive states corresponding to modes of the five-dimensional metric: $g_{\mu\nu}$, $g_{\mu 4}$ and g_{44}. Our principal interest is in the massless states, which arise from modes that are independent of y (we will need to refine this identification shortly). We expect to find a four-dimensional metric tensor, $g_{\mu\nu}$, a field which transforms as a vector of the four-dimensional Lorentz group, $g_{4\mu}$, and a scalar, g_{44}. There are various ways in which we can rewrite the five-dimensional fields in terms of four-dimensional fields. The physics is independent of this choice, but clearly some choices will be more helpful than others. The most sensitive choice is that of the gauge field; we would like to choose this field in such a way that its gauge transformation properties are simple. The general coordinate invariance associated with transformations of the fifth dimension, $x_4 = x_4 + \epsilon_4(x)$, is given by

$$g_{\mu 4} = g_{\mu 4} + \partial_\mu\epsilon_4(x). \tag{25.5}$$

This looks just like the transformation of a gauge field. So, we adopt the conventions

$$g_{\mu 4} = A_\mu, \quad g_{44}(x) = e^{2\sigma(x)}, \quad g_{\mu\nu} = g_{\mu\nu}. \tag{25.6}$$

Note we are defining, here, a reference metric and are measuring distances relative to that; we can take the basic distance to be the Planck length. Substituting this ansatz back into the five-dimensional action, one can proceed very straightforwardly, working out the Christoffel symbols and from these the various components of the curvature. Gauge invariance significantly constrains the possible terms. One obtains

$$\mathcal{L} = \frac{2\pi R}{2\kappa^2}\sqrt{g}e^{\sigma}R + \frac{1}{4}e^{-\sigma}F_{\mu\nu}^2. \tag{25.7}$$

So the theory at low energies consists of a $U(1)$ gauge field, the graviton and a scalar. The Lagrangian is not quite in the canonical form; usually one writes the action for general relativity in a form where the coefficient of the Ricci scalar (the "Einstein term") is field

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independent. One can achieve this by performing an overall rescaling of the metric, known as a Weyl rescaling,

$$g_{\mu\nu} \to e^{-\sigma} g_{\mu\nu}. \tag{25.8}$$

This introduces a kinetic term for the scalar:

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[R + \frac{3}{2}(\partial\phi)^2\right]. \tag{25.9}$$

The scalar field here is particularly significant. As it corresponds to g_{55}, giving it an expectation value amounts to changing the radius of the internal space. In the Lagrangian there is no potential for σ so, at this level, nothing determines this expectation value. As in our four-dimensional examples, σ is said to be a modulus. We now show that quantum mechanical effects generate a potential for σ even at one loop. This potential falls to zero rapidly as the radius becomes large. If there is a minimum of the potential, it occurs at radii of order one, where the computation is certainly not reliable.

The calculation is equivalent to a Casimir energy computation in quantum field theory; one can think of the system as sitting in a periodic box of size $2\pi R$ and can ask how the energy depends on the size of the box. We can guess the form of the answer before doing any calculation. Since this is a one-loop computation, the result is independent of the coupling. On dimensional grounds the energy density is proportional to $1/R^4$.

To simplify matters, we will treat the gravitational field as a scalar field. At one loop,

$$\Gamma = \mathrm{Tr}\ln\left(-\partial^2 + \frac{n^2}{R^2}\right), \tag{25.10}$$

where we can do the calculation in Euclidean space. We can obtain a more manageable expression by differentiating with respect to R. The trace can be interpreted now as a sum over the possible momentum states in four Euclidean dimensions, in a box of volume VT. Replacing the sum by an integral gives an explicit factor of VT; the coefficient is the energy per unit volume:

$$\frac{\partial V}{\partial R} = \int \frac{d^4p}{(2\pi)^4 R^3}\sum_n \frac{n^2}{p^2 + (n^2/R^2)}. \tag{25.11}$$

This can be evaluated using the same trick as one uses to compute the partition function in finite-temperature field theory (this is described in Appendix C). One first converts the sum into a contour integral, by introducing a function with simple poles located at the integers:

$$\frac{\partial V}{\partial R} = \int \frac{d^4p}{(2\pi)^3}\oint \frac{dz}{2\pi i}\frac{1}{z^2+p^2}\frac{1}{1-e^{2\pi iRz}}\frac{z^2}{R}. \tag{25.12}$$

The contour consists of one line running slightly above the real axis and one line running slightly below it. Now deform the contour in such a way that the upper line encircles the pole at $z = ip$ and the lower line encircles the pole at $z = -ip$. The resulting expression is

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divergent, but we can separate off a term independent of R and a convergent, R-dependent, term:

$$\frac{\partial V}{\partial R} = \frac{1}{R}\int \frac{d^4p}{(2\pi)^4}\frac{p^2}{2p}\left(1+\frac{1}{e^{2\pi pR}-1}\right)$$
$$= \frac{24\zeta[5]}{(2\pi)^4R^5} + R\text{-independent, term;} \tag{25.13}$$

the zeta function was defined in Eq. (22.33).

25.1.1 Generalizations and limitations of the Kaluza–Klein program

So far we have considered the compactification of a five-dimensional theory on a circle, but one can clearly consider compactifications of more dimensions on more complicated manifolds. It is possible to obtain, in this way, non-Abelian groups. So, one might hope to understand the interactions of the Standard Model. The principal obstacle to such a program turns out to be obtaining chiral fermions in suitable representations. The existence of chiral fermions in a particular compactification is a topological question, as can readily be seen. As one smoothly varies the size and shape of the manifold, it is possible that some fields will become massless; equivalently, massless fields can become massive. However, fields which gain mass must come in vector-like pairs; the chiral structure of a theory will not change as one continuously changes the parameters of the compactification.

That chirality is special follows from the observation that spinors in higher dimensions decompose as left–right symmetric pairs with respect to four dimensions. For compactification manifolds with non-trivial topology, it is indeed possible to obtain chiral fermions. However, it turns out to be impossible to obtain chiral fermions in the required representations of the Standard Model group. We will see, though, that string theory can generate both gauge groups and chiral fermions upon compactification.

25.2 Closed strings on tori

So far we have considered compactifications of field theories in higher dimensions, but general higher-dimensional field theories are non-renormalizable and must be viewed as low-energy limits of some other structure. The only sensible structure we know in higher dimensions is string theory. At the same time, if string theory is to have anything to do with the world around us then it must be compactified to four dimensions.

It is not complicated to repeat the field theory analysis for the case of closed strings on circles, or more generally on tori. Consider first compactifying one dimension, X^9, on a circle of radius $2\pi R$. We require that states be invariant under translations by $2\pi R$. This means that the momenta, as in the field theory case, are quantized,

$$p^9 = \frac{n}{R}. \tag{25.14}$$

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But now there is a new feature. Because of the identification of points, the string fields themselves (X^9) need not be strictly periodic. Instead, we now have the mode expansion

$$X^9 = x^9 + p^9\tau + 2mR\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^9 e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^9 e^{-in(\tau+\sigma)}\right), \tag{25.15}$$

where m is an integer. The states with non-zero m are called *winding modes*. They correspond to the possibility of a string winding around, or wrapping, the extra dimension. Now the mass operator, in addition to including a contribution $(p^9)^2 = n^2/R^2$, includes a contribution from the windings, m^2R^2 (if there is no momentum). If R is large compared with the string scale, these states are very heavy. At small R, however, they become light while the momentum (Kaluza–Klein) states become heavy. This reciprocity often corresponds, as we will see, to a symmetry between compactification at large and at small radius.

Let us focus on the various superstring theories. It is convenient to break up X^9 in terms of left- and right-moving fields:

$$X_{\rm L}^9 = \frac{x^9}{2} + \left(\frac{n}{2R} + mR\right)(\tau - \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^9 e^{-in(\tau-\sigma)}, \tag{25.16}$$

$$X_{\rm R}^9 = \frac{x^9}{2} + \left(\frac{n}{2R} - mR\right)(\tau + \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^9 e^{-in(\tau+\sigma)}. \tag{25.17}$$

It is then natural to define left- and right-moving momenta:

$$p_{\rm L} = \frac{n}{2R} + mR, \quad p_{\rm R} = \frac{n}{2R} - mR. \tag{25.18}$$

The world-sheet fermions are untouched by this compactification. The mass operators are essentially as before, with p replaced by $p_{\rm L}$ for the left movers and by $p_{\rm R}$ for the right movers:

$$L_0 = \frac{1}{2}p_{\rm L}^2 + N, \quad \tilde{L}_0 = \frac{1}{2}p_{\rm R}^2 + \tilde{N}. \tag{25.19}$$

Suppose we compactify on a simple product of circles, whose coordinates are labeled X^I. The left- and right-moving momenta form a lattice:

$$p_{\rm L}^I = \frac{n^I}{R^I} + 2m^I R^I, \quad p_{\rm R}^I = \frac{n}{R^I} - 2m^I R^I. \tag{25.20}$$

Now we will determine the spectrum, focusing on the light states. Consider, first, the heterotic string; to simplify the formulas, we take the $O(32)$ case. The $O(32)$ symmetry is unbroken. The original ten-dimensional gauge bosons,

$$\left|A_M^{AB}\right\rangle = \lambda_{-1/2}^A \lambda_{-1/2}^B \psi_{M\,-1/2}|p\rangle, \tag{25.21}$$

now decompose into a set of four-dimensional gauge bosons, corresponding (in light cone gauge) to $M = 2$ and 3, and six scalars corresponding to $M = I$. The graviton, scalar and antisymmetric tensor field now decompose as a set of scalars, g_{IJ}, B_{IJ}, vectors $g_{\mu i}, B_{\mu I}$, a four-dimensional graviton $g_{\mu\nu}$, an antisymmetric tensor, $b_{\mu\nu}$, and a scalar, ϕ.

In order to understand space–time fermions, we will work in light cone gauge and return to our description of $O(8)$ spinors. Group the γ-matrices into a set associated with the

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internal six dimensions and a set associated with the (transverse) Minkowski directions. In other words, instead of the four creation and annihilation operators $a^i, a^{\bar{i}}$, we group these into one set of three (labeled a^i, where now $i = 1, 2, 3$) and b, together with their conjugates). So the 8_s, which previously consisted of the states

$$|0\rangle, \quad a^{i\dagger} a^{j\dagger} |0\rangle, \quad a^{1\dagger} a^{2\dagger} a^{3\dagger} a^{4\dagger} |0\rangle, \tag{25.22}$$

now decomposes as

$$|0\rangle, \quad a^{i\dagger} a^{j\dagger} |0\rangle, \quad b^\dagger a^{j\dagger} |0\rangle, \quad b^\dagger a^{1\dagger} a^{2\dagger} a^{3\dagger} |0\rangle. \tag{25.23}$$

There are four states with no bs and four with one b. These groups have opposite *four-dimensional* helicity. They can also be classified according to their transformation properties under $O(6)$. The group $O(6)$ is isomorphic to $SU(4)$. We have just seen that $8_s = 4 + \bar{4}$. We can also see that, under the $SU(3)$ subgroup of $SU(4)$, the spinor decomposes as

$$8 = 3 + \bar{3} + 1 + 1. \tag{25.24}$$

Now consider how the gravitino in ten dimensions decomposes under $O(3,1) \times SU(4)$. We see that it consists of a set of spin-3/2 particles in the four-dimensional representation of $SU(4)$ and their antiparticles. So, from the perspective of four dimensions, this is a theory with $N = 4$ supersymmetry. This is not really surprising since the ten-dimensional theory is a theory with 16 supercharges, and none of these is touched by this reduction to four dimensions.

Because of the high degree of susy, one cannot write a potential for the scalar fields g_{IJ}, b_{IJ} etc.; they are exactly flat directions. If we redo our Casimir energy calculation then we will find that, because there is a fermionic state degenerate with every bosonic state, there are cancelations.

To what do these moduli correspond? Those which arise from the diagonal components of the metric correspond to the fact that the radii are not fixed. There is a string solution for any value of the R^I. The off-diagonal components are related to the fact that the general torus in six dimensions is not simply a product of circles; there can be non-trivial angles.

The massless scalars arising from the gauge bosons A^I are also moduli. For constant values of these fields there is no associated field strength, so they carry zero energy. But there are non-trivial Wilson lines:

$$U_I = \exp\left(i \int_0^{2\pi R_I} dx^I A_I\right). \tag{25.25}$$

Because of the periodicity these are gauge invariant and correspond to distinct physical states. These moduli are often themselves called Wilson lines.

The periodicities of a general N-dimensional torus can be characterized in terms of N basis vectors e_a^I, $a = 1, \ldots, N$. The theory is defined by the identifications

$$X^I = X^I + 2\pi n^a e_a^I. \tag{25.26}$$

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The set of integers defines a lattice. To determine the allowed momenta we define the dual lattice, with unit vector $\tilde{e}_a^I$, satisfying

$$\tilde{e}_a^I e_b^I = \delta_{a,b}. \tag{25.27}$$

In terms of these, we can write the momenta for the general torus:

$$p^I = n^a \tilde{e}_a^I, \tag{25.28}$$

while the windings are

$$w^I = m^a e_a^I. \tag{25.29}$$

We can break these into left-moving and right-moving parts:

$$p_{\mathrm{L}}^I = (p^I/2 + w^I), \quad p_{\mathrm{R}}^I = (p^I/2 - w^I). \tag{25.30}$$

The lattice of left- and right-moving momenta $(p_{\mathrm{L}}, p_{\mathrm{R}})$ has some interesting features. Considered as a Lorentzian lattice, it is even and self-dual. The term "even" refers to the fact that the inner product of a vector with itself,

$$p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 = 2nm, \tag{25.31}$$

is even. The self-duality means that the basis vectors of the lattice and the dual are the same (Eq. (25.27)).

In bosonic or Type II theories, these are are the most general four-dimensional compactifications with $N = 8$ supersymmetry. The different possible choices of torus define a moduli space of such theories. These moduli space correspond to varying the metric and antisymmetric tensor fields. In the heterotic case, the four dimensional theory has $N = 4$ supersymmetry. Additional moduli arise from Wilson lines. As in the case of the simple compactification on a circle, these are essentially constant gauge fields. A constant gauge field is almost a pure gauge transformation (take I fixed, for simplicity),

$$A^I = ie^{ix_I A^I} \partial^I e^{-ix_I A^I} = ig\, \partial^I g^\dagger, \tag{25.32}$$

but the gauge transformation is only periodic if $A^I = 1/R_I$. In this case the Wilson line is unity. But we can do a redefinition of all the charged fields which eliminates the A^Is,

$$\phi = g\phi'. \tag{25.33}$$

With this choice, the charged fields are no longer periodic but obey boundary conditions

$$\phi'(X^I) = e^{2\pi i R_I A^I} \phi'. \tag{25.34}$$

This means that the momenta are shifted:

$$p^I = \frac{n}{R_I} + A^I. \tag{25.35}$$

Shortly, we will see how all the different momentum lattices can be understood in terms of constant background fields.

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25.3 Enhanced symmetries and T-duality

For large radius, the spectrum of the toroidally compactified string theory is very similar to that expected from Kaluza–Klein field theories. The principal new feature, the winding states, is not important. At smaller radius, however, these states introduce startling new phenomena. We focus first on the compactification of just one dimension. Examining the momenta

$$p_{\rm L} = \frac{m}{2R} + nR, \quad p_{\rm R} = \frac{m}{2R} - nR \tag{25.36}$$

we see that these are symmetric under $R \to 1/(2R)$. This symmetry is often called *T-duality*. It means that there is no sense in which one can take the compactification radius to be arbitrarily small; it is our first indication that there is some sort of fundamental length scale in the theory. The T-duality symmetry is *not* a feature of the compactification of field theory; the string windings are critical.

What is the physical significance of this symmetry? The answer depends on which string theory we study. Consider the heterotic string. We first ask whether duality is truly a symmetry or just a feature of the spectrum of that theory. To settle this we can check that it has a well-defined action on all vertex operators. Alternatively, we can note that there is a self-dual point, $R_{\rm sd} = 1/\sqrt{2}$. Examining Eq. (25.19) we see that, at this radius, various states can become massless. These include both scalars (from the point of view of the non-compact dimensions) and gauge bosons:

$$\psi^{I,\mu}_{-1/2}|n = \pm 1, m = \mp 1\rangle. \tag{25.37}$$

Together with the $U(1)$ gauge boson, the spin-1 particles form the adjoint of an $SU(2)$ group. We can check this by studying the operator product expansions of the associated vertex operators (see the exercises at the end of this chapter).

Now we can understand the $R \to 1/R$ symmetry. At the fixed point the symmetry is an unbroken symmetry. It transforms as follows:

$$p_{\rm L} \to -p_{\rm L}, \quad p_{\rm R} \to p_{\rm R}. \tag{25.38}$$

In world-sheet terms this corresponds to a change of sign of $\partial X_{\rm L}$,

$$\partial X_{\rm L} \to -\partial X_{\rm L}, \quad \partial X_{\rm R} \to \partial X_{\rm R}. \tag{25.39}$$

From (25.37) $X_{\rm L}$ is the third component of isospin, T_3, so $T_3 \to -T_3$ under T-duality.

This transformation corresponds to a 90° rotation about the 1 or 2 axis in the $SU(2)$ space, i.e. it is a gauge transformation! This means that the large and small radii do not merely exhibit the same physics, they *are* the same. It also means that, provided the theory makes sense, the symmetry is an exact symmetry of the theory, in perturbation theory and beyond. As for any gauge symmetry, any violation of this symmetry would signal an inconsistency.

Returning to the self-dual point, the momentum lattice at this point can be thought of as a group lattice, with the $p_{\rm L}$s labeling the $SU(2)$ charges. Much larger symmetry groups can

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be obtained by making special choices of the torus, Wilson lines and antisymmetric tensor fields.

In other string theories the symmetry has a different significance. Consider the Type II theories; take the case of IIA for definiteness. Then, since $\psi^9_{\rm R} \rightarrow -\psi^9_{\rm R}$, the GSO projection in the right-moving Ramond sectors is flipped. So this transformation takes the Type IIA theory to the Type IIB theory. In other words, *the IIA theory at large R is equivalent to the IIB theory at small R*.

25.4 Strings in background fields

The possibilities for string compactification are not limited to tori, they are much richer. We will explore them in this and the next chapter. We can approach the problem in two ways, each of which is very useful. First, we can examine the low-energy effective field theory which describes the massless modes of the string in ten dimensions and look for solutions corresponding to large compactified (i.e. internal) spaces. The effective action can be organized into terms with more and more derivatives. The spaces must be large in order that this use of the low-energy effective action makes sense. Alternatively, we can look for more direct ways to construct classical solutions in string theory. Both approaches have turned out to have great value.

We will first formulate the string problem in a more general way. We want to ask: how do we describe a string propagating in a background which is not flat? The background might be described by a metric, G_{MN}, but it might also include an antisymmetric tensor, B_{MN}, a dilaton, ϕ, and, in the case of the heterotic string, gauge fields. We first focus on the metric. Start with the bosonic string. It is natural, as we saw in the previous chapter, to generalize the string action

$$\frac{1}{2\pi} \int d^2\sigma \; \partial_\alpha X^M \partial^\alpha X^N \eta_{MN} \tag{25.40}$$

to

$$\frac{1}{2\pi} \int d^2\sigma \; \partial_\alpha X^M \partial^\alpha X^N G(X)_{MN}. \tag{25.41}$$

From a world-sheet point of view, we have replaced a simple free-field theory with a non-trivial, interacting-field, theory: a two-dimensional non-linear sigma model. We can think of the X^Ms as fields which propagate on a manifold with metric G_{MN}. Often this space is called the *target space* of the theory; the Xs then provide a mapping from two-dimensional space–time to this target space.

This looks plausible, and we can give some evidence that in fact it is the correct prescription. Suppose, in particular, we consider a metric which is nearly that of flat space:

$$G_{MN} = \eta_{MN} + h_{MN}. \tag{25.42}$$

Substitute this form in the action, and examine the path integral for the field theory:

$$Z[h] = \int [dX^M] \exp\left(iS_0 + \frac{1}{2\pi}\int d^2\sigma\, \partial_\alpha X^M \partial^\alpha X^N h(X)_{MN}\right). \tag{25.43}$$

Differentiating with respect to h brings down a vertex operator for the graviton. In other words, the path integral for this action is the generating functional for the graviton S-matrix.

This observation suggests a general treatment for backgrounds for the massless particles

$$I = \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma\,(g_{IJ}\partial_\alpha X^I + \epsilon^{\alpha\beta} B_{IJ}\partial_\alpha X^I \partial_\beta X^J). \tag{25.44}$$

The corresponding path integral generates the S-matrix elements for both the graviton and the antisymmetric tensor field. But we would like to consider configurations which are not close to the flat metric with vanishing B_{MN}. We can ask: what are acceptable backgrounds for string propagation? To answer this question, we need to remember that, for the free string, *conformal invariance* was the crucial feature to the consistency of the picture. It was conformal invariance which guaranteed Lorentz invariance and unitarity. So we need to look for interacting two-dimensional field theories which are conformally invariant.

25.4.1 The beta function

Field theories of the type we have just encountered are called non-linear sigma models. In $1+1$ dimensions these are renormalizable theories: g_{IJ}, B_{IJ} etc. are dimensionless. A priori, however, they are general functions of the fields, and there are an infinite – continuously infinite – set of possible couplings.

Physically, the statement that these theories must be conformally invariant is equivalent to the statement that their beta functions must vanish. To get some feeling for what this means, let us consider a special situation. Suppose that B_{IJ} vanishes and that the metric is close to the flat-space metric η_{MN}:

$$g_{MN} = \eta_{MN} + \int d^D k\, h_{MN}(k) e^{ik\cdot x}. \tag{25.45}$$

The action is then

$$I = \frac{1}{2\pi}\int d^2\sigma \left(\eta_{IJ}\,\partial_\alpha X^I \partial^\alpha X^J + \sum_k h_{IJ}(k) e^{ik\cdot x}\,\partial_\alpha X^I \partial^\alpha X^J\right). \tag{25.46}$$

We can treat the term involving h as a perturbation. Working to second order, we have

$$\Big\langle \int d^2 z_1 \left[h_{\mu\nu}(k) e^{ik\cdot X(z_1)}\, \partial X(z_1)^\mu \partial X(z_1)^\nu\right] \times \int d^2 z_2 \left[h_{\rho\sigma}(k') e^{ik'\cdot X(z_2)}\, \partial X(z_2)^\rho \partial X(z_2)^\sigma\right]\Big\rangle. \tag{25.47}$$

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We will write this simply as

$$\int d^2z \int d^2z' \, h_1 \mathcal{O}_1(z_1) h_2 \mathcal{O}_2(z_2). \tag{25.48}$$

Ultraviolet divergences will arise in this integral when $z_1 \to z_2$. In this limit, we can use the operator product expansion

$$\mathcal{O}_1(z_1)\mathcal{O}_2(z_2) = \frac{c_{12j}}{|z_1 - z_2|^2} \mathcal{O}_j(z_2) + \cdots. \tag{25.49}$$

The integral over z_2 is ultraviolet divergent. If we cut it off at scale Λ^{-1} then we have the correction to the world-sheet Lagrangian:

$$\int d^2z \, h_1 h_2 c_{12j} \, \mathcal{O}_j \, \ln \Lambda. \tag{25.50}$$

There is another divergence associated with the couplings h_1 and h_2; this comes from normal ordering. In the case of the graviton vertex operator, if we simply expand the exponential factors and contract the xs, we obtain

$$\int d^2z \, h_1(x) k^2 \ln \Lambda. \tag{25.51}$$

Requiring, then, that the beta function for the coupling h_1 should vanish gives

$$k^2 h_1 + h_2 h_3 \, c_{123} = 0. \tag{25.52}$$

Recall now that c_{ijk} is the three-point coupling for the three fields. So this is just the equation of motion to quadratic order in the fields.

This result is general. At higher orders, one encounters divergences of two types. First, there are terms involving a single logarithm of the cutoff times more powers of the fields. Second, there are terms involving higher powers of logarithms. These higher powers are, from a renormalization perspective, associated with iterations of lowest-order divergences, and they are systematically subtracted in computing the beta functions. From a space–time point of view, these correspond to the appearance of massless intermediate states, which must be subtracted in constructing the effective action or equations of motion.

This procedure can be used to recover Einstein's equations. A more elegant and efficient approach is to apply the background-field method. For a general gravitational background, one can view X as a fixed background which solves the two-dimensional equations of motion and study fluctuations about it. For a suitable choice of coordinates, the metric is second order in the fluctuations. One can include in this analysis the background antisymmetric tensor fields and a background dilaton. The antisymmetric tensor can be analyzed along the lines of our analysis of $h_{\mu\nu}$. The dilaton Φ is more subtle. In our action above we omitted one possible coupling: the two-dimensional curvature. The dilaton couples to the world-sheet fields through

$$\int d^2\sigma \, \Phi \mathcal{R}^{(2)}; \tag{25.53}$$

here $\mathcal{R}^{(2)}$ is the two-dimensional curvature scalar.

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The full analysis leads to the equations of motion:

$$\beta_{\mu\nu} = 0 = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu^{\lambda\omega}, \tag{25.54}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi H_{\omega\mu\nu} + \mathcal{O}(\alpha')^2, \tag{25.55}$$

$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H^{\mu\nu\lambda} H_{\mu\nu\lambda}. \tag{25.56}$$

It is possible to extend these methods to describe quantum corrections to the equations, at least in the case of supersymmetric compactifications.

25.4.2 More general toroidal compactification

As a first application, we consider the heterotic string theory in the case of more general toroidal compactification.

For general metrics and backgrounds for both the antisymmetric tensor and gauge fields, one obtains a somewhat more involved expression for the momenta. A particularly elegant way to derive this is to argue that constant background fields should affect only slow modes of the string. In the presence of a background, a constant metric and antisymmetric tensor fields, the action is

$$I = \frac{1}{2\pi} \int d\tau d\sigma \int_0^\pi (g_{IJ}\, \partial_\alpha X^I \partial^\alpha X^j + \epsilon^{\alpha\beta} B_{IJ}\, \partial_\alpha X^I \partial_\beta X^J). \tag{25.57}$$

To realize the notion of slowly varying fields, one makes the ansatz

$$X^I = q^I(\tau) + 2\sigma m^I, \tag{25.58}$$

where the second term allows for the possibility of winding. Substituting this back in the action and performing the integral over σ:

$$I = \int d\tau \left(\frac{1}{2} g_{IJ} \dot{q}^I \dot{q}^J + 2B_{IJ} \dot{q}^I m^J - 2g_{IJ} n^I n^J \right). \tag{25.59}$$

Now we can read off the canonical momenta:

$$P_I = g_{IJ} \dot{q}^J + 2B_{IJ} m^J. \tag{25.60}$$

In quantum mechanics it is the canonical momenta which act by differentiation on wave functions, so it is the canonical momenta which must be quantized for a periodic system:

$$P_I = n_I, \tag{25.61}$$

where n_I is an integer. In terms of q^I this gives

$$\dot{q}^I = g^{IJ} m_J - 2B^I_J n^J. \tag{25.62}$$

Finally, integrating this equation and substituting back into X^I:

$$X^I = q^I + 2\sigma m^I + \tau \left(g^{IJ} n_J - 2B^I_J m^J \right). \tag{25.63}$$

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From this, we can read off the left- and right-moving momenta:

$$p_L^I = m^I + \frac{1}{2} g^{IJ} n_J - g^{IJ} B_{JK} m^K,$$
$$p_R^I = -m^I + \frac{1}{2} g^{IJ} n_J - g^{IJ} B_{JK} m^K. \quad (25.64)$$

Once again, $p_L p_L' - p_R p_R'$ is an integer; the lattice, thought of as a Lorentzian lattice, is even and self-dual.

Including Wilson lines is slightly more subtle, because of their asymmetric coupling between left and right movers. For small A, the modification is essentially what we guessed above. There is also a modification of the internal, E_8-charge, lattice.

25.5 Bosonic formulation of the heterotic string

We have seen that, in toroidal compactifications of string theory, new unbroken gauge symmetries can arise at particular radii. We have also seen that a toroidal compactification can be described by a lattice. So far, in describing the heterotic string we have worked in what is known as the fermionic formulation. There is an alternative formulation, in which the 32 left-moving fermions are replaced by 16 left-moving bosons.

It is an old result that two-dimensional fermions are equivalent to bosons; more precisely, two real left-moving fermions are equivalent to a single real boson, and vice versa. The correspondence, for a complex fermion, λ, is

$$\lambda(z) = e^{i\phi(z)}, \quad (25.65)$$

where ϕ is a left-moving boson. The equal sign here is subtle; at finite volume care is required with the zero modes, as we will see. To be convinced that this equivalence is plausible, consider correlation functions at infinite volume. From our previous analyses of two-dimensional Green's functions, we have

$$\langle \lambda(z)\lambda(w) \rangle = \left\langle e^{i\phi(z)} e^{i\phi(w)} \right\rangle \sim \frac{1}{z-w}. \quad (25.66)$$

This suggests that in the case of, say, the $SO(32)$ heterotic string, we can replace the 32 left-moving fermions by 16 left-moving bosons. Note that this means, loosely, that we have 26 left-moving coordinates, as in the bosonic string (but still only 10 right-moving bosons). At finite volume (i.e. $0 < \sigma < \pi$), we can write the usual mode expansions for these fields:

$$X_L^A = \frac{1}{2} p_L^A + \frac{i}{2} \sum_n \frac{1}{n} \tilde{\alpha}_n^A e^{-in(\tau+\sigma)}. \quad (25.67)$$

Now the p_Ls are elements of the group lattice. Modular invariance requires that the lattice be even and self-dual. In 16 dimensions there are two such lattices, those of $O(32)$ and $E_8 \times E_8$.

The bosonization of fermions which we have described here is useful for the right-moving fields as well, and also for the fermions of the Type II theories. We have avoided

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discussing space–time supersymmetry in the RNS formalism because the fermion vertex operators and the supersymmetry generators must change the boundary conditions on two-dimensional fields. But, in this bosonized form, this problem is simpler. Once again, we have relations of the form

$$\psi_i \sim e^{i\phi_i}. \tag{25.68}$$

The ϕs live on a torus, whose "momenta" describe both N and RS states. Operators of the form $e^{i\phi/2}$ change NS to R states, i.e. they connect bosons to fermions. This connection allows the construction of fermion vertex operators and supersymmetry generators.

25.6 Orbifolds

Toroidal compactifications of string theory are simple; they involve free two-dimensional field theories. But they are also unrealistic. Even in the case of the heterotic string they have far too much supersymmetry, and their spectra are not chiral. There is a simple construction which reduces the amount of supersymmetry, yielding models with interesting gauge groups and a chiral structure. The corresponding world-sheet theories are still free, so explicit computations are straightforward. These constructions are also interesting in other ways. They correspond to particular submanifolds of the moduli space of larger classes of solutions. They exhibit interesting features such as discrete symmetries and subtle cancelations of four-dimensional anomalies. At low orders it is a simple matter to work out their low-energy effective actions. Through a combination of world-sheet and space–time methods, one can understand their perturbative, and in some cases non-perturbative, dynamics.

Here, we will work out one example in some detail. Other examples can be studied in a similar way. We will also mention some other free-field constructions of interesting string solutions.

We start with a toroidal compactification on a particular lattice, a product of three tori as shown in Fig. 25.1. It is convenient to introduce complex coordinates,

$$z^1 = x^1 + ix^2, \quad z^2 = x^3 + ix^4, \quad z^3 = x^5 + ix^6. \tag{25.69}$$

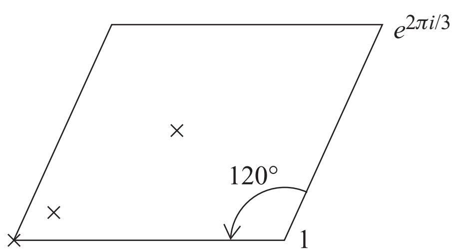

Fig. 25.1 A torus that admits a Z_3 symmetry, allowing an orbifold construction.

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This lattice is invariant under the Z_3 symmetry

$$z^i \to e^{2\pi i/3} z^i. \tag{25.70}$$

This can be seen by examining the figure carefully. The lattice vector $(1, 0)$, for example, in the original Cartesian coordinates is rotated into the lattice vector $(-1/2, 1/\sqrt{2})$. This is related by a lattice vector translation to $(1/2, 1/\sqrt{2})$.

Now we identify points under the symmetry, i.e. two points related by a symmetry transformation are considered to be the same point. The result is almost a manifold, but not quite. There are three particular points which are invariant under the symmetry. These are called *fixed points*. They are the points

$$(0, 0), \quad (1/2, \sqrt{3}/2), \quad (1, \sqrt{3}). \tag{25.71}$$

The geometry near each of these points is singular. If one parallel-transports about, say, the point at the origin then after 120° one has returned to where one started. The space is said to have a deficit angle (a *conical singularity*). It is as if there were an infinite amount of curvature located at each of the points. Such a space is called an *orbifold*.

In quantum mechanics, requiring such an identification of points under a symmetry means requiring that states be invariant under the quantum mechanical operator which implements the symmetry. Consider the various states of the original ten-dimensional theory. In the Type II theory, for example, in the NS–NS sector we have the following states, before making any identifications:

$$\tilde{\psi}^{\mu}_{-1/2}\psi^{\nu}_{-1/2}|0\rangle, \quad \tilde{\psi}^{\bar{j}}_{-1/2}\psi^{i}_{-1/2}|0\rangle, \quad \tilde{\psi}^{j}_{-1/2}\psi^{\bar{i}}_{-1/2}|0\rangle, \tag{25.72}$$

$$\tilde{\psi}^{j}_{-1/2}\psi^{i}_{-1/2}|0\rangle, \quad \tilde{\psi}^{\bar{i}}_{-1/2}\psi^{\bar{j}}_{-1/2}|0\rangle. \tag{25.73}$$

After the identifications, the first set of states is invariant; the latter two are not. These states all have simple interpretations. The first three are the four-dimensional graviton, the antisymmetric tensor and the dilaton. The second two are the moduli of the torus. The parts symmetric under $i \to \bar{j}$ correspond to the metric components $g_{i\bar{j}}$ in the original theory. The antisymmetric parts correspond to the corresponding components of $B_{i\bar{j}}$.

The diagonal components, $g_{i\bar{i}}$, are easily understood. Changing the value of these components slightly corresponds to changing the overall radius of the ith torus. This does not change the symmetry properties. The off-diagonal components, $g_{1\bar{2}}$ etc., correspond to deformations which mix up the three planes but leave a lattice with an overall Z_3 symmetry.

To understand what happens to the supersymmetries, we will focus on the gravitino. It is convenient to work in light cone gauge and to decompose the spinors as we did earlier. To determine how the spinors transform under the Z_3, we need to decide how the state we called $|0\rangle$ transforms under the symmetry. Consider a rotation, say in the 12 plane, by 120°. The rotation generator is

$$S_{12} = \frac{i}{4}(\gamma_1\gamma_2 - \gamma_2\gamma_1) = a^{1\dagger}a^1 + \frac{1}{2}. \tag{25.74}$$

So, the rotation of the state $|0\rangle$ is described by

$$e^{(2\pi i/6)s_{12}}|0\rangle = e^{2\pi i/6}|0\rangle. \tag{25.75}$$

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The transformations of the other states can then be read off from the transformation laws of the a^is:

$$|0\rangle \rightarrow e^{-2\pi i/6}|0\rangle, \quad a^{\bar{i}}|0\rangle \rightarrow e^{2\pi i/6}a^{\bar{i}}|0\rangle. \tag{25.76}$$

Now we have to be a bit more precise about the orbifold action. This is a product of Z_3s for each plane. But we see that, acting on fermions, the separate transformations are Z_6s. In order that the group action be a sensible Z_3 we need to take, for example:

$$Z^1 \rightarrow e^{2\pi i/3}Z^1, \quad Z^2 \rightarrow e^{2\pi i/3}Z^2, \quad Z^3 \rightarrow e^{-4\pi i/3}Z^3. \tag{25.77}$$

With this definition the fermion component 0, which we will write as $|0\rangle$, is invariant under the orbifold projection. The components i , which we will write as $a^{\bar{i}}|0\rangle$, are not.

We can label the gravitinos

$$\psi^\mu_{0,\alpha}, \quad \tilde{\psi}^\mu_{\bar{0},\alpha}, \quad \psi^\mu_{i,\alpha}, \quad \tilde{\psi}^\mu_{i,\alpha}. \tag{25.78}$$

After the projection, instead of eight gravitinos, as in the toroidal case, there are only two; we have $N = 2$ supersymmetry in four dimensions.

In addition to projecting out states we need to consider a new class of states. We can consider closed strings which sit at the fixed points. More precisely, in addition to the strict periodic boundary condition we can consider strings which satisfy

$$X^i(\sigma + \pi) = e^{2\pi i/3}X^i(\sigma). \tag{25.79}$$

These boundary conditions do not permit the usual bosonic zero modes. Instead, we have a mode expansion

$$X^i = x^i_{(a)} + \frac{i}{2}\sum_n \left(\alpha^i_{n-1/3}e^{2i(n-1/3)(\sigma-\tau)} + \tilde{\alpha}^i_{n-1/3}e^{2i(n-1/3)(\sigma+\tau)}\right). \tag{25.80}$$

The mode numbers are now fractional; the absence of a momentum term indicates that the strings sit at fixed points (labeled by a). In this case there are 27 fixed points. For the fermions, we again have to distinguish the Ramond and Neveu–Schwarz sectors. In the NS sectors the fermions have modes which differ from integers by multiples of $1/2 - 1/3 = 1/6$:

$$\psi^i = \sum \psi_{n-1/6}e^{-2i(n-1/6)(\tau-\sigma)}, \tag{25.81}$$

with a similar expansion for $\tilde{\psi}$.

We can readily work out the normal-ordering constant, using a formula that we wrote down earlier (Eq. (22.30)). We have, in the NS–NS sector:

$$a = 6 \times \frac{1}{4}\left(\frac{1}{3} \times \frac{2}{3}\right) - 6 \times \frac{1}{4}\left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right) - 4 \times \frac{1}{4}\left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = 0. \tag{25.82}$$

So, the ground state is massless in the twisted sectors. Again, because of the $N = 2$ supersymmetry there can be no potential for this field. So there is a modulus in each twisted sector. Unlike the moduli in the untwisted sector, this modulus does not correspond to a simple change in the features of the torus which defines the orbifold. Instead, it represents a deformation which, from a space–time viewpoint, smooths out the orbifold singularity.

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The resulting smooth space is an example of a Calabi–Yau manifold, of a type that we will discuss in the next chapter.

We now turn to the heterotic string theory on this orbifold. We will take the same projector on the spatial coordinates X^i as before. As a result there is only one gravitino; the four-dimensional theory has $N = 1$ supersymmetry. The moduli are in one-to-one correspondence with the scalars of the NS–NS sector of the $N = 2$ theory: $g_{i\bar{j}}, B_{i\bar{j}}, \phi$. We can also make a projection on to the world-sheet gauge degrees of freedom. There are many possible choices of this gauge transformation; the principal restriction comes from the requirement of modular invariance. A particularly simple one is almost symmetrical between the left and right movers. In the fermionic formulation it works as follows. Take $E_8 \times E_8$ for definiteness. Of the 16 fermions in the first E_8, single out six, and rewrite them in terms of three complex fermions, λ^i. Call the remaining ten fermions λ^a. Now, in the projection, require invariance under

$$Z^i \to e^{2\pi i/3} Z^i, \quad \psi^i \to e^{2\pi i/3} \psi^i, \quad \lambda^i \to e^{2\pi i/3} \lambda^i. \tag{25.83}$$

In the untwisted sector this projection has no effect on the graviton or the moduli which we have identified previously. But consider the various gauge fields. In ten dimensions these were vectors in the adjoint of the two E_8s and their fermionic partners. The fields with space–time indices in the internal dimensions now appear as four-dimensional scalars. In order that they be invariant under the full projection, it is necessary to choose their gauge quantum numbers appropriately. In the NS sector, for each E_8, the invariant states include the following.

1. *A set of fields in the adjoints of E_6 and E_8 and an $SU(3)$* Of these, an $O(10)$ subgroup of the E_6 is manifest in the NS–NS–NS sector, as well as an $O(16)$ subgroup of E_8. Correspondingly, the gauge bosons are

$$\lambda^a_{-1/2} \lambda^b_{-1/2} \psi^\mu_{-1/2} |0\rangle \tag{25.84}$$

in $O(10)$,

$$\lambda^A_{-1/2} \lambda^B_{-1/2} \psi^\mu_{-1/2} |0\rangle \tag{25.85}$$

in $O(16)$ and, in $SU(3) \times U(1)$,

$$\lambda^i_{-1/2} \lambda^{\bar{i}}_{-1/2} \psi^\mu_{-1/2} |0\rangle. \tag{25.86}$$

Note that all these states are invariant. The $U(1)$ is actually an E_6 generator. The group E_6 has an $O(10) \times U(1)$ subgroup under which the adjoint representation, which is 78-dimensional, decomposes as follows:

$$78 = 45_0 + 1_0 + 16_{-1/2} + \overline{16}_{1/2}. \tag{25.87}$$

The remaining E_6 gauge bosons are found in the R–NS–NS sector. The left-moving normal ordering constant vanishes. The ground states in this sector are spinors of $O(10)$, the 16 and $\overline{16}$ above. The 248-dimensional representation of the second E_8 is filled out as in the uncompactified theory.

2. *Matter fields* These lie in the fundamental representation of E_6, the 27 under $O(10)$. The 27 decomposes as follows:

$$27 = 1_{-2} + 10_1 + 16_{-1/2}. \tag{25.88}$$

There are nine 10s in the untwisted sectors, corresponding to the states

$$\lambda^a_{-1/2}\lambda^i_{-1/2}\psi^{\bar{j}}_{-1/2}|0\rangle. \tag{25.89}$$

Each of these is one real scalar; we can use the conjugate fields to form nine more real scalars or eight complex scalars. There are nine singlets of charge -2:

$$\lambda^{\bar{i}}_{-1/2}\lambda^{\bar{j}}_{-1/2}\psi^{\bar{k}}_{-1/2}|0\rangle. \tag{25.90}$$

The 16s come from the R–NS–NS sector.

So, we have nine 27s from the twisted sectors, and no $\overline{27}$s; the theory is chiral.

Let us turn now to the twisted sectors. In the Type II case we found moduli in each sector. Here we will find moduli, additional 27s and more. We first need to compute the normal-ordering constants. For the right movers the calculation is exactly as in the Type II theory and gives zero. For the left movers in the NS–NS sector, we have

$$\begin{aligned} a = &-\frac{8}{24} + \frac{6}{4} \times \frac{1}{3} \times \frac{2}{3} - \frac{16}{4} \times \left(-\frac{1}{24} + \frac{1}{4}\right) \\ &+ \frac{16}{24} - \frac{10}{4} \times \frac{1}{4} - \frac{6}{4} \times \frac{1}{6} \times \frac{5}{6} \\ = &-1/2, \end{aligned} \tag{25.91}$$

where the first two terms come from the bosons, the next two from the fermions in the unbroken E_8 and the last two from the fermions in the broken E_8. So we can make massless states in a variety of ways:

1. *ten-dimensional representations of* $O(10)$,

$$\lambda^a_{-1/2}|0\rangle_{\text{twist}} \tag{25.92}$$

(note that E_6 invariance requires that this state have $U(1)$ charge $+1$);
2. *a singlet of* $O(10)$ *with* $U(1)$ *charge* -2,

$$\lambda^{\bar{1}}_{-1/6}\lambda^{\bar{2}}_{-1/6}\lambda^{\bar{3}}_{-1/6}|0\rangle_{\text{twist}} \tag{25.93}$$

(together with a set of spinorial states from the R–NS sector, this completes a 27 of E_6);
3. *moduli, other gauge singlets*,

$$\alpha^i_{-1/3}\lambda^{\bar{j}}_{-1/6}|0\rangle \tag{25.94}$$

(if we contract the i and $\bar{j}$ indices, we find the analog of the twisted sector modulus we had in the Type II theory; the other states represent additional singlets).

All together, then, we have found $9 + 27 = 36$ copies of the 27 of E_6, and 36 moduli. Each 27 comfortably accommodates a generation of the Standard Model plus an additional vector-like set of fields. So, while this example is hardly realistic, it is interesting: it predicts a particular number of Standard Model generations, plus additional fields. Whether variants of these ideas can lead to something more realistic is an important question, which we will postpone for the time being.

25.6.1 Discrete symmetries

An unappealing feature of supersymmetric models as theories of nature is the need to postulate discrete symmetries in order to have a sensible phenomenology. This seems rather ad hoc. One aspect of the orbifold construction we have just described is that a variety of discrete symmetries appear naturally. This phenomenon is common in string constructions, as we will see. Here it is particularly easy to exhibit the symmetries.

We have, for simplicity, considered a particular form for the torus – a particular point in the moduli space at which the six-dimensional torus is a product of three two-dimensional tori. But at this point (which is really a surface), there is a large symmetry. First, there is a separate Z_3 symmetry for each plane. (You can check that each plane in fact admits a Z_6 symmetry.) Because of the orbifold projection, one of these symmetries acts trivially on all states but two are non-trivial. If we take the size of each of the three two-dimensional tori to be the same then we also have a permutation symmetry, S_3, among the tori.

The Z_3s are R symmetries. We have already seen that the spinor with index 0 rotates by a phase $e^{2\pi i/6}$ under such a symmetry. By definition, this is an R transformation. This has significant consequences for the low-energy theory, greatly restricting the form of the superpotential.

As an example of the far-reaching consequences of such symmetries, one can show that there are exactly flat directions involving the matter fields. Consider the untwisted moduli. One can give expectation values to the $O(10)$ ten-dimensional and one-dimensional representations in one multiplet in a way which respects the supersymmetry. Specifically, consider the field ϕ given by

$$\phi = \lambda^a_{-1/2}\psi^{\bar{1}}_{-1/2}|0\rangle \tag{25.95}$$

and the corresponding singlet. Both of these are neutral under the rotation in the second plane. So, one cannot construct any superpotential term involving ϕ alone. One can give an expectation value to the singlet and to the 10 in such a way as to cancel the D terms for E_6. The main danger, then, is a superpotential term of the form

$$W = \Psi\phi^2 \tag{25.96}$$

with Ψ some other 27. This is E_6 invariant (in terms of $O(10)$ representations, it involves a product of a singlet and two 10s). But no such term is allowed by the discrete symmetries.

This simple argument shows that the moduli space is even larger than we might have thought. Such symmetries, as they forbid not only certain dimension-four but also certain dimension-five operators, might also be important for understanding the problem of proton stability and other important phenomenological issues.

The model possesses other symmetries as well. There is Z_3 symmetry, under which the twisted sector states transform but the untwisted sector states do not. We will not derive this here but it is plausible, and can be shown readily if one constructs the vertex operators for the twisted states. Many discrete symmetries of the model are subgroups of the Lorentz symmetry of the original higher-dimensional theory. As such they can probably be thought of as gauge symmetries. This is less obvious for other symmetries, but it is generally believed that the discrete symmetries of string theory all have this character. Searches for anomalies in discrete symmetries, for example, have yielded no examples.

One could ask: why would nature choose a point in the moduli space of some string theory at which there is an unbroken discrete symmetry? At the moment our understanding of how to connect string theory to nature is not good enough to give a definite answer to this question but, at the very least, such points are *necessarily* stationary points of the effective potential for the moduli; at the symmetric point, the symmetry forbids linear terms in the action for the charged moduli.

25.6.2 Modular invariance, interactions in orbifold constructions

As in our original string theory constructions, there seems much which is arbitrary in the choices we made above. Also, we have not spelled out what are the appropriate GSO projectors. As for the simple ten-dimensional constructions, the possible GSO projections are constrained by modular invariance. We will leave for the exercises the checking of some particular cases, but the basic result is easy to state. One can project by any transformation, provided that it has a sensible action on fermions and on spinor representations of the gauge group and provided that one has “level matching” in all the twisted sectors. This means that one must be able to construct an infinite tower of states in each sector. To understand the significance of this statement, consider a different choice of group action from that we considered above. Instead of twisting by $(1/3, 1/3, -2/3)$, project by $(1/3, -1/3, 0)$. In this case, for example, in the NS–NS–NS sector, the left-moving normal-ordering constant is $-13/18$. As a result, one cannot construct any states in the twisted sector which satisfy the level-matching condition.

There are other constructions of compactifications with $N = 1$ supersymmetry based on free fields. These include models based purely on free fermions. These models are believed to be equivalent to orbifold models in which one *mods out* (performs projections) asymmetrically on the left- and right-moving fields. The latter, “asymmetric orbifold”, models are interesting in that they potentially have very few moduli. In order to have sensible, unbroken, discrete symmetries acting on the left and right, typically the original lattice must sit at a self-dual point. So, many moduli are fixed – they are projected out by the orbifold transformation. It is not difficult, in this way, to construct models where there are no moduli that are neutral under space–time symmetries except for the dilaton.

25.7 Effective actions in four dimensions for orbifold models

While string theory provides a very explicit set of computational rules, at least for low orders of perturbation theory, these rules are complicated and rather cumbersome.

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Moreover, except in some special circumstances we lack a non-perturbative formulation of the theory. Effective-field-theory methods have proven extremely useful in understanding the dynamics of string theory, both perturbative and non-perturbative. In this section we will work out the effective action for the orbifold models introduced above. More precisely, we work out the Lagrangian for a subset of the fields, up to and including terms with two derivatives. Many features of these Lagrangians will be relevant to the more intricate Calabi–Yau compactifications that we will encounter shortly.

In principle, to calculate the effective action we should calculate the string S-matrix and write down an action for the massless fields which yields the same scattering amplitudes. Alternatively, we can calculate the equations of motion from the beta function and look for an action which reproduces these. But, for low-order terms in the derivative (α') expansion, for the fields in the untwisted sector there is a simpler procedure. We know the form of the ten-dimensional effective action; we can simply truncate the theory to four dimensions. To do this, we start by setting all the charged fields to zero (this includes the gauge fields). We also work at a point with a large discrete symmetry: $Z_3^3/Z_3 \times S_3$. We set all the fields which transform under these symmetries to zero. This includes all the moduli except the one that determines the overall size of the torus and its superpartners. We then write the metric as

$$g_{i\bar{j}}(x^\mu) = g_{\bar{j}i}(x^\mu) = e^{\sigma(x^\mu)}\delta_{i\bar{j}}. \tag{25.97}$$

With this parameterization we are describing the size of the space with respect to a reference metric. We make a similar ansatz for the antisymmetric tensor:

$$b_{i\bar{j}}(x^\mu) = -b_{\bar{j}i}(x^\mu) = b(\sigma(x^\mu))\delta_{i\bar{j}}. \tag{25.98}$$

We must keep also the four-dimensional metric components $g_{\mu\nu}$, the scalar field ϕ and the antisymmetric tensor $B_{\mu\nu}$. We take them all to be functions of x^μ, the uncompactified coordinates, only. Substituting these fields into the ten-dimensional Lagrangian, Eq. (24.8), the integral over the six internal coordinates is easy since all fields are independent of the coordinates. One simply obtains $e^{3\sigma(x)}$ from the $\sqrt{g}$ factor. This is just the volume of the internal space, if σ is constant. There are additional factors $e^{-\sigma}$ coming from the factors of the inverse metric: one from the four-dimensional contribution to the Ricci curvature; one from the kinetic term for ϕ; and three from the $H_{\mu\nu\rho}$ terms. The ten-dimensional curvature term also gives derivative terms in σ. After a short computation we obtain

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}e^{3\sigma}R^{(4)} - 3e^{3\sigma}\partial_\mu\sigma\partial^\mu\sigma - \frac{9}{16}e^{3\sigma}\frac{\partial_\mu\phi\partial^\mu\phi}{\phi^2} \\ &- \frac{9}{2}e^{\sigma}\phi^{-3/2}\partial_\mu b\partial^\mu b - \frac{3}{4}\phi^{-3/2}H^{\mu\nu\rho}H_{\mu\nu\rho}.\end{aligned} \tag{25.99}$$

It is customary to rescale the metric so that the Einstein term has the standard form

$$g_{\mu\nu} = e^{-3\sigma}g'_{\mu\nu}. \tag{25.100}$$

After this Weyl rescaling, the action becomes

$$\mathcal{L} = -\frac{1}{2}R^{(4)} - 3\partial_\mu\sigma\partial^\mu\sigma - \frac{9}{16}\frac{\partial^\mu\phi\partial_\mu\phi}{\phi^2} - \frac{3}{2}e^{-2\sigma}\phi^{-3/2}\,\partial_\mu b^2 - \frac{3}{4}\phi^{-3/2}e^{6\sigma}H^2_{\mu\nu\rho}. \tag{25.101}$$

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It should be possible to cast this Lagrangian as a standard four-dimensional, $N = 1$ supergravity Lagrangian, with a particular Kahler potential. Having set to zero all the fields except for a few moduli, there is no superpotential. To determine the Kahler potential we first note that, in four dimensions, an antisymmetric tensor field is equivalent to a scalar. This follows from counting degrees of freedom; with our usual rules, an antisymmetric tensor in four dimensions has only one degree of freedom. To make this explicit, one performs a "duality transformation" (the term is starting to seem a bit overused!)

$$\phi^{-3/2} e^{6\sigma} H_{\mu\nu\rho}(x) = \epsilon_{\mu\nu\rho\sigma} \partial^{\sigma} a(x). \tag{25.102}$$

The field a is often called the model-independent axion because it couples like an axion and its features do not depend on the details of the compactification. Then we define two chiral superfields, whose scalar components are

$$S = e^{3\sigma} \phi^{-3/4} + 3i\sqrt{2}a \tag{25.103}$$

and

$$T = e^{\sigma} \phi^{3/4} - i\sqrt{2}b. \tag{25.104}$$

Choosing the Kahler potential

$$K = -\ln(S + S^*) - 3\ln(T + T^*) \tag{25.105}$$

reproduces all the terms in Eq. (25.101). The reader may want to check the terms in this equation carefully, but at the least it is a good idea to make sure one understands how the σ^{-1} and ϕ^{-} dependences are reproduced.

Let's now return to the ten-dimensional gauge field terms, Eq. (24.9). This will allow us to include the matter fields as well as the gauge fields. Rather than consider the full set of fields, we can restrict ourselves to the set which is invariant under each separate Z_3 in combination with three separate Z_3s in the gauge group ($\lambda^i \to e^{2\pi k_i i/3}\lambda^i$). This leaves us with three complex scalars C^i corresponding to the states

$$C^i \leftrightarrow \lambda^i_{-1/2} \lambda^a_{-1/2} \psi^{\bar{i}}_{-1/2} |0\rangle \tag{25.106}$$

(here i is not summed). From the point of view of ten dimensions, these are the $A^{ia}_{\bar{i}}$. We also need to include the four-dimensional gauge fields A^{ab}_{μ}. In this way we obtain the additional terms, after Weyl rescaling,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\phi^{-3/4} e^{3\sigma} F^2_{\mu\nu} - 3e^{-\sigma}\phi^{-3/4} D_{\mu} C^{*\bar{i}} D^{\mu} C^i + \cdots. \tag{25.107}$$

This can still be put into the standard supergravity form. First we need to remember that, in the duality transformation, $H_{\mu\nu\rho}$ now includes the Chern–Simons terms. Then it is necessary to modify the definition of T to include a contribution from the C fields, so that now

$$T = e^{\sigma} \phi^{3/4} - i\sqrt{2}b + C^{*\bar{i}} C^i, \tag{25.108}$$

and to modify the Kahler potential to

$$K = -\ln(S + S^*) - 3\ln(T + T^* - C^* C). \tag{25.109}$$

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There is also a coupling of the field S to the gauge fields:

$$\mathcal{L}_S = -\frac{1}{4} S W_\alpha^2. \tag{25.110}$$

This includes a coupling of ϕ and σ to $F_{\mu\nu}^2$, already apparent in Eq. (24.9). The $aF\tilde{F}$ coupling arises from the Chern–Simons term in Eq. (24.10). Recall that

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} - \omega_{\mu\nu\rho}. \tag{25.111}$$

So $\int d^4x\, H^2$, using the definition of a and integrating by parts, gives an $aF\tilde{F}$ coupling. Finally, there is a superpotential that is cubic in the C fields.

25.7.1 Couplings and scales

It is worth pausing to note the connections between the couplings and scales in different dimensions. We will focus first on the heterotic string. We see from Eq. (25.110) that S determines the gauge coupling: $S = 1/g^2$. This is as we would naively expect. The ten-dimensional gauge coupling: is essentially $1/g_s^2$; when we reduce to four dimensions, the four-dimensional gauge fields correspond to modes which are constant on the internal manifold, so that

$$\frac{1}{g_4^2} = \frac{1}{g_s^2} V M_s^6. \tag{25.112}$$

In terms of the fields we defined above, $V = e^{3\sigma}$.

These simple formulas pose a serious problem for the application of weakly coupled heterotic string phenomenology. If we simply identify S with the four-dimensional coupling then the string coupling satisfies

$$g_s^2 = g_4^2 V M_s^6. \tag{25.113}$$

So, we see that at large volume, the limit in which an α' expansion is valid, there is a conflict with small g_s if g_4 is fixed. We can also write a relation between the string scale and the Planck scale in four dimensions:

$$M_{\mathrm{p}}^2 = M_s^8 V g_s^{-2}. \tag{25.114}$$

Solving for M_s and substituting in the previous expression gives an expression for g_s which is incompatible with weak coupling, if we assume that $V = M_{\mathrm{gut}}^{-6}$.

Later, we will sharpen this strong coupling problem and consider possible solutions.

25.8 Non-supersymmetric compactifications

So far, we have considered compactifications that are supersymmetric. This is not a necessary restriction, but we will see that non-supersymmetric compactifications raise new conceptual and technical problems.

Perhaps the simplest non-supersymmetric compactification is *Scherk–Schwarz compactification*. Here one compactifies the theory (this can be Type I, Type II or heterotic) on a torus. In one direction, say the ninth direction, one imposes the requirement that bosons should obey periodic boundary conditions and fermions anti-periodic ones. One can describe this by taking the radius of the extra dimension to be $2 \times 2\pi R$ and performing a projection

$$P = (-1)^F e^{i(2\pi i)Rp_9}. \tag{25.115}$$

This projection eliminates, for example, the massless gravitinos; there is no supersymmetry and no Bose–Fermi degeneracy in the spectrum. Indeed, in the simplest version there are no massless fermions at all.

As a result, the usual Fermi–Bose cancelation of supersymmetry does not take place and, at one loop, there is a non-zero vacuum energy. More precisely there is a potential for the classical modulus R. The calculation of this potential is just the Casimir calculation we encountered earlier. Only the massless ten-dimensional fields contribute; the massive string states give effects which are exponentially suppressed for large R. To see this one can return to our earlier calculation with a massive state (an oscillator excitation of the string). Replacing the sum over integers by an integral in the complex plane and deforming the contour, as in Eqs. (25.11)–(25.13), yields a term exponentially small in the mass. The detailed results depend on the particular model, but typically the potential is negative and goes to zero at large R. In other words, at one loop the dynamics tends to drive the system to small R. It is not well understood how to study the system beyond one loop.

One can obtain non-supersymmetric theories in four dimensions in many other ways. The Scherk–Schwarz construction can be understood as modding out a supersymmetric compactification by an R symmetry. With this viewpoint, one can simply enumerate the R symmetries of a particular construction and mod out, subject to conditions of modular invariance.

Suggested reading

An introduction to Kaluza–Klein theory prior to the development of string theory is provided in the text *Modern Kaluza–Klein Theories* by Appelquist *et al*. (1985). More thorough discussions of aspects of string compactification are provided by the texts of Green *et al*. (1987) and Polchinski (1998). Some original papers, particularly the orbifold papers, are highly readable; see, for example, Dixon *et al*. (1986). There are many topics here thay we have only touched on in this chapter. We gave an argument that the vanishing of the beta function of the two-dimensional sigma model is equivalent to the equations of motion in space–time, but readers may wish to work through the background field analysis which leads to Einstein's equations. This is described in Polchinski's book and elsewhere. The bosonic formulation of the heterotic string is also well described there, but the original papers are quite readable (Gross *et al*., 1985, 1986). Bosonization and space–time supersymmetry in the RNS formulation are thoroughly discussed by Polchinski (1998); a clear, but rather brief, introduction, is provided by Peskin's 1996 TASI lectures (Peskin, 1997). The non-supersymmetric compactification described here was introduced by Rohm (1984).

Exercises

(1) Derive the gauge terms in the Lagrangian of Eq. (25.7). You can do this by taking the metric to be flat.

(2) Derive the scalar kinetic terms of Eq. (25.8). You can do this by at first taking the four-dimensional metric to be flat, and allowing only σ to be a function of x.

(3) Verify, by studying the OPEs of the vertex operators for the different massless fields, that the enhanced symmetry of the bosonic string at the point $R = 1/\sqrt{2}$ is $SU(2) \times SU(2)$. Explain why, in the heterotic string, the symmetry is only $SU(2)$. What is the symmetry in the IIA theory?

(4) For the orbifold model, work out the spectrum in the untwisted sectors in greater detail, paying particular attention to spinorial representations of the O groups and to the space–time spinors. In particular, make sure that you are clear that the 27s are chiral, i.e. all the states in the 27s have one four-dimensional chirality and all those in $\overline{27}$s have the opposite chirality.

(5) Derive the term in Eq. (25.99) involving $\partial\sigma^2$.

(6) Verify that the Kahler potential of Eq. (25.109) properly reproduces the kinetic terms of the matter fields.

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