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# NATURALLY REDUCTIVE HOMOGENEOUS RIEMANNIAN MANIFOLDS

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**1. Introduction.** The simple algebraic and geometric properties of naturally reductive metrics make them useful as examples in the study of homogeneous Riemannian manifolds. (See for example **[2]**, **[3]**, **[15]** ). The existence and abundance of naturally reductive left-invariant metrics on a Lie group $G$ or homogeneous space $G/L$ reflect the structure of $G$ itself. Such metrics abound on compact groups, exist but are more restricted on noncompact semisimple groups, and are relatively rare on solvable groups. The goals of this paper are

(i) to study all naturally reductive homogeneous spaces of $G$ when $G$ is either semisimple of noncompact type or nilpotent and

(ii) to give necessary conditions on a Riemannian homogeneous space of an arbitrary Lie group $G$ in order that the metric be naturally reductive with respect to some transitive subgroup of $G$.

In Sections 2 and 3, we show that every naturally reductive Riemannian manifold may be realized as a homogeneous space $G/L$ of a Lie group of the form $G = G_{\text{nc}}G_{\text{c}}N$ where $G_{\text{nc}}$ is a noncompact semisimple normal subgroup, $G_{\text{c}}$ is compact semisimple, and $N$ is the nilradical of $G$. $N \cap L = \{e\}$ and the induced metrics on each of $G_{\text{nc}}/(G_{\text{nc}} \cap L)$, $G_{\text{c}}/(G_{\text{c}} \cap L)$ and $N(= N/(N \cap L))$ are naturally reductive. Thus the study of naturally reductive metrics is partially reduced to the cases in which $G$ is semisimple of either compact or noncompact type or $G$ is nilpotent.

D'Atri and Ziller [**3**] have studied the compact case extensively obtaining a complete classification of naturally reductive left-invariant metrics on $G$ when $G$ is compact and simple (and $L$ is trivial). Our treatment of the noncompact semisimple case in Section 5 is motivated by their work. The complete classification is possible in this case without the additional assumptions of simplicity of $G$ or triviality of $L$, due to the relative sparcity of naturally reductive metrics on noncompact as opposed to compact homogeneous spaces.

We study the nilpotent case in Section 4. Any Riemannian manifold which admits a (necessarily simply) transitive nilpotent group $G$ of isometries is called a homogeneous nilmanifold. If $M$ is naturally

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reductive, we show that $G$ is at most two-step nilpotent. We then give necessary and sufficient conditions for a two-step homogeneous nilmanifold to be naturally reductive. Our methods generalize those of Kaplan [**11**] who studied a special class of homogeneous nilmanifolds.

In Section 6 we pull together all the results of the previous sections in order to study problem (ii).

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**2. Naturally reductive submanifolds of** $M$. Let $M$ be a connected homogeneous Riemannian manifold and let $\tilde{G} = I_0(M)$ be the connected component of the identity in the full isometry group of $M$. $\tilde{G}$ acts transitively and effectively on $M$, and the isotropy subgroup $\tilde{L}$ of $\tilde{G}$ at $p \in M$ is compact. If $G$ is any transitive subgroup of $\tilde{G}$ and $L = G \cap \tilde{L}$, then $M$ is naturally identified with the coset space $G/L$ with a left-invariant metric. Recall that $\mathfrak{l}$ is compactly embedded in $\mathfrak{g}$; i.e., $\mathfrak{g}$ admits an inner product relative to which the operators $\mathrm{ad}_{\mathfrak{g}}X$, $X \in \mathfrak{l}$, are skew-symmetric. (We will always denote the Lie algebra of a Lie group by the corresponding gothic letter.) We may choose a complement $\mathfrak{q}$ of $\mathfrak{l}$ in $\mathfrak{g}$ with $\mathrm{Ad}(L)\mathfrak{q} \subset \mathfrak{q}$. $\mathfrak{q}$ is identified with the tangent space $T_p(M)$ via the mapping

$$X \rightarrow \frac{d}{dt} \exp tX \cdot p_{|t=0},$$

and the Riemannian structure induces an $\mathrm{Ad}(L)$-invariant inner product $\langle , \rangle$ on $\mathfrak{q}$.

(2.1) *Definition.* $M$ is said to be *naturally reductive* (with respect to $G$ and the decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$) if

$$\langle [X, Y]_{\mathfrak{q}}, Z \rangle = -\langle Y, [X, Z]_{\mathfrak{q}} \rangle$$

for all $X, Y, Z \in \mathfrak{q}$ where $U_{\mathfrak{l}}$ and $U_{\mathfrak{q}}$ denote the $\mathfrak{l}$ and $\mathfrak{q}$ components of $U \in \mathfrak{g}$. Equivalently for each $X \in \mathfrak{q}$, the map $Y \rightarrow [X, Y]_{\mathfrak{q}}$ is skew-symmetric on $(\mathfrak{q}, \langle , \rangle)$.

We caution that we will frequently say that a metric on a homogeneous space $G/L$ is naturally reductive even though it is not naturally reductive with respect to the particular transitive group $G$ (see for example Lemma 2.3).

(2.2) *Remark.* By a theorem of Kostant (see [**3**], p. 4), if $M$ is naturally reductive with respect to $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$, then $\bar{\mathfrak{g}} = \mathfrak{q} + [\mathfrak{q}, \mathfrak{q}]$ is a $\mathfrak{g}$-ideal, the corresponding connected subgroup $\bar{G} \subset G$ is transitive on $M$, and there exists a unique $\mathrm{Ad}(\bar{G})$-invariant symmetric non-degenerate bilinear form

$Q$ on $\mathfrak{g}$ such that

$$Q(\bar{\mathfrak{g}} \cap \mathfrak{l}, \mathfrak{q}) = 0$$

and $Q_{|\mathfrak{q}}$ is the inner product induced by the Riemannian metric. Conversely if $M = G/L$ with $G$ connected, then for an $\mathrm{Ad}(G)$-invariant, symmetric, bilinear form $Q$ on $\mathfrak{g}$, which is non-degenerate on both $\mathfrak{g}$ and $\mathfrak{l}$ and positive-definite on $\mathfrak{q} = \mathfrak{l}^{\perp}$, the metric on $M$ defined by $\langle , \rangle = Q_{|\mathfrak{q}}$ is naturally reductive. (Kostant actually stated this theorem for $M$ compact; D'Atri and Ziller pointed out that compactness is not needed.)

(2.3) *Notation*. For any connected Lie group $G$, we will denote by $G = G_1G_2$ a Levi decomposition of $G$. i.e., $G_1$ is a maximal connected semisimple subgroup of $G$, unique up to conjugacy, and $G_2$ is the solvable radical of $G$. $G_1$ can be further decomposed $G_1 = G_{\mathrm{nc}}G_{\mathrm{c}}$ where $G_{\mathrm{nc}}$ and $G_{\mathrm{c}}$, the noncompact and compact parts of $G_1$, are the products of all noncompact, respectively compact, simple connected normal subgroups of $G_1$. Thus $G_{\mathrm{nc}}$ and $G_{\mathrm{c}}$ are maximal connected semisimple subgroups of noncompact and compact type in $G$.

(2.4) *Definition*. Let $G$ be a connected transitive group of isometries of a Riemannian manifold $M$ and let $L$ be the isotropy subgroup at $p \in M$. A semisimple Levi factor $G_1$ is said to be *compatible* with $L$ if $G_1L$ is a reductive subgroup of $G$. (Recall that a Lie group $H$ is said to be reductive if the radical $H_2$ is central in $H$. Equivalently, $[\mathfrak{h}, \mathfrak{h}]$ is semisimple and $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the center of $\mathfrak{h}$. Note that the semisimple Levi factor of $H$ is unique and has Lie algebra $[\mathfrak{h}, \mathfrak{h}]$.) In particular if $G_1$ is compatible with $L$, then $G_1$, being a maximal semisimple subgroup of $G$ and hence of $G_1L$, is the unique semisimple Levi factor of $G_1L$. $\mathfrak{g}_1 + \mathfrak{l} = \mathfrak{g}_1 \oplus \mathfrak{t}$ for some abelian subalgebra commuting with $\mathfrak{g}_1$. $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}$. (See [**8**].)

(2.5) LEMMA. *Let $G$ be a transitive connected group of isometries of a Riemannian manifold $M$. Given $p \in M$, there exists a semisimple Levi factor $G_1$ of $G$ compatible with the isotropy subgroup of $G$ at $p$. Conversely, given any semisimple Levi factor $G_1$, there exists $p \in M$ such that $G_1$ is compatible with the isotropy subgroup at $p$.*

*Proof*. The first statement is proved in [**8**]. For the second, choose $q \in M$ and a Levi factor $G_1'$ compatible with the isotropy subgroup $L_q$ of $G$ at $q$. There exists $g \in G$ such that $p = g \cdot q$. $L_p = gL_qg^{-1}$, and hence the Levi factor $G_1 \equiv gG_1'g^{-1}$ is compatible with $L_p$.

(2.6) *Remark*. Suppose $G_1$ is compatible with $L$ and define $\mathfrak{t}$ as in 2.4. Let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{g}_1 + \mathfrak{l}$ relative to the Killing form $B$ of $\mathfrak{g}$. Then

$$\mathrm{nilrad}\,(\mathfrak{g}) \subset \mathfrak{s} \subset \mathfrak{g}_2.$$

Since $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}$, $B$ is negative semi-definite on $\mathfrak{t}$ and $\mathfrak{g}_2 = \mathfrak{t} + \mathfrak{s}$ with $\mathfrak{t} \cap \mathfrak{s}$ central in $\mathfrak{g}$.

(2.7) LEMMA. *Suppose $M = G/L$ is naturally reductive with respect to a transitive subgroup of $G$. Let $H$ be a subgroup of $G$ containing $L$. Then the submanifold $H/L$ of $M$ with the induced Riemannian structure is naturally reductive and totally geodesic.*

*Proof.* First assume $M$ is naturally reductive with respect to $G$ and the decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$. Let $L_0$ be the largest normal subgroup of $H$ contained in $L$. $H/L_0$ is a transitive effective group of isometries of $N \equiv H/L$ with Lie algebra $\mathfrak{h}/\mathfrak{l}_0$. Denote elements of $\mathfrak{h}/\mathfrak{l}_0$ by $\bar{X}$ with $X \in \mathfrak{h}$. Set $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{l} + \mathfrak{p}$ and $[X, Y]_{\mathfrak{q}} \in \mathfrak{p}$ whenever $X, Y \in \mathfrak{h}$. The map $X \to \bar{X}$ is injective on $\mathfrak{p}$, $\bar{\mathfrak{p}} = T_p(N)$ under identification and the Riemannian metric on $\bar{\mathfrak{p}}$ is given by $(\bar{X}, \bar{Y}) = \langle X, Y \rangle$ where $\langle , \rangle$ is the Riemannian metric on $\mathfrak{q}$. Since $[\bar{X}, \bar{Y}]_{\bar{\mathfrak{p}}} = \overline{[X, Y]_{\mathfrak{q}}}$, it follows easily that $N$ is naturally reductive with respect to $\mathfrak{h}/\mathfrak{l}_0 = \bar{\mathfrak{l}} + \bar{\mathfrak{p}}$. Viewing $\mathfrak{p}(\cong \bar{\mathfrak{p}})$ as the tangent space of $N$, the induced metric is just the restriction of $\langle , \rangle$ to $\mathfrak{p}$. Since $(\exp tX) \cdot p$, $X \in \mathfrak{p}$, is the geodesic in $M$ through $p$ with initial tangent vector $X$ (see **[12]**), $N$ is totally geodesic at $p$. By homogeneity, $N$ is totally geodesic.

For the general case, suppose $M$ is naturally reductive with respect to the transitive subgroup $G'$ of $G$. Let $L' = G' \cap L$, $H' = H \cap G'$. Then $L' \subset H'$ and $H/L = H'/L'$. Hence the first part of the proof applies.

Note that if $A$ is any subgroup of $G$ normalized by $L$, then Lemma 2.7 implies that $A/(A \cap L) = AL/L$ with the induced Riemannian metric is naturally reductive. In particular we have:

(2.8) PROPOSITION. *Suppose $M = G/L$ is naturally reductive with respect to a transitive subgroup of $G$. Choose a semisimple Levi factor $G_1$ of $G$ compatible with L and write $G_1 = G_{\mathrm{nc}}G_{\mathrm{c}}$ as in* 2.3. *Let $N =$* nilrad($G$). *Then $N \cap L = \{e\}$ and the submanifolds $G_{\mathrm{nc}}/(G_{\mathrm{nc}} \cap L)$, $G_{\mathrm{c}}/(G_{\mathrm{c}} \cap L)$ and $N$ $(= N/(N \cap L))$ with the induced Riemannian metrics are naturally reductive and totally geodesic.*

*Proof.* By 2.4, $G_{\mathrm{nc}}$ and $G_{\mathrm{c}}$ as well as $N$ are normalized by $L$. We are left only to prove $N \cap L = \{e\}$. For this we do not need natural reductivity but only the condition that $G$ act effectively on $M$. Suppose $x \in N \cap L$. Since the group exponential map of any nilpotent Lie group is surjective, $x = \exp X$ for some $X \in \mathfrak{n} = \mathrm{nilrad}(\mathfrak{g})$. $\mathrm{Ad}_G(x) = e^{\mathrm{ad}_{\mathfrak{g}} X}$ is a unipotent operator. But since $x \in L$, $\mathrm{Ad}(x)$ acts orthogonally on $(\mathfrak{q}, \langle , \rangle)$ where $\mathfrak{q}$ is an $\mathrm{Ad}(L)$-invariant subspace of $\mathfrak{g}$ and $\langle , \rangle$ is the Riemannian inner product. Hence $\mathrm{Ad}(x)_{|\mathfrak{q}}$ is the identity operator and $x$ acts as the identity on $M$. Since $G$ acts effectively, $x = e$.

**3. Structural conditions on the isometry groups.** We now obtain some necessary structural conditions on the isometry groups of a homogeneous Riemannian manifold $M$ in order that $M$ be naturally reductive.

(3.1) THEOREM. *Suppose $M = G/L$ is naturally reductive with respect to a transitive subgroup of $G$ and let $N = \text{nilrad}(G)$. Then $G_1N$ acts transitively on $M$ for every semisimple Levi factor $G_1$ of $G$.*

*Proof.* Since any two Levi factors $G_1$ and $G_1'$ are conjugate by an element of $N$, $G_1N = G_1'N$. Hence it suffices to prove transitivity of $G_1N$ when $G_1$ is chosen to be compatible with $L$ (see 2.5). Define $\mathfrak{s}$ as in Remark 2.6. Since $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{l} + \mathfrak{s}$ and $\mathfrak{s}$ is a $\mathfrak{g}$-ideal, $G = (G_1S)L$ and $G_1S$ is transitive on $M$. $N \subset S$; we prove $N = S$. Note that $\mathfrak{s} \cap \mathfrak{l} = \{0\}$ since the Killing form is negative-definite on $\mathfrak{l}$.

Let $M$ be naturally reductive with respect to the subgroup $H$ of $G$ and decomposition $\mathfrak{h} = \mathfrak{k} + \mathfrak{q}$ with $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{l}$. Then $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$ with $\mathfrak{l} \cap \mathfrak{q} = \{0\}$. Let $X_{\mathfrak{l}}$ and $X_{\mathfrak{q}}$ denote the $\mathfrak{l}$ and $\mathfrak{q}$ components of $X \in \mathfrak{g}$. For $X, Y \in \mathfrak{q}$, $[X, Y] \in \mathfrak{h}$, so $[X, Y]_{\mathfrak{q}}$ is also the $\mathfrak{q}$-component of $[X, Y]$ in $\mathfrak{h}$ and $[X, \cdot]_{\mathfrak{q}}$ is skew-symmetric on $\mathfrak{q}$ relative to the Riemannian inner product $\langle , \rangle$ on $\mathfrak{q}$. Since $\mathfrak{s} \cap \mathfrak{l} = \{0\}$, $X \rightarrow X_{\mathfrak{q}}$ is injective on $\mathfrak{s}$ and induces an inner product on $\mathfrak{s}$ given by $(X, Y) = \langle X_{\mathfrak{q}}, Y_{\mathfrak{q}} \rangle$. The operators ad $U_{|\mathfrak{s}}$, $U \in \mathfrak{l}$, are skew relative to $(,)$. For $\mathfrak{a} \subset \mathfrak{g}$, we will write

$$(\mathfrak{a})_{\mathfrak{q}} \equiv \{X_{\mathfrak{q}} : X \in \mathfrak{a}\}.$$

Since $\mathfrak{s}$ is a $\mathfrak{g}$-ideal,

(1) $\quad [X_{\mathfrak{q}}, Y_{\mathfrak{q}}]_{\mathfrak{q}} \in ([\mathfrak{s} + \mathfrak{l}, \mathfrak{s} + \mathfrak{l}])_{\mathfrak{q}} \subset ([\mathfrak{g}, \mathfrak{s}])_{\mathfrak{q}} \quad$ for $X, Y \in \mathfrak{s}$.

Let $\mathfrak{b}$ be the orthogonal complement of $[\mathfrak{g}, \mathfrak{s}]$ in $\mathfrak{s}$ relative to $(,)$. By (1) and skew-symmetry of $[X_{\mathfrak{q}}, \cdot]_{\mathfrak{q}}$,

$$([X_{\mathfrak{q}}, (\mathfrak{b})_{\mathfrak{q}}])_{\mathfrak{q}} = \{0\},$$

i.e.,

(2) $\quad [X_{\mathfrak{q}}, Y_{\mathfrak{q}}] \in \mathfrak{l}$ for $X \in \mathfrak{s}, \quad Y \in \mathfrak{b}$.

Since $[\mathfrak{l}, \mathfrak{s}] \subset [\mathfrak{g}, \mathfrak{s}]$, the skew-symmetry of ad $\mathfrak{l}_{|\mathfrak{s}}$ implies $[\mathfrak{l}, \mathfrak{b}] = \{0\}$. Hence for $X \in \mathfrak{s}$, $Y \in \mathfrak{b}$, $[X_{\mathfrak{l}}, Y] = 0$ and

(3) $\quad [X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X, Y] + [Y_{\mathfrak{l}}, X] + [X_{\mathfrak{l}}, Y_{\mathfrak{l}}]$.

By (2) and (3),

$$[X, Y] + [Y_{\mathfrak{l}}, X] \in \mathfrak{s} \cap \mathfrak{l} = \{0\}.$$

Thus

(4) $\quad \text{ad } Y_{|\mathfrak{s}} = \text{ad } Y_{\mathfrak{l}|\mathfrak{s}} \quad$ for all $Y \in \mathfrak{b}$.

But $\mathfrak{s}$ was defined to be orthogonal to $\mathfrak{l}$ relative to the Killing form of $\mathfrak{g}$.

Therefore

$$0 = \mathrm{tr}\ \mathrm{ad}\ Y_{\mathfrak{l}}\ \mathrm{ad}\ Y = \mathrm{tr}\ \mathrm{ad}\ Y_{\mathfrak{l}}\ \mathrm{ad}\ Y_{|\mathfrak{s}}$$

since $\mathfrak{s}$ is a $\mathfrak{g}$-ideal. (4) then implies

$$\mathrm{tr}(\mathrm{ad}\ Y_{\mathfrak{l}})^2{}_{|\mathfrak{s}} = 0.$$

By the skew-symmetry of ad $Y_{\mathfrak{l}}$ and another application of (4),

$$[Y_{\mathfrak{l}}, \mathfrak{s}] = 0 = [Y, \mathfrak{s}] \quad \text{for all } Y \in \mathfrak{b}.$$

Hence $\mathfrak{b}$ is central in $\mathfrak{s}$ and $\mathfrak{s} = \mathfrak{b} \oplus [\mathfrak{g}, \mathfrak{s}]$. Since $[\mathfrak{g}, \mathfrak{s}] \subset \mathfrak{n}$, it follows that $\mathfrak{s}$ is nilpotent. Hence $\mathfrak{s} = \mathfrak{n}$.

(3.2) THEOREM. *Suppose $M = G/L$ is naturally reductive with respect to a transitive subgroup of G. Let $\widetilde{G} = I_0(M)$ and let $G_1$ and $\widetilde{G}_1$ be semisimple Levi factors of G and $\widetilde{G}$ with $G_1 \subset \widetilde{G}_1$. Then the noncompact parts of $G_1$ and $\widetilde{G}_1$ coincide, i.e., $G_{\mathrm{nc}} = \widetilde{G}_{\mathrm{nc}}$.*

*Proof.* The transitive (by 3.1) group $G_1N$ has nilpotent radical and semisimple Levi factor $G_1$. We now apply Theorem 2.2 of [**6**] which asserts: given that $G$ is a transitive subgroup of the full connected isometry group $\widetilde{G}$ of a Riemannian manifold, that rad($G$) is nilpotent and that $G_1 \subset \widetilde{G}_1$ are semisimple Levi factors of $G$ and $\widetilde{G}$, then $G_{\mathrm{nc}} = \widetilde{G}_{\mathrm{nc}}$.

(3.3) THEOREM. *Suppose $M = G/L$ is naturally reductive with respect to a transitive subgroup of G and let $G_{\mathrm{nc}}$ be the noncompact part of a semisimple Levi factor of G. Then $G_{\mathrm{nc}}$ is normal in the full isometry group $I(M)$.*

(3.4) *Remark.* Theorem 3.3 asserts in particular that $G_{\mathrm{nc}}$ is normal in $G$. By the conjugacy of semisimple Levi factors, $G_{\mathrm{nc}}$ is the noncompact part of every semisimple Levi factor of $G$; i.e., $G_{\mathrm{nc}}$ is the unique maximal connected semisimple subgroup of noncompact type in $G$. By 3.2 and 3.3, $\widetilde{G}_{\mathrm{nc}} = G_{\mathrm{nc}} \lhd \widetilde{G}$ for $\widetilde{G} = I_0(M)$, so $\widetilde{G}_{\mathrm{nc}}$ satisfies an analogous uniqueness property in $\widetilde{G}$.

*Proof of Theorem* 3.3. By Remark (3.4), it suffices to prove that a fixed choice of $G_{\mathrm{nc}}$ is normal in $I(M)$. Let $\bar{G}$ be the transitive subgroup of $G$ defined by Kostant's Theorem (see 2.2). Choose Levi factors $\bar{G}_1$, $G_1$ and $\widetilde{G}_1$ of $\bar{G}$, $G$ and $\widetilde{G}$ with $\bar{G}_1 \subseteq G_1 \subseteq \widetilde{G}_1$. By Theorem 3.2, the noncompact parts satisfy $\bar{G}_{\mathrm{nc}} = G_{\mathrm{nc}} = \widetilde{G}_{\mathrm{nc}}$. Thus we may replace $G$ by $\bar{G}$, i.e., we assume the existence of a symmetric non-degenerate form $Q$ on $\mathfrak{g}$ and a $Q$-orthogonal decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}$ as in Kostant's Theorem.

Choose a Levi factor $\mathfrak{g}_1$ of $\mathfrak{g}$ compatible with $\mathfrak{l}$. Then $\mathfrak{g}_1 + \mathfrak{l} = \mathfrak{g}_1 \oplus \mathfrak{t}$ where $\mathfrak{t}$ is an abelian subalgebra of the radical of $\mathfrak{g}$ and $[\mathfrak{g}_1, \mathfrak{t}] = \{0\}$. In particular $\mathfrak{g}_{\mathrm{nc}}$ is an ideal in $\mathfrak{g}_1 + \mathfrak{l}$. Let $\mathfrak{q}' = \mathfrak{q} \cap (\mathfrak{g}_1 + \mathfrak{l})$ and let $\mathfrak{q}''$ be the $Q$-orthogonal complement of $\mathfrak{q}'$ in $\mathfrak{q}$. $\mathfrak{g}_1 + \mathfrak{l} = \mathfrak{l} + \mathfrak{q}'$, so

$$Q(\mathfrak{g}_1 + \mathfrak{l}, \mathfrak{q}'') = 0.$$

Since the operators of $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_{\mathrm{nc}}$ are skew relative to $Q$,

$$[\mathfrak{g}_{\mathrm{nc}}, \mathfrak{q}''] \subset \mathfrak{q}''.$$

Thus a representation $\rho$ of $\mathfrak{g}_{\mathrm{nc}}$ on $\mathfrak{q}''$ is defined by

$$\rho(X) = \mathrm{ad}\, X_{|\mathfrak{q}''}.$$

The operators $\rho(X)$ are all skew-symmetric relative to the positive-definite form $Q_{|\mathfrak{q}''}$. But the only representation of a semisimple Lie algebra of noncompact type by skew-symmetric operators is the trivial representation. Therefore

$$\rho \equiv 0 \quad \text{and} \quad [\mathfrak{g}_{\mathrm{nc}}, \mathfrak{q}''] = \{0\}.$$

Since $\mathfrak{g} = \mathfrak{l} + \mathfrak{q} = (\mathfrak{g}_1 + \mathfrak{l}) + \mathfrak{q}''$, it follows that $\mathfrak{g}_{\mathrm{nc}}$ is a $\mathfrak{g}$-ideal i.e., $G_{\mathrm{nc}}$ is a normal subgroup of $G$.

A theorem of **[7]** states that if $G$ is a transitive group of isometries of a Riemannian manifold $M$, then every semisimple normal subgroup of noncompact type in $G$ is also normal in $I_0(M)$. Thus $G_{\mathrm{nc}}$ is normal in $\widetilde{G} \equiv I_0(M)$. As noted in (3.4), it follows that $G_{\mathrm{nc}}$ is the unique maximal connected semisimple subgroup of noncompact type in $\widetilde{G}$; hence $G_{\mathrm{nc}}$ is invariant under every automorphism of $\widetilde{G}$. Therefore $G_{\mathrm{nc}}$ is normal in the full isometry group $I(M)$.

**4. Naturally reductive nilmanifolds.** A connected Riemannian manifold which admits a transitive nilpotent group $N$ of isometries is called a homogeneous nilmanifold. The action of $N$ is necessarily simply transitive assuming it is effective (see the proof of 2.8). Hence the manifold may be identified with the group $N$ endowed with a left-invariant metric.

(4.1) *Notation.* (i) As discussed in **[16]**, a homogeneous nilmanifold can be specified by a data triple $(\mathfrak{n}, \langle,\rangle, L)$ where $\mathfrak{n}$ is a nilpotent Lie algebra; $\langle,\rangle$ an inner product on $\mathfrak{n}$ and $L$ is a lattice (i.e., discrete vector subgroup) in the center of $\mathfrak{n}$. For $\widetilde{N}$ the simply-connected Lie group with Lie algebra $\mathfrak{n}$ and $\exp:\mathfrak{n} \to \widetilde{N}$ the group exponential, exp is a diffeomorphism, so $\exp(L)$ is a discrete central subgroup of $\widetilde{N}$, $N \equiv \widetilde{N}/\exp(L)$ is a nilpotent Lie group, and $\langle,\rangle$ defines a left-invariant metric on $N$. Two data triples $(\mathfrak{n}^{(i)}, \langle,\rangle_i, L_i)$, $i = 1, 2$ are said to be *equivalent* if there exists a Lie algebra isomorphism $\phi:\mathfrak{n}^{(1)} \to \mathfrak{n}^{(2)}$ such that $\phi(L_1) = L_2$ and

$$\langle \phi(X), \phi(Y) \rangle_2 = \langle X, Y \rangle, \quad \text{for all } X, Y \in \mathfrak{n}^{(1)}.$$

(ii) Given an inner product space $(V, \langle,\rangle)$, we denote by $so(V)$ the Lie algebra of skew-symmetric operators on $V$. For $\mathfrak{n}$ a Lie algebra, $\mathrm{Der}(\mathfrak{n})$ will denote the Lie algebra of derivations of $\mathfrak{n}$. For $L$ a lattice in $\mathfrak{n}$, $\mathrm{Ann}(L)$ denotes the annihilator of $L$ in the dual space $\mathfrak{n}^*$.

(4.2) LEMMA. (**[16]**). (i) *The full isometry group $\widetilde{G}$ of a homogeneous* *nilmanifold contains a unique simply transitive nilpotent subgroup N; N coincides with the nilradical of G.*

(ii) *There is a one-to-one correspondence between equivalence classes of data triples as in* 4.1 *and isometry classes of homogeneous nilmanifolds.*

(iii) *The full isometry algebra* $\tilde{\mathfrak{g}}$ *of the homogeneous nilmanifold with data triple* $(\mathfrak{n}, \langle,\rangle, L)$ *is the vector space direct sum* $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$ *where the isotropy algebra* $\mathfrak{l}$ *is given by*

$$\tilde{\mathfrak{l}} = so(\mathfrak{n}) \cap \mathrm{Der}(\mathfrak{n}) \cap \mathrm{Ann}(L).$$

(4.3) THEOREM. *Every naturally reductive nilmanifold is at most 2-step nilpotent.*

*Proof.* Let $(\mathfrak{n}, \langle,\rangle), L)$ be a data triple for $M$. Suppose $\mathfrak{n}$ is $n$-step nilpotent and let $C^{(k)}(\mathfrak{n})$, $k = 0, \ldots, n$, be the $k^{\text{th}}$ term in the lower central series of $\mathfrak{n}$. The full isometry algebra is given by $\tilde{\mathfrak{g}} = \tilde{\mathfrak{l}} + \mathfrak{n}$ as in 4.2 (iii).

Let $M$ be naturally reductive with respect to the subgroup $G$ of $\tilde{G}$ and the decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$ with $\mathfrak{l} = \mathfrak{g} \cap \tilde{\mathfrak{l}}$.

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{l}} + \mathfrak{q} = \tilde{\mathfrak{l}} + \mathfrak{n} \quad \text{with } \tilde{\mathfrak{l}} \cap \mathfrak{q} = \{0\} = \tilde{\mathfrak{l}} \cap \mathfrak{n}.$$

There exists a linear map $\rho:\mathfrak{n} \to \tilde{\mathfrak{l}}$ such that

$$\mathfrak{q} = \{X + \rho(X):X \in \mathfrak{n}\}.$$

Define

$$\psi(X) = X + \rho(X) \quad \text{for } X \in \mathfrak{n}.$$

Each of $\mathfrak{n}$, $\mathfrak{q}$ is naturally identified with $T_p(M)$. Relative to the induced inner products $\langle,\rangle_{\mathfrak{n}}$ and $\langle,\rangle_{\mathfrak{q}}$, $\psi$ is an isometry.

For $i = 0, \ldots, n - 1$, define

$$(1) \qquad \mathfrak{n}^{(i)} = C^{(i)}(\mathfrak{n}) \ominus C^{(i+1)}(\mathfrak{n}) \text{ and } \mathfrak{q}^{(i)} = \psi(\mathfrak{n}^{(i)})$$

where $\ominus$ denotes the orthogonal difference relative to $\langle,\rangle_{\mathfrak{n}}$. Since $\mathfrak{n}$ is a $\tilde{\mathfrak{g}}$-ideal,

$$[\tilde{\mathfrak{l}}, C^{(i)}(\mathfrak{n})] \subset C^{(i)}(\mathfrak{n}).$$

The operators ad $X_{|\mathfrak{n}}$, $X \in \tilde{\mathfrak{l}}$, are skew-symmetric relative to $\langle,\rangle_{\mathfrak{n}}$, so

$$[\tilde{\mathfrak{l}}, \mathfrak{n}^{(i)}] \subset \mathfrak{n}^{(i)}, \quad i = 0, \ldots, n - 1.$$

For $X \in \mathfrak{n}^{(i)}$, $Y \in \mathfrak{n}^{(j)}$,

$$(2) \qquad [\psi(X), \psi(Y)] = [X, Y] + [\rho(X), Y] - [\rho(Y), X] + [\rho(X), \rho(Y)].$$

The four terms on the right-hand-side of (2) lie in

$$C^{(i+j+1)}(\mathfrak{n}) = \bigoplus_{k=i+j+1}^{n-1} \mathfrak{n}^{(k)},$$

$\mathfrak{n}^{(j)}$, $\mathfrak{n}^{(i)}$ and $\tilde{\mathfrak{l}}$, respectively. Thus the $\mathfrak{q}$-component of $[\psi(X), \psi(Y)]$ in $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$ is given by

(3) $$[\psi(X), \psi(Y)]_{\mathfrak{q}} = \psi[X, Y] + \psi[\rho(X), Y] - \psi[\rho(Y), X] \in \left(\bigoplus_{k=i+j+1}^{n-1} \mathfrak{q}^{(k)}\right) \oplus \mathfrak{q}^{(j)} \oplus \mathfrak{q}^{(i)}.$$

Hence for $X \in \mathfrak{n}^{(i)}$,

$$[\psi(X), \bigoplus_{k=i}^{n-1} \mathfrak{q}^{(k)}]_{\mathfrak{q}} \subset \bigoplus_{k=i}^{n-1} \mathfrak{q}^{(k)}$$

and by skew-symmetry

(4) $$[\psi(X), \bigoplus_{k=0}^{i-1} \mathfrak{q}^{(k)}]_{\mathfrak{q}} \subset \bigoplus_{k=0}^{i-1} \mathfrak{q}^{(k)}.$$

For $Y \in \mathfrak{n}^{(j)}$, $j < i$, (3) and (4) imply $\psi[X, Y] = 0$ and therefore $[X, Y] = 0$. i.e.,

(5) $$[\mathfrak{n}^{(i)}, \mathfrak{n}^{(j)}] = \{0\} \quad \text{for } i \neq j.$$

Hence

$$C^{(1)}(\mathfrak{n}) = [\mathfrak{n}, \mathfrak{n}] = \sum_{i=0}^{n-1} [\mathfrak{n}^{(i)}, \mathfrak{n}^{(i)}]$$

and

(6) $$C^{(2)}(\mathfrak{n}) = \sum_{i=0}^{n-1} [\mathfrak{n}, [\mathfrak{n}^{(i)}, \mathfrak{n}^{(i)}]].$$

But for $X \in \mathfrak{n}$, $Y, Z \in \mathfrak{n}^{(i)}$,

(7) $$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \in [\mathfrak{n}^{(i)}, C^{(i+1)}(\mathfrak{n})].$$

Since

$$C^{(i+1)}(\mathfrak{n}) = \bigoplus_{k=i+1}^{n-1} \mathfrak{n}^{(k)},$$

(5) and (7) imply

$$[\mathfrak{n}, [\mathfrak{n}^{(i)}, \mathfrak{n}^{(i)}]] = \{0\}.$$

By (6), $C^{(2)}(\mathfrak{n}) = \{0\}$, i.e., $\mathfrak{n}$ is at most 2-step nilpotent.

Our task now is to classify the naturally reductive 2-step nilmanifolds. Kaplan **[11]** classified the naturally reductive manifolds among a certain class of 2-step nilmanifolds said to be "of type $H$". Our arguments and notation parallel those of Kaplan.

(4.4) *Notation*. (i) Let $(\mathfrak{n}, \langle,\rangle, L)$ be a data triple as in 4.1 and assume $\mathfrak{n}$ is 2-step nilpotent. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and set $\mathfrak{a} = \mathfrak{z}^{\perp}$. Note that $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}$. Denote by $\langle,\rangle_{\mathfrak{a}}$ and $\langle,\rangle_{\mathfrak{z}}$ the restrictions of $\langle,\rangle$ to $\mathfrak{a}$ and $\mathfrak{z}$, respectively. Define $j:\mathfrak{z} \to so(\mathfrak{a})$ by

(8) $\quad j(Z)X = (\mathrm{ad}\ X)^*Z.$

(ii) We will say $(\mathfrak{n}, \langle,\rangle, L)$ is a *naturally reductive data triple* if the associated nilmanifold is naturally reductive.

(4.5) *Remark*. All two-step homogeneous nilmanifolds can be constructed as follows: Let $(\mathfrak{a}, \langle,\rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle,\rangle_{\mathfrak{z}})$ be inner product spaces, $j:\mathfrak{z} \to so(\mathfrak{a})$ a linear map and $L$ a lattice in $\mathfrak{z}$. Let $(\mathfrak{n}, \langle,\rangle)$ be the direct sum of $(\mathfrak{a}, \langle,\rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle,\rangle_{\mathfrak{z}})$. The skew-symmetric bilinear map $[,]:\mathfrak{n} \times \mathfrak{n} \to \mathfrak{z}$ defined by

$$[X, Z] = 0 \quad \text{for } X \in \mathfrak{n}, Z \in \mathfrak{z} \quad \text{and}$$

$$\langle [X, Y], Z\rangle = \langle j(Z)X, Y\rangle \quad \text{for } X, Y \in \mathfrak{a}, Z \in \mathfrak{z}$$

defines a Lie algebra structure on $\mathfrak{n}$, so $(\mathfrak{n}, \langle,\rangle, L)$ is a data triple for a 2-step nilmanifold. By 4.1, two such triples $(\mathfrak{n}^{(i)}, \langle,\rangle_i, L_i)$, $i = 1, 2$ are equivalent if and only if there exist inner product space isomorphisms $\phi:\mathfrak{a}^{(1)} \to \mathfrak{a}^{(2)}$ and $\psi:\mathfrak{z}^{(1)} \to \mathfrak{z}^{(2)}$ such that

$$\psi(L_1) = L_2 \quad \text{and} \quad j_2(\psi(Z)) = \phi \circ j_1(Z) \circ \phi^{-1} \quad \text{for all } Z \in \mathfrak{z}.$$

The nilmanifolds of type $H$ studied by Kaplan are those for which

$$|j(Z)X| = |Z|\,|X| \quad \text{for all } Z \in \mathfrak{z}, X \in \mathfrak{a}.$$

(4.6) PROPOSITION. *In the notation of* 4.4, *let* $\mathfrak{n}' = \mathfrak{a} + [\mathfrak{a}, \mathfrak{a}]$, *let* $\langle,\rangle'$ *be the restriction of* $\langle,\rangle$ *to* $\mathfrak{n}$, *and let* $L'$ *denote the orthogonal projection of* $L$ *in* $\mathfrak{n}'$. *Then*

(i) $\mathfrak{n} = \mathfrak{n}' \oplus \ker(j)$, *orthogonal direct sum of ideals.* $\mathrm{Ker}(j) = \{0\}$ *if and only if there is no Euclidean factor in the De Rham decomposition of the simply-connected covering of the nilmanifold associated with* $(\mathfrak{n}, \langle,\rangle, L)$.

(ii) $(\mathfrak{n}, \langle,\rangle, L)$ *is naturally reductive if and only if* $(\mathfrak{n}', \langle,\rangle', L')$ *is naturally reductive.*

*Proof*. (i) is easily verified since the Riemannian metric on $N$ is left-invariant. Using either (i) or 4.2 (iii), one checks that the full isometry algebra $\tilde{\mathfrak{g}}$ of $(\mathfrak{n}, \langle,\rangle, L)$ is a direct sum of ideals $\tilde{\mathfrak{g}} = \mathfrak{g}' \oplus \mathfrak{g}''$ with $\mathfrak{n}' \subset \mathfrak{g}'$ and $\ker(j) \subset \mathfrak{g}''$, and (ii) follows easily.

(4.7) LEMMA. *Let* $(\mathfrak{n}, \langle,\rangle, L)$ *be a data triple for a* 2-*step homogeneous nilmanifold and let* $\tilde{\mathfrak{l}}$ *be the isotropy algebra given by* 4.2 (iii). *We use the* *notation* 4.4 *and assume* $j$ *is injective. Then*

(i) $\tilde{\mathfrak{l}}$ *leaves each of* $\mathfrak{a}$ *and* $\mathfrak{z}$ *invariant.*

(ii) *For* $\phi \in \tilde{\mathfrak{l}}$,

$$\phi_{|\mathfrak{z}} = j^{-1} \circ \mathrm{ad}_{so(\mathfrak{a})} \phi_{|\mathfrak{a}} \circ j.$$

*In particular the map* $\phi \to \phi_{|a}$ *is an isomorphism of* $\tilde{\mathfrak{l}}$ *onto a subalgebra of* $so(\mathfrak{a})$.

(iii) *Let* $\phi \in so(\mathfrak{a})$. *Then* $\phi$ *extends to an element of* $\tilde{\mathfrak{l}}$ *if and only if* $[\phi, j(\mathfrak{z})] \subset j(\mathfrak{z})$ *and*

$$j^{-1} \circ \mathrm{ad}_{so(\mathfrak{a})} \phi \circ j \in so(\mathfrak{z}) \cap \mathrm{Ann}(L).$$

*Proof.* (i) is easily checked. We prove (ii) and (iii) simultaneously. Let $\alpha \in so(\mathfrak{a})$ and $\beta \in so(\mathfrak{z})$. By 4.2, the linear map $\phi$ which agrees with $\alpha$ and $\beta$ on $\mathfrak{a}$ and $\mathfrak{z}$, respectively, lies in $\tilde{\mathfrak{l}}$ if and only if $\beta(L) = 0$ and for all $X, Y \in \mathfrak{a}$, $Z \in \mathfrak{z}$,

$$\begin{aligned}(9) \qquad 0 &= \langle Z, [\alpha(X), Y] + [X, \alpha(Y)] - \beta[X, Y] \rangle \\ &= \langle Y, j(Z)\alpha(X) - \alpha(j(Z)X) + j(\beta(Z))X\rangle.\end{aligned}$$

(The second equality uses the skew-symmetry of $\alpha$ and $\beta$.) (9) is equivalent to

$$[\alpha, j(Z)] = j(\beta(Z)) \quad \text{or} \quad \beta = j^{-1} \circ \mathrm{ad}_{so(\mathfrak{a})}\alpha \circ j.$$

(4.8) THEOREM. *Let* $M$ *be a two-step homogeneous nilmanifold and* $(\mathfrak{n}, \langle,\rangle, L)$ *an associated data triple. We use notation* 4.4 *and set* $\mathfrak{z}' = \mathfrak{z} \ominus \ker \mathfrak{z}$, *where* $\ominus$ *denotes orthogonal difference. Then M is naturally reductive if and only if both of the following hold*:

(i) $j(\mathfrak{z})$ *is a subalgebra of* $so(\mathfrak{a})$ *and*

(ii) $(j_{|\mathfrak{z}'})^{-1} \circ \mathrm{ad}_{so(\mathfrak{a})} j(Z) \circ j \in so(\mathfrak{z}) \cap \mathrm{Ann}(L)$

*for all* $Z \in \mathfrak{z}$.

*Proof.* (i) and (ii) hold if the analogous conditions hold on the triple $(\mathfrak{n}', \langle,\rangle', L')$ defined in 4.6. Hence by 4.6, we may assume that $\mathfrak{n} = \mathfrak{n}'$, i.e., that $j$ is injective and $\mathfrak{z} = \mathfrak{z}'$.

Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{l}} + \mathfrak{n}$ be the full isometry algebra of $M$. First assume $M$ is naturally reductive with respect to a transitive subalgebra $\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ and decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$ with $\mathfrak{l} = \mathfrak{g} \cap \tilde{\mathfrak{l}}$. Define $\rho:\mathfrak{n} \to \tilde{\mathfrak{l}}$ so that

$$\mathfrak{q} = \{X + \rho(X):X \in \mathfrak{n}\}.$$

Viewing $\rho(X)$ as a linear operator on $\mathfrak{n}$ as in 4.2 (iii), we write $\rho(X)Y$ for $[\rho(X), Y]$ when $X, Y \in \mathfrak{n}$. The condition for natural reductivity

$$\begin{aligned}&\langle [X + \rho(X), Y + \rho(Y)], U + \rho(U) \rangle_{\mathfrak{q}} \\ &= -\langle Y + \rho(Y), [X + \rho(X), U + \rho(U)] \rangle_{\mathfrak{q}}\end{aligned}$$

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(where $\langle , \rangle_{\mathfrak{q}}$ is the Riemannian inner product on $\mathfrak{q}$) can be interpreted on $\mathfrak{n}$ as

$$(10)\quad \langle [X, Y] + \rho(X)Y - \rho(Y)X, U\rangle = -\langle Y, [X, U] + \rho(X)U - \rho(U)X\rangle.$$

Since $\rho(X) \in so(\mathfrak{n})$, the terms involving $\rho(X)$ cancel and (10) yields

$$(11)\quad (\text{ad } Y)^*U + (\text{ad } U)^*Y = \rho(Y)U + \rho(U)Y \quad \text{for all } Y, U \in \mathfrak{n}.$$

((11) was obtained in [**11**].) Moreover since both $\mathfrak{q}$ and $\mathfrak{n}$ are normalized by $\rho(\mathfrak{n})$, we have

$$[\rho(X), Y + \rho(Y)] = \rho(X)Y + [\rho(X), \rho(Y)] \in \mathfrak{q}$$

and therefore

$$(12)\quad \rho(\rho(X)Y) = [\rho(X), \rho(Y)] \quad \text{for all } X, Y \in \mathfrak{n}.$$

When $U \in \mathfrak{z}$ and $Y \in \mathfrak{a}$, $(\text{ad } U)^*Y = 0$ and (11) says

$$(13)\quad j(U)Y = \rho(Y)U + \rho(U)Y.$$

But $\rho(Y)U \in \mathfrak{z}$ and $\rho(U)Y \in \mathfrak{a}$ by 4.7 (i) so (13) implies

$$\rho(U)_{|\mathfrak{a}} = j(U).$$

By 4.7 (iii), it then follows that

$$[j(U), j(\mathfrak{z})] \subset j(\mathfrak{z}) \quad \text{and}$$

$$j^{-1} \circ \text{ad}_{so(\mathfrak{a})} j(U) \circ j \in so(\mathfrak{z}) \cap \text{Ann}(L) \quad \text{for all } U \in \mathfrak{z}.$$

This proves the necessity of (i) and (ii).

Conversely if (i) and (ii) hold, then by 4.7, $j(Z)$ extends to an element $\rho(Z)$ of $\tilde{\mathfrak{l}}$ with $\rho(Z)_{|\mathfrak{z}}$ given by the left-hand side of (ii). Extend $\rho$ to a linear map $\rho:\mathfrak{n} \to \tilde{\mathfrak{l}}$ by setting $\rho_{|\mathfrak{a}} = 0$. We claim

$$(14)\quad \rho(\rho(X)Y) = [\rho(X), \rho(Y)] \quad \text{for all } X, Y \in \mathfrak{n}.$$

(14) holds trivially if at least one of $X, Y \in \mathfrak{a}$. If $X, Y \in \mathfrak{z}$, then

$$\rho(\rho(X)Y)_{|\mathfrak{a}} = j(j^{-1}[j(X), j(Y)]) = [j(X), j(Y)]$$

and therefore (14) follows from 4.7 (ii). Define

$$(15)\quad \mathfrak{l} = \rho(\mathfrak{n}), \mathfrak{q} = \{X + \rho(X):X \in \mathfrak{n}\} \quad \text{and} \quad \mathfrak{g} = \mathfrak{l} + \mathfrak{q}.$$

By (14), $\mathfrak{l}$ is a subalgebra of $\tilde{\mathfrak{l}}$ and $[\mathfrak{l}, \mathfrak{q}] \subset \mathfrak{q}$. Moreover since $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$ and $\mathfrak{n}$ is a $\tilde{\mathfrak{g}}$-ideal, $\mathfrak{g}$ is a subalgebra of $\tilde{\mathfrak{g}}$.

We next claim that (11) is valid. (11) is easily verified whenever at least one of $Y, U \in \mathfrak{a}$. If both $Y, U \in \mathfrak{z}$, the left-hand size of (11) is zero. The right-hand side lies in $\mathfrak{z} \cap \ker(\rho)$ by (14). But

$$\mathfrak{z} \cap \ker(\rho) = \mathfrak{z} \cap \ker(j) = \{0\}$$

by our assumption that $j$ is injective. This proves (11). Tracing the argument preceding (11) backwards, we see that $M$ is naturally reductive with respect to $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$.

(4.9) *Examples.* If $M$ is simply-connected, then $L = \{0\}$ and the right-hand side of (ii) in Theorem 4.8 is just $so(\mathfrak{z})$.

Recall that the Heisenberg groups are characterized as the two-step nilpotent groups with one-dimensional centers. They are odd-dimensional and, up to isomorphism, there is a unique simply-connected Heisenberg group of dimension $2n + 1$ for each $n \geqq 1$. Theorem 4.8 trivially implies that every left-invariant metric on a simply-connected Heisenberg group is naturally reductive. (J. D'Atri in unpublished work independently proved this result for the three-dimensional Heisenberg group.) If $\mathfrak{n}$ is the Lie algebra of a simply-connected Heisenberg group $N$, two data triples $(\mathfrak{n}, \langle,\rangle_i, \{0\})$, $i = 1, 2$ associated with left-invariant metrics on $N$ are equivalent if and only if $j_1(Z_1)$ and $j_2(Z_2)$ have the same eigenvalues counted with multiplicities, where $Z_i$ is an element of $\mathfrak{z}$ of norm one relative to $\langle,\rangle_i$, $i = 1, 2$. The Heisenberg manifold associated with $(\mathfrak{n}, \langle,\rangle, \{0\})$ is of type $H$ as defined by Kaplan if and only if the only eigenvalues of $j(Z)$ for $|Z| = 1$ are $\pm\sqrt{-1}$.

Using Theorem 4.8, one can show that the only simply-connected homogeneous nilmanifolds of dimension $\leqq 5$ without Euclidean factors are the Heisenberg manifolds of dimension 3 and 5.

**5. Naturally reductive homogeneous spaces of noncompact semisimple Lie groups.** Let $G$ be a connected semisimple Lie group of noncompact type. We will use the results of Section 3 to classify all naturally reductive Riemannian metrics on homogeneous spaces $G/L$ of $G$. Throughout this section we drop the assumption that $G$ act effectively on $G/L$ and require only that $G$ act almost effectively.

(5.1) We recall some general properties of connected semisimple Lie groups of noncompact type and of their Riemannian homogeneous spaces.

(i) Let $\mathfrak{k}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$ (unique up to conjugacy). There exists a unique, necessarily connected, subgroup $K$ of $G$ with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \ldots \oplus \mathfrak{g}_{(n)}$ be the decomposition of $\mathfrak{g}$ into simple ideals.

$$\mathfrak{k} = \bigoplus_{i=1}^{n} \mathfrak{k} \cap \mathfrak{g}_{(i)}$$

and the center of $\mathfrak{k} \cap \mathfrak{g}_{(i)}$ is at most one-dimensional for each $i$.

Let $\mathfrak{p} = \mathfrak{k}^{\perp}$ relative to the Killing form $B$ of $\mathfrak{g}$. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. $B$ is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$. If $G$ is simple, $\mathrm{Ad}(K)$ acts irreducibly on $\mathfrak{p}$. See **[10]** for further details.

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(ii) (See [5].) Suppose $M = G/L$ is a Riemannian homogeneous space. Choose a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing $\mathfrak{l}$ and let $K$ be the corresponding subgroup of $G$. Denote by $N_K(L)$ the normalizer of $L$ in $K$. For $g \in G$, $h \in N_K(L)$ let $L_g$ and $R_h$ denote left translation by $g$ and right translation by $h^{-1}$ on $G/L$, and let $W$ be the identity component in $\{u \in N_K(L):R_u$ is an isometry$\}$. Then

$$\tilde{G} \equiv I_0(M) = \{L_g \circ R_u : g \in G, u \in W\}.$$

$\mathfrak{w}$ may be decomposed into a direct sum of ideals $\mathfrak{w} = \mathfrak{l} \oplus \mathfrak{u}$. Let $U$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{u}$. Then $\tilde{G} \cong G \times U/D$ where $D$ is the discrete effective kernel of the action $(g, u) \to L_g \circ R_u$. $\tilde{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathfrak{u}$ with isotropy algebra $(\mathfrak{l}, 0) + \Delta(\mathfrak{u})$ where $\Delta(\mathfrak{u})$ is the diagonal subalgebra of $\mathfrak{u} \oplus \mathfrak{u}$.

(5.2) THEOREM. *A left-invariant Riemannian metric on a homogeneous space $G/L$ of a connected semisimple Lie group $G$ of noncompact type is naturally reductive if and only if the following three conditions are satisfied*:

(i) *Let $\mathfrak{k}$ be any maximal compactly embedded subalgebra of $\mathfrak{g}$ containing $\mathfrak{l}$. Then $L$ is a normal subgroup of the corresponding subgroup $K$ of $G$.*

(ii) *The Riemannian metric is* $\mathrm{Ad}(K)$*-invariant for $K$ as in* (i).

(iii) *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition and let $\mathfrak{f}$ be a $\mathfrak{k}$-ideal complementary to $\mathfrak{l}$. (Note that $\mathfrak{f} + \mathfrak{p}$ is naturally identified with the tangent space of $G/L$ at the base point.) Relative to the inner product $\langle , \rangle$ on $\mathfrak{f} + \mathfrak{p}$ induced by the Riemannian metric, $\mathfrak{f} \perp \mathfrak{p}$.*

*When these conditions hold, the metric is naturally reductive with respect to $I_0(M) \cong G \times F/D$, where $F$ is the connected subgroup of $G$ with Lie algebra $\mathfrak{f}$ and $D$ is discrete. The metric is not naturally reductive with respect to any proper subgroup of $I_0(M)$.*

Before proving Theorem 5.2, we reformulate conditions (ii) and (iii).

(5.3) THEOREM. *We use the notation of Theorem 5.2 and assume that $L$ satisfies condition* (i). *Let $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \ldots \oplus \mathfrak{g}_{(n)}$ be the decomposition of $\mathfrak{g}$ into simple ideals, let*

$$\mathfrak{p}_{(j)} = \mathfrak{p} \cap \mathfrak{g}_{(j)}, \quad j = 1, \ldots, n$$

*and let*

$$\mathfrak{f} = \mathfrak{f}_{(1)} \oplus \ldots \oplus \mathfrak{f}_{(r)} \oplus z(\mathfrak{f})$$

*be the decomposition of $\mathfrak{f}$ into simple ideals and center $z(\mathfrak{f})$. A left-invariant Riemannian metric on $G/L$ satisfies conditions* (ii) *and* (iii) *if and only if $\mathfrak{f}_{(i)}$, $(1 \leqq i \leqq r)$, $\mathfrak{p}_{(j)}$, $(1 \leqq j \leqq n)$ and $z(\mathfrak{f})$ are pairwise orthogonal and*

$$\langle , \rangle = -\alpha_1 B_{|\mathfrak{f}_{(1)}} - \ldots - \alpha_r B_{|\mathfrak{f}_{(r)}} + \beta_1 B_{\mathfrak{p}_{(1)}} + \ldots + \beta_n B_{|\mathfrak{p}_{(n)}} + A_{|z(\mathfrak{f})}$$

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*where B is the Killing form of* $\mathfrak{g}$, $\alpha_i$ *and* $\beta_j$ *are any positive constants and A is any inner product on* $z(\mathfrak{f})$.

*Proof.* $\mathrm{Ad}(K)$-invariance of $\langle , \rangle$ together with condition (iii) guarantee that the $\mathfrak{f}_{(i)}$, $\mathfrak{p}_{(j)}$ and $z(\mathfrak{f})$ are orthogonal. Since $\mathrm{Ad}(K)$ acts trivially on $z(\mathfrak{f})$, every inner product on $z(\mathfrak{f})$ is $\mathrm{Ad}(K)$-invariant. For each $i$ and $j$, $\mathrm{Ad}(K)$ acts irreducibly on $\mathfrak{f}_{(i)}$ and $\mathfrak{p}_{(j)}$ and the only $\mathrm{Ad}(K)$-invariant metrics on $\mathfrak{f}_{(i)}$ and $\mathfrak{p}_{(j)}$ are multiples of $B$. Since $B$ is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$, the theorem follows.

*Proof of Theorem* 5.2. Suppose the left-invariant metric $\langle , \rangle$ on $G/L$ is naturally reductive with respect to a subgroup $A$ of $\widetilde{G} = I_0(M)$. Choose a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing $\mathfrak{l}$. As in 5.1 (ii), the full isometry algebra is given by $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is a subalgebra of $\mathfrak{k}$ commuting with $\mathfrak{l}$, and the isotropy subalgebra of $\widetilde{\mathfrak{g}}$ is $(\mathfrak{l}, 0) + \Delta(\mathfrak{u})$.

Choose a transitive subgroup $H \subset A$ and a bilinear form $Q$ on $\mathfrak{h}$ satisfying the conclusions of Kostant's Theorem (see 2.2). By Theorem 3.2, $\mathfrak{h}$ contains the maximal semisimple subgroup of noncompact type in $\widetilde{\mathfrak{g}}$, i.e., $\mathfrak{g} \subset \mathfrak{h}$. Hence

$$\mathfrak{h} = (\mathfrak{g}, 0) \oplus (0, \mathfrak{f}) \quad \text{with } \mathfrak{f} \subset \mathfrak{u}.$$

$\mathfrak{h} \cap \widetilde{\mathfrak{l}} = (\mathfrak{l}, 0) + \Delta(\mathfrak{f})$. The $\mathrm{Ad}(H)$-invariance of $Q$ implies that

$$Q( (\mathfrak{g}, 0), (0, \mathfrak{f}) ) = 0.$$

If $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \ldots \oplus \mathfrak{g}_{(n)}$ is the decomposition of $\mathfrak{g}$ into simple ideals,

$$Q( (\mathfrak{g}_{(i)}, 0), (\mathfrak{g}_{(j)}, 0) ) = 0 \quad \text{when } i \neq j$$

and $Q_{|(\mathfrak{g}_{(i)}, 0)}$ is a multiple $\beta_i B$ of the Killing form $B$ of $\mathfrak{g}$. (Here $\mathfrak{g}$ is identified with $(\mathfrak{g}, 0)$.) Hence if $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition.

$$Q( (\mathfrak{k}, 0), (\mathfrak{p}, 0) ) = 0.$$

Thus

$$Q(\mathfrak{h} \cap \widetilde{\mathfrak{l}}, (\mathfrak{p}, 0) ) = 0,$$

i.e., $(\mathfrak{p}, 0)$ lies in the $Q$-orthogonal complement $\mathfrak{q}$ of $\mathfrak{h} \cap \widetilde{\mathfrak{l}}$. Since $Q$ and $B$ are positive-definite on $\mathfrak{q}$ and $\mathfrak{p}$, respectively, $\beta_i > 0$ for $i = 1, \ldots, n$. Now let $(\mathfrak{b}, 0)$ be the $Q$-orthogonal complement of $(\mathfrak{l} + \mathfrak{f}, 0)$ in $(\mathfrak{k}, 0)$.

$$Q( (\mathfrak{b}, 0), \widetilde{\mathfrak{l}} \cap \mathfrak{h}) = 0,$$

so $(\mathfrak{b}, 0) \subset \mathfrak{q} \cap (\mathfrak{k}, 0)$. But $Q_{|(\mathfrak{k},0)}$ is negative-definite since $\beta_i B$ is negative definite on $\mathfrak{k} \cap \mathfrak{g}_{(i)}$. Hence $\mathfrak{q} \cap (\mathfrak{k}, 0) = \{0\}$, $\mathfrak{b} = \{0\}$ and $\mathfrak{k} = \mathfrak{l} + \mathfrak{f}$, proving (i). Moreover it follows that $\mathfrak{f} = \mathfrak{u}$, $\widetilde{\mathfrak{g}} = (\mathfrak{g}, 0) \oplus (0, \mathfrak{f})$ and $K = LF$ where $F$ is the connected subgroup corresponding to $\mathfrak{f}$. Hence $\mathfrak{h} = \widetilde{\mathfrak{g}}$ and $H = A = I_0(M)$.

By the definition of $\mathfrak{u}$ in 5.1 (ii), $R_h$ is an isometry of $G/L$ for all $h \in F$

and trivially for all $h \in L$. Thus the metric is Ad($K$)-invariant. For (iii), define $\tau:\mathfrak{k} + \mathfrak{p} \to \mathfrak{q}$ by $\tau(X) = \pi(X, 0)$ where $\pi$ is the $Q$-orthogonal projection of $(\mathfrak{k} + \mathfrak{p}, 0)$ onto $\mathfrak{q}$. $\tau$ is an isometry relative to the inner products $\langle , \rangle$ on $\mathfrak{k} + \mathfrak{p}$ and $Q$ on $\mathfrak{q}$. $\tau(\mathfrak{p}) = (\mathfrak{p}, 0)$ and

$$\tau(\mathfrak{k}) \subset (\mathfrak{k}, 0) + (\tilde{\mathfrak{l}} \cap \mathfrak{h}) \subset (\mathfrak{k}, 0) + (\tilde{\mathfrak{l}} \cap \mathfrak{h}).$$

Hence

$$Q(\tau(\mathfrak{k}), \tau(\mathfrak{p})) = 0$$

and $\langle \mathfrak{k}, \mathfrak{p} \rangle = 0$.

For the converse, if $L$ satisfies (i) and the metric satisfies (ii) and (iii), then the metric is of the form described in Theorem 5.3. That every such metric is naturally reductive is proved by an argument analogous to that given by D'Atri-Ziller [3], p. 9-11 for compact groups. We sketch the argument here. By 5.1 (ii), the full isometry algebra is

$$\tilde{\mathfrak{g}} = (\mathfrak{g}, 0) \oplus (0, \mathfrak{k}).$$

Since $\mathfrak{k} \subset \mathfrak{g}$, we may extend the Killing form $B$ of $\mathfrak{g}$ to an invariant form $B'$ on $\tilde{\mathfrak{g}}$ by defining

$$B'((X_1, Y_1), (X_2, Y_2)) = B(X_1, X_2) + B(Y_1, Y_2).$$

We need to construct a bilinear form $Q$ on $\mathfrak{g} \oplus \mathfrak{k}$ of the form

$$\begin{aligned} Q = {} & \beta_1 B'_{|(\mathfrak{g}_{(1)},0)} + \dots + \beta_n B'_{|(\mathfrak{g}_{(n)},0)} \\ & - \gamma_1 B'_{|(0,\mathfrak{k}_{(1)})} - \dots - \gamma_r B'_{|(0,\mathfrak{k}_{(r)})} \\ & + C_{|(0,z(\mathfrak{k}))} \end{aligned} \tag{1}$$

(with $\beta_i$ given by 5.3) such that both $Q$ and $Q_{|\tilde{\mathfrak{l}}}$ are non-degenerate and such that for

$$\tau:(\mathfrak{k} + \mathfrak{p}, 0) \to \mathfrak{q} = \tilde{\mathfrak{l}}^{\perp}$$

the $Q$ orthogonal projection,

$$Q(\tau(X, 0), \tau(Y, 0)) = \langle X, Y \rangle$$

where $\langle , \rangle$ is given by 5.3. For each $i = 1, \dots, r$, $\mathfrak{k}_{(i)} \subset \mathfrak{g}_{(j)}$ for a unique $j$, and we set

$$\gamma_i = \frac{\beta_j \alpha_i}{\beta_j + \alpha_i}. \tag{2}$$

Let $\bar{A}$ and $\bar{D}$ be the symmetric endomorphisms of $z(\mathfrak{k})$ given by

$$A(X, Y) = B(\bar{A}X, Y) \quad \text{and} \quad Q((X, 0), (Y, 0)) = B(\bar{D}X, Y).$$

(Note that $\bar{D}$ is diagonal with eigenvalues lying in $\{\beta_1, \dots, \beta_n\}$.)

Since $B$ is negative-definite on $z(\mathfrak{k})$, $A$ is negative-definite whereas $\bar{D}$ is positive-definite. Define

$$\bar{C} = (\bar{D}^{-1} - \bar{A}^{-1})^{-1}$$

and let

$$C((0, X), (0, Y)) = -B(\bar{C}X, Y)$$

in (1). Then $\bar{C}$ is positive-definite and $Q$ is non-degenerate. $Q$ is also non-degenerate on $\tilde{\mathfrak{l}}$ since $\beta_j - \gamma_i \neq 0$ when $\mathfrak{k}_{(i)} \subset \mathfrak{g}_{(j)}$ and $\bar{D} - \bar{C}$ $(= -\bar{D}^{-1}\bar{A}^{-1}\bar{C})$ is nonsingular.

The isomorphism $\tau:(\mathfrak{k} + \mathfrak{p}, 0) \rightarrow \mathfrak{q}$ is given by

$$\tau(X, 0) = \begin{cases} (X, 0) & x \in \mathfrak{p} \\ \left(\dfrac{\gamma_i}{\gamma_i - \beta_j} X, \dfrac{\beta_j}{\gamma_i - \beta_j} X\right) & x \in \mathfrak{k}_{(i)} \subset \mathfrak{g}_{(j)} \\ (E(X), \bar{C}^{-1}\bar{D}E(X)) & x \in z(\mathfrak{k}) \end{cases}$$

where $E = (I - \bar{C}^{-1}\bar{D})^{-1}$. (The non-degeneracy of $I - \bar{C}^{-1}\bar{D}$ follows from that of $\bar{D} - \bar{C}$.) Now

$$Q(\tau(X), \tau(Y)) = \beta_j B(X, Y) \quad \text{for } X, Y \in \mathfrak{p}_{(j)}.$$

For $X, Y \in \mathfrak{k}_{(i)} \subset \mathfrak{g}_{(j)}$,

$$Q(\tau(X), \tau(Y)) = \frac{\gamma_i\beta_j}{\gamma_i - \beta_j} B(X, Y) = -\alpha_i B(X, Y)$$

by (2). For $X, Y \in Z(\mathfrak{k})$,

$$\begin{aligned} Q(\tau(X), \tau(Y)) &= B(\bar{D}E(X), E(Y)) - B(\bar{D}E(X), \bar{C}^{-1}\bar{D}E(Y)) \\ &= B(\bar{D}(I - \bar{C}^{-1}\bar{D})E(X), E(Y)) \\ &= B(\bar{D}(X), E(Y)) \end{aligned}$$

by definition of $E$

$$\begin{aligned} &= B(X, \bar{D}E(Y)) \\ &= B(X, (\bar{D}^{-1} - \bar{C}^{-1})^{-1}Y) \\ &= B(X, \bar{A}Y). \end{aligned}$$

Thus

$$Q(\tau(X, 0), \tau(Y, 0)) = \langle X, Y \rangle \quad \text{for all } X, Y \in \mathfrak{k} + \mathfrak{p}.$$

(5.4) *Remark*. In [**3**] p. 64, D'Atri and Ziller incorrectly assert that a class of left-invariant metrics on $G$, which properly contains those of Theorem 5.3 with $L$ trivial, are naturally reductive. Setting $\mathfrak{b} = \{0\}$ in their notation, we obtain the metrics of 5.3.

We conclude this section with a study of the geometrical properties of naturally reductive metrics on $G/L$.

(5.5) THEOREM. *Let $M = G/L$ be a naturally reductive homogeneous space of a semisimple Lie group $G$ of noncompact type. Choose a subgroup $K$ of $G$ containing $L$ such that $\mathfrak{k}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$. Under the obvious choice of left-invariant metric on $G/K$, the projection*

$$\pi : M \to G/K$$

*is a Riemannian submersion of $M$ onto the symmetric space $G/K$ of non-positive curvature. The fibres are Lie groups with bi-invariant Riemannian metrics forming totally geodesic symmetric spaces. Every two-plane which intersects the vertical space non-trivially has non-negative sectional curvature, while every horizontal two-plane has non-positive sectional curvature. In the notation of Theorem* 5.3, *the sectional curvature $K$ of $\langle , \rangle$ for orthonormal vectors $X$ and $Y$ is given by*:

$$K(X, Y) = -\sum_{i=1}^{r}\sum_{m=1}^{n}\left(\frac{3}{4} + \frac{\beta_m}{\alpha_i}\right)\| [X, Y]_{(i)}\|^2 - \sum_{m=1}^{n} \beta_m B([X_m, Y_m]_{\mathfrak{l}}, [X_m, Y_m]_{\mathfrak{l}})$$

*for $X, Y \in \mathfrak{p}$ and*

$$K(X, Y) = \frac{1}{4}\sum_{i=1}^{r} \| [X_{(i)}, Y_{(i)}] \|^2 + \sum_{i,j=1}^{r}\sum_{m=1}^{n} \frac{\alpha_i\alpha_j}{4\beta_m^2} \langle [X_{(i)}, Y_m], [X_{(j)}, Y_m] \rangle + \sum_{m=1}^{n} \frac{1}{2\beta_m} \left\langle [\bar{A} X_z, Y_m], \left[ -\left(1 + \frac{1}{2\beta_m}\bar{A}\right)X_z + \sum_{i=1}^{r} \frac{\alpha_i}{\beta_m} X_{(i)}, Y_m \right] \right\rangle .$$

*for $X \in \mathfrak{k}$, $Y \in \mathfrak{k} + \mathfrak{p}$ where $U_{\mathfrak{l}}$, $U_{(i)}$, $U_z$ and $U_m$ denote the $\mathfrak{l}$, $\mathfrak{k}_{(i)}$, $z(\mathfrak{k})$ and $\mathfrak{p}_{(m)}$ components of $U \in \mathfrak{g}$.*

*Proof.* It is immediate from Theorems 5.2 and 5.3 that $\pi$ is a submersion. By Cartan's theory, $G/K$ is a symmetric space of non-positive curvature, and by O'Neill's curvature formulas for submersions ( [**14**], Corollary 1), every horizontal two-plane in $M$ has non-positive curvature. The fibre over $eK$ is $K/L$, a Lie group with Lie algebra $\mathfrak{k}$. The metric on $K/L$ is bi-invariant so $K/L$ is a symmetric space of non-negative curvature and by Lemma 2.7, $K/L$ is totally geodesic. Since each $X \in \mathfrak{k}$ is a Killing vector field, $K(X, \cdot) \geqq 0$ ( [**13**], p. 296). The curvatures can be computed as follows: for $X \in \mathfrak{k} + \mathfrak{p}$, let $X^*$ be the fundamental vector field given by

$$X^*_{(q)} = \frac{d}{dt} \exp tX \cdot q_{|t=0}.$$

For $X, Y \in \mathfrak{k} + \mathfrak{p}$, define

$$A_x Y = (\nabla_{X^*} Y^* - [X^*, Y^*])(\mathfrak{p}).$$

$A_x$ is a skew-symmetric operator and may be computed by the formula ( [**12**], pp. 188-204)

$$(3) \qquad 2\langle A_x Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle.$$

For $X_{\mathfrak{l}}$ and $X_{\mathfrak{k}+\mathfrak{p}}$ the $\mathfrak{l}$ and $\mathfrak{k} + \mathfrak{p}$ components of $X \in \mathfrak{g}$, the curvature tensor at $p$ is given by ( [**12**], p. 192):

$$(4) \qquad R_p(X, Y) = [A_X, A_Y] - A_{[X,Y]_{\mathfrak{k}+\mathfrak{p}}} - \rho([X, Y]_{\mathfrak{l}})$$

where $\rho$ denotes the isotropy action of $\mathfrak{l}$,

$$\rho([X, Y]_{\mathfrak{l}}) \equiv \operatorname{ad}[X, Y]_{\mathfrak{l}|\mathfrak{k}+\mathfrak{p}}.$$

Using (3) we obtain

$$(5) \qquad A_X Y = \begin{cases} \frac{1}{2}[X, Y] & X, Y \in \mathfrak{k}_{(1)} \oplus \ldots \oplus \mathfrak{k}_{(r)} \\ \frac{1}{2}[X, Y]_{\mathfrak{k}} & X, Y \in \mathfrak{p} \\ \left(1 + \frac{\alpha_i}{2\beta_m}\right)[X, Y] & X \in \mathfrak{k}_{(i)}, Y \in \mathfrak{p}_{(m)} \\ -\frac{\alpha_i}{2\beta_m}[X, Y] & X \in \mathfrak{p}_{(m)}, Y \in \mathfrak{k}_{(i)} \\ 0 & X \in Z(\mathfrak{k}), Y \in \mathfrak{k} \\ \left[\left(1 - \frac{1}{2\beta_m}\bar{A}\right)X, Y\right] & X \in Z(\mathfrak{k}), Y \in \mathfrak{p}_{(m)} \\ \frac{1}{2\beta_m}[X, \bar{A}Y] & X \in \mathfrak{p}_{(m)}, Y \in Z(\mathfrak{k}). \end{cases}$$

A tedious computation using (4), (5) and the fact that

$$K(X, Y) = -\langle R(X, Y)X, Y\rangle$$

for orthonormal $X$ and $Y$ yields the curvature formulas.

**6. Conclusions.** The following theorem summarizes the necessary conditions for natural reductivity established in Sections 2-5.

(6.1) THEOREM. *Suppose* $M = G/L$ *is naturally reductive with respect to a transitive subgroup of* $G$. *Choose a maximal connected semisimple subgroup* $G_1$ *of* $G$ *compatible with* $L$ *and let* $G_{nc}$ *and* $G_c$ *be the noncompact and compact parts of* $G_1$. *Let* $N = \text{nilrad}(G)$. *Then*:

(1) $G_{nc}$ *is a maximal connected semisimple subgroup of noncompact type in the full isometry group* $I(M)$ *and is normal in* $I(M)$.

(2) $N$ *is at most* 2-*step nilpotent.*

(3) $G_1N$ *acts transitively on* $M$.

(4) *The image of* $\mathfrak{l}$ *under the homomorphic projection of* $\mathfrak{g}$ *onto* $\mathfrak{g}_{nc}$ *is a maximal compactly embedded subalgebra* $\mathfrak{k}$ *of* $\mathfrak{g}_{nc}$. $\mathfrak{k} \cap \mathfrak{l}$ *is a* $\mathfrak{k}$-*ideal.*

(5) *The submanifolds* $G_{nc}/(G_{nc} \cap L)$, $G_c/(G_c \cap L)$ *and* $N(= N/(N \cap L))$ *with the induced Riemannian metrics are totally geodesic and naturally reductive. In particular, the metric on* $G_{nc}/(G_{nc} \cap L)$ *is of the type defined in Theorem* 5.3, *and the data* $(\mathfrak{n}, \langle,\rangle, L)$ *associated with* $N$ *as in* 4.1 *satisfies conditions* (i) *and* (ii) *of Theorem* 4.8.

(6) $M$ *admits a transitive group* $H \subset G$ *of isometries which contains no noncompact semisimple subgroups.*

*Proof.* (1), (2), (3) and (5) are proven in Sections 2-5. For (4), suppose $M$ is naturally reductive with respect to the subgroup $H$ of $G$. Choose a Levi factor $H_1$ of $H$ compatible with $H \cap L$. By Remark 3.4, $H_{nc} = G_{nc}$. Hence by Theorem 3.3,

$$\mathfrak{h} = \mathfrak{g}_{nc} \oplus (\mathfrak{h}_c + \mathfrak{h}_2) \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_{nc} \oplus (\mathfrak{g}_c + \mathfrak{g}_2),$$

direct sums of ideals with $\mathfrak{h}_c + \mathfrak{h}_2 \subset \mathfrak{g}_c + \mathfrak{g}_2$. In particular, the homomorphic projection $\pi_{nc}:\mathfrak{h} \to \mathfrak{g}_{nc}$ is the restriction to $\mathfrak{h}$ of the projection $\mathfrak{g} \to \mathfrak{g}_{nc}$. It therefore suffices to show that $\pi_{nc}(\mathfrak{l} \cap \mathfrak{h})$ is a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}_{nc}$ and $\mathfrak{k} \cap \mathfrak{l}$ $(= \mathfrak{k} \cap (\mathfrak{l} \cap \mathfrak{h}))$ is a $\mathfrak{k}$-ideal. Thus we may assume for simplicity that $\mathfrak{g} = \mathfrak{h}$; i.e., $M$ is naturally reductive with respect to $G$.

$\pi_{nc}(\mathfrak{l})$ lies in some maximal compactly embedded subalgebra $\mathfrak{k}$. Let $L_0$ be the largest normal subgroup of $G_{nc}L$ contained in $L$. By the proof of Lemma 2.7 $G_{nc}/(G_{nc} \cap L)$ is naturally reductive with respect to $G_{nc}L/L_0$. $\pi_{nc}(\mathfrak{l}_0) = \{0\}$ since $\mathfrak{g}_{nc}$ contains no compact ideals. $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{m}$ for some $\mathfrak{l}$-ideal $\mathfrak{m}$ and $\pi_{nc}(\mathfrak{l}) = \pi_{nc}(\mathfrak{m})$. $(G_{nc}L)/L_0$ has Lie algebra isomorphic to $\mathfrak{g}_{nc} + \mathfrak{m}$. By Theorem 5.2, $\pi_{nc}(\mathfrak{m}) = \mathfrak{k}$ and $\mathfrak{l} \cap \mathfrak{g}_{nc} = \mathfrak{m} \cap \mathfrak{g}_{nc}$ is a $\mathfrak{k}$-ideal.

For (6), let $K$ be the subgroup of $G_{nc}$ corresponding to $\mathfrak{k}$ in (4). Recall that $G_{nc}$ admits an Iwasawa decomposition $G_{nc} = KS$ where $S$ is solvable and $K \cap S = \{e\}$ (see **[10]**). Let $H = SG_cG_2$. (Here $G_2 = \text{rad}(G)$.) The reductive subalgebra $\mathfrak{g}_1 + \mathfrak{l}$ equals $\mathfrak{g}_1 + t$ where $t = (\mathfrak{g}_1 + \mathfrak{l}) \cap \mathfrak{g}_2$ is central in $\mathfrak{g}_1 + \mathfrak{l}$. (See 2.4.) By (4),

$$\mathfrak{k} \subset \mathfrak{l} + \mathfrak{g}_c + \mathfrak{t}.$$

Hence $K \subset G_cTL \subset G_cG_2L$. But then

$$G = SKG_cG_2 \subset SG_cG_2L = HL.$$

Thus $H$ acts transitively on $M$.

(6.2) *Remark*. One can use Theorem 6.1 to obtain a new proof of a theorem of Deloff [**4**] stating that every naturally reductive homogeneous Riemannian manifold of non-positive sectional curvature is symmetric. We omit the details here as a more general theorem (with sectional curvature replaced by Ricci curvature) is proved in [**9**]. The proof of the latter theorem uses both a result of Wang-Ziller [**15**] and Theorem 6.1.

## References


**1.** R. Azencott and E. N. Wilson, *Homogeneous manifolds with negative curvature I*, Trans. Amer. Math. Soc. *215* (1976), 323-362.

**2.** J. E. D'Atri, *Geodesic spheres and symmetries in naturally reductive spaces*, Mich. Math. J. *22* (1975), 71-76.

**3.** J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc. *18*, No. 215 (1979).

**4.** E. D. Deloff, *Naturally reductive metrics and metrics with volume preserving geodesic symmetries on* NC *algebras*, thesis, Rutgers (1979).

**5.** C. S. Gordon, *Riemannian isometry groups containing transitive reductive subgroups*, Math. Ann. *248* (1980), 185-192.

**6.** ——— *Transitive Riemannian isometry groups with nilpotent radicals*, Ann. Inst. Four., Grenoble *31* (1981), 193-204.

**7.** ——— *Semisimple normal subgroups of transitive Riemannian isometry groups*, Osaka J. Math. *19* (1982), 283-286.

**8.** C. S. Gordon and E. N. Wilson, *The fine structure of transitive Riemannian isometry groups*, Part I, to appear in Trans. Amer. Math. Soc.

**9.** C. S. Gordon and W. Ziller, *Naturally reductive metrics of non-positive Ricci curvature*, to appear in Proc. Amer. Math. Soc.

**10.** S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, New York, 1978).

**11.** A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. *15* (1983), 35-42.

**12.** S. Kobayashi and K. Nomizu, *Foundations of differential geometry II* (Interscience, New York, 1969).

**13.** J. Milnor, *Curvatures of left-invariant metrics on Lie groups*, Adv. in Math. *21* (1976), 293-329.

**14.** B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. *13* (1966), 459-469.

**15.** M. Wang and W. Ziller, *On normal homogeneous Einstein metrics*, preprint.

**16.** E. N. Wilson, *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata *12* (1982), 337-346.


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