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DIMENSION ESTIMATE OF THE EXPONENTIAL ATTRACTOR FOR THE CHEMOTAXIS–GROWTH SYSTEM*

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Abstract. In this paper, we study an upper bound of the fractal dimension of the exponential attractor for the chemotaxis–growth system in a two-dimensional domain. We apply the technique given by Eden, Foias, Nicolaenko and Temam. Our results show that the bound is estimated by polynomial order with respect to the chemotactic coefficient in the equation similar to our preceding papers.

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1. Introduction. This paper is concerned with the following initial value problem for a quasi-linear parabolic system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\triangle u - \nabla\cdot\{u\nabla\chi(\rho)\} + f(u) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \rho}{\partial t} = b\triangle\rho - c\rho + du & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{CG}$$

This system was presented by Mimura and Tsujikawa [**22**] as a mathematical model describing aggregating patterns by some biological individuals. Here, $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance, respectively, at a position $x\in\Omega\subset\mathbb{R}^n$, $n = 1, 2$, and a time $t\in[0,\infty)$. The mobility of individuals consists of two effects: one is random walking, and the other is the directed movement in a sense that they have a tendency to move

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towards higher concentration of the chemical substance. This is called chemotaxis in biology (for detail see [**5**, **23**]). The constants $a > 0$ and $b > 0$ are diffusion rates of u and ρ, respectively, while $c > 0$ and $d > 0$ are degradation and production rates of ρ, respectively. We denote by $\chi(\rho)$ a sensitivity function of u with respect to ρ, and by $f(u)$ a growth term of u.

In this paper, we consider the two-dimensional case. $\Omega \subset \mathbb{R}^2$ is assumed to be a bounded domain of class $\mathcal{C}^3$. For simplicity, $\chi(\rho)$ is assumed to be linear,

$$\chi(\rho) = \nu\rho, \tag{1.1}$$

with a chemotactic coefficient $\nu > 0$, and $f(u)$ is assumed to be a cubic function

$$f(u) = f u^2(1 - u), \tag{1.2}$$

with a growth coefficient $f > 0$.

In order to study aggregating patterns due to chemotaxis and growth, there are several contributions not only from experiments but also from mathematical analysis. Budrene and Berg [**5**] experimentally observed that *Escherichia coli* bacteria form complex spatio-temporal colony patterns. In order to understand such a chemotactic pattern formation, theoretically several models have been proposed, e.g., models presented in [**3**, **15**, **21**, **24**, **30**]. Mimura and Tsujikawa presented in [**22**] a model (CG), which is rather simple in the sense that it is characterized by only four effects: diffusion, chemotaxis, production of a chemical substance and growth. In the absence of the growth term $f(u)$, (CG) reduce to the Keller–Segel equations [**17**] modelling the initiation of aggregating patterns of slime mold.

The formation of the colony patterns by chemotaxis is considered as to be a prototype of various phenomena of self-organization, cf. [**16**, **26**]. According to Haken [**16**], the chemical substance plays the role of a conductor which leads the individuals and is itself produced by them cooperatively.

In this paper, we are mainly interested in the long-time behaviour of solutions (or equivalently dynamical system) generated by (CG). It is well-known that the long-time behaviour of a dynamical system can be described in terms of the global attractor. More precisely, assuming that the system is globally well-posed, we can define the family of solution operators

$$S_t : u_0 \mapsto u(t, u_0), \quad t \geq 0,$$

acting on a metric space V (with the metric ρ_V), which maps the initial datum u_0 to the solution at time t. This family of operators satisfies

$$S_0 = Id, \qquad S_{t+s} = S_t \circ S_s, \quad t, s \geq 0,$$

where Id denotes the identity operator, and we say that it forms a *semi-flow* on the phase space V. Then a non-empty compact subset $\mathfrak{A}$ of V is called the *global attractor* for $\{S_t\}$ in V if it is invariant under $\{S_t\}$, i.e. $S_t\mathfrak{A} = \mathfrak{A}$ for every $t \geq 0$, and it attracts each bounded subset B of V in the following sense:

$$\lim_{t\to\infty} d_V(S_tB, \mathfrak{A}) = 0,$$

where d_V denotes the Hausdorff pseudo-distance between subsets of V, defined by

$$d_V(A, B) = \sup_{a\in A} \inf_{b\in B} \rho_V(a, b).$$

It follows from its definition that the global attractor, if it exists, is unique (although it is not a smooth manifold, in general, and can have a very complicated geometric structure). If one proves that the global attractor has finite dimension (in the sense of covering dimensions such as the fractal dimension), even though the initial phase space is infinite-dimensional, the dynamics, reduced to the global attractor, is in some specific sense finite-dimensional and can be described by a finite number of parameters (see [**4**, **28**]). It thus follows that the global attractor appears as a suitable object in view of the study of the long-time behaviour of the system. We refer the reader to [**4**, **20**, **28**].

Several authors have already studied the system (CG), and it is well known [**1**, **2**, **27**] that the asymptotic behaviour of solutions of (CG) is described by the dynamical system $(S_t, \mathfrak{X}, X)$ in the universal space $X = L^2(\Omega) \times H^1(\Omega)$, where the phase space $\mathfrak{X}$ is a bounded set of $H^2_N(\Omega) \times H^3_N(\Omega)$ (the $H^2 \times H^3$-space of functions with zero-boundary flux) and, hence, a compact subset of X, and S_t is a non-linear semi-group acting on $\mathfrak{X}$ which is continuous in the X-norm. Therefore, the dynamical system $(S_t, \mathfrak{X}, X)$ possesses a global attractor $\mathfrak{A} = \bigcap_{0\le t<\infty} S_t\mathfrak{X}$.

Observe that the global attractor may present some defects. For instance, it may attract the trajectories slowly (cf. [**18**]) or it may be sensitive to perturbations. Also, in some situations, the global attractor may fail to capture important transient behaviours. This can be observed, e.g., in models of pattern formation equations in chemotaxis (see [**2**]). Therefore it should be useful to have a possibly larger object which contains the global attractor, attracts the trajectories at a fast rate, is still finite-dimensional and is more robust under perturbations. Such an object called an *exponential attractor* was proposed by Eden et al. in Hilbert spaces [**7**]. Its first construction was based on the so-called squeezing property which, roughly speaking, says that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially. The contraction of the exponential attractor is also valid for Banach spaces (see [**6**, **9**]).

Osaki et al. [**27**] have proved also the existence of the exponential attractors $\mathfrak{M}$ for the system without estimating its fractal dimension with respect to the parameters of the system (see also [**1**, **14**]). Aida et al. [**2**] showed, with some numerical simulations, that the dimensions of attractors for this system increase as the chemotactic coefficient ν increases. Kuto et al. [**19**] have obtained by local bifurcation theory that, as ν increases, the number of hexagonal pattern solutions which bifurcate from the homogeneous solution would increase. They suggest that the structure of attractors would become more complicated as ν becomes larger.

The aim of this paper is to estimate from the above the fractal dimension $\dim \mathfrak{M}$ of the exponential attractor $\mathfrak{M}$ in terms of the coefficients a, b, c, d, f and ν in the equations of (CG). We will apply the technique given by Eden et al. [**7**, Chapter 3] to obtain the upper bound of $\dim \mathfrak{M}$.

The authors have already established in the previous papers [**11**, **12**] the upper and lower estimate

$$C_1(\nu d - 1) \le \dim \mathfrak{A} \le C_2((\nu d)^2 + 1), \tag{1.3}$$

C_1 and C_2 being some positive constants, of the fractal dimension $\dim \mathfrak{A}$ of the global attractor $\mathfrak{A}$ for (CG). Here we state main result of the paper.

THEOREM 1.1. *The dimension of exponential attractors* $\mathfrak{M}$ *satisfy the estimate*

$$C_1(\nu d - 1) \leq \dim \mathfrak{A} \leq \dim \mathfrak{M} \leq C_3((\nu d)^{22} + 1), \tag{1.4}$$

with some constant $C_3 > 0$.

REMARK 1.1. In this case the fractal dimension of the attractor corresponds to a reduction of the degrees of freedom in the process of pattern formation which is called the slaving principle [**16**].

The paper is organized as follows: in Section 2 we recall some known facts on the exponential attractors. Section 3 is devoted to show a priori estimates of solutions to (CG) also for preliminary of the succeeding section. Then, in Section 4, we present the upper estimate of $\dim \mathfrak{M}$.

REMARK 1.2. In the preceding papers [**10–13**] we allow the domain Ω to be convex but non-smooth, for example, a bounded convex polygonal domain. In such cases, we must replace the function space $H^3_N(\Omega)$ by the domain $\mathcal{D}((-\triangle + 1)^{3/2})$ of the fractional power of the Laplace operator (see Section 2); the phase space $H^2_N(\Omega) \times H^3_N(\Omega)$ should be replaced by $H^2_N(\Omega) \times \mathcal{D}((-\triangle + 1)^{3/2})$, and some additional revisions arise in the calculation, but it may not make the principal part of estimates worse.

2. Preliminaries. As was shown in [**1**, **2**, **27**], the system (CG) possesses an exponential attractor. In the subsequent section we present upper estimate for dimension of the exponential attractor. To this end we follow [**7**, **28**] and recall some basic facts.

Let X be a Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $\|\cdot\|_X$, $\mathfrak{X}$ a compact set of X, and consider a continuous dynamical system $(S_t, \mathfrak{X}, X)$. According to [**28**], $\mathfrak{A} = \bigcap_{0 \leq t < \infty} S_t \mathfrak{X}$ is a global attractor of $(S_t, \mathfrak{X}, X)$.

The exponential attractor $\mathfrak{M}$ is defined as follows (see [**7**]):

DEFINITION 2.1. A subset $\mathfrak{M} \subset \mathfrak{X}$ is called the exponential attractor for $(S_t, \mathfrak{X}, X)$ if (i) $\mathfrak{A} \subset \mathfrak{M} \subset \mathfrak{X}$; (ii) $\mathfrak{M}$ is a compact subset of X and is a positively invariant set for S_t, that is, $S_t\mathfrak{M} \subset \mathfrak{M}$; (iii) $\mathfrak{M}$ has finite fractal dimension $\dim \mathfrak{M}$; (iv) there exist positive constants c_0 and c_1 such that $h(S_t\mathfrak{X}, \mathfrak{M}) \leq c_0 \exp(-c_1 t)$ holds for all $t \geq 0$, where $h(A, B)$ denotes the Hausdorff pseudo-distance of two sets A and B.

By virtue of [**7**, Theorem 3.1], we have the following theorem.

THEOREM 2.1. *Let* S_{t^*} *with a fixed* $t^* > 0$ *satisfy the squeezing property* [**7**, *Definition 2.2*]*: for some* $\delta^* \in (0, 1/8)$ *there exists an orthogonal projection P of finite rank* N^* *such that either*

$$\|S_{t^*}U - S_{t^*}V\|_X \leq \delta^* \|U - V\|_X \tag{2.1}$$

or

$$\|(I - P)(S_{t^*}U - S_{t^*}V)\|_X \leq \|P(S_{t^*}U - S_{t^*}V)\|_X \tag{2.2}$$

holds for each pair $U, V \in \mathfrak{X}$. *Moreover, let the mapping* $G(t, U) = S_t U$ *from* $[0, T] \times \mathfrak{X}$ *into* $\mathfrak{X}$ *satisfy the Lipschitz condition*

$$\|S_t U_0 - S_s V_0\|_X \leq C_T (|t - s| + \|U_0 - V_0\|_X), \quad t, s \in [0, T], \ U_0, V_0 \in \mathfrak{X}, \tag{2.3}$$

for each $T > 0$. Then, the dynamical system $(S_t, \mathfrak{X}, X)$ admits an exponential attractor $\mathfrak{M}$ whose fractal dimension can be estimated by

$$\dim \mathfrak{M} \le 1 + N^* \frac{\log(1 + (2L^*/\delta^*))}{\log(1/4\delta^*)}. \tag{2.4}$$

Here, $L^* > 0$ is the Lipschitz constant of S_{t^*} on $\mathfrak{X}$

$$\|S_{t^*} U_0 - S_{t^*} V_0\|_{\mathfrak{X}} \le L^* \|U_0 - V_0\|_{\mathfrak{X}}, \quad U_0, V_0 \in \mathfrak{M}. \tag{2.5}$$

In the second half of this section we will list the well-known results in the theories of function spaces and linear operators [**25**, **27**]. Here Ω is a bounded domain in $\mathbb{R}^2$ of class $\mathcal{C}^3$.

For $0 \le s_0 < s < s_1 < \infty$, $H^s(\Omega)$ is the interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]$ between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$, where $s = (1-\theta)s_0 + \theta s_1$, with the estimate

$$\|\cdot\|_{H^s} \le C \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta}. \tag{2.6}$$

For the Laplace operator $A_0 = -\Delta + 1$ in $L^2(\Omega)$ with Neumann boundary condition, the domain of which is $H^2_N(\Omega)$, it holds that

$$\mathcal{D}(A_0^\theta) = \begin{cases} H^{2\theta}(\Omega), & 0 \le \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega), & \frac{3}{4} < \theta \le \frac{3}{2}, \end{cases} \tag{2.7}$$

with norm equivalence. Here and in what follows, $H^s_N(\Omega)$, $\theta > \frac{3}{2}$, denotes a closed subspace of $H^s(\Omega)$ such that $H^s_N(\Omega) = \{u \in H^s(\Omega);\ \partial u/\partial n = 0 \text{ on } \partial\Omega\}$. Indeed, $A_0 u \in \mathcal{D}(A_0^{1/2})$ means that $\Delta u \in H^1(\Omega)$ with $\partial u/\partial n = 0$ on $\partial\Omega$; since $\partial\Omega$ is of class $\mathcal{C}^3$, these then imply that $u \in H^3(\Omega)$.

When $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $\frac{1}{p} = \frac{1-s}{2}$, with

$$\|\cdot\|_{L^p} \le C_s \|\cdot\|_{H^s}. \tag{2.8}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $1 \le q < \infty$ with

$$\|\cdot\|_{L^q} \le C_{q,p} \|\cdot\|_{H^1}^{1-\frac{p}{q}} \|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{2.9}$$

where $1 \le p \le q < \infty$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with

$$\|\cdot\|_{\mathcal{C}} \le C_s \|\cdot\|_{H^s}. \tag{2.10}$$

The norms of a product of two functions are estimated as follows. Let $\varepsilon \in (0, 1]$ denote an arbitrary exponent. From above inequalities, it is seen that

$$\|uv\|_{L^2} \le \begin{cases} C_\varepsilon \|u\|_{L^2} \|v\|_{H^{1+\varepsilon}}, & u \in L^2(\Omega),\ v \in H^{1+\varepsilon}(\Omega), \\ C_\varepsilon \|u\|_{H^\varepsilon} \|v\|_{H^1}, & u \in H^\varepsilon(\Omega),\ v \in H^1(\Omega), \end{cases} \tag{2.11}$$

$$\|uv\|_{H^1} \le C_\varepsilon \|u\|_{H^1} \|v\|_{H^{1+\varepsilon}}, \quad u \in H^1(\Omega),\ v \in H^{1+\varepsilon}(\Omega). \tag{2.12}$$

$$\|uv\|_{H^2} \le C \|u\|_{H^2} \|v\|_{H^2}, \quad u, v \in H^2(\Omega). \tag{2.13}$$

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It is also verified that

$$\|\nabla\cdot(u\nabla v)\|_{L^2}\le\begin{cases}C_\varepsilon\|u\|_{H^1}\|v\|_{H^{2+\varepsilon}}, & u\in H^1(\Omega),\ v\in H^{2+\varepsilon}(\Omega),\\ C_\varepsilon\|u\|_{H^{1+\varepsilon}}\|v\|_{H^2}, & u\in H^{1+\varepsilon}(\Omega),\ v\in H^2(\Omega),\end{cases}\tag{2.14}$$

$$\|\nabla\cdot(u\nabla v)\|_{H^1}\le C\|u\|_{H^2}\|v\|_{H^3},\quad u\in H^2(\Omega),\ v\in H^3(\Omega).\tag{2.15}$$

3. A priori estimates of the solutions. In a similar manner as in [**12**, **27**], we can establish the following a priori estimates for the solutions in the exponential attractor $\mathfrak{M}\subset H^2_N(\Omega)\times H^3_N(\Omega)$ for (CG):

$$\|u(t)\|_{L^1}=\int_\Omega u(t,x)dx\le 2|\Omega|;\tag{3.1}$$

$$\int_s^t\|u(\tau)\|_{L^2}^2\,d\tau\le|\Omega|\left((t-s)+\frac{4}{f}\right);\tag{3.2}$$

$$\int_s^t\|u(\tau)\|_{L^3}^3\,d\tau\le|\Omega|\left((t-s)+\frac{6}{f}\right);\tag{3.3}$$

$$\|\rho(t)\|_{L^2}^2\le\frac{2d^2}{c^2}|\Omega|;\tag{3.4}$$

$$\|\nabla\rho(t)\|_{L^2}^2\le\frac{d^2}{bc}|\Omega|;\tag{3.5}$$

$$\int_s^t\|\triangle\rho(\tau)\|_{L^2}^2\,d\tau\le\frac{d^2}{b^2}|\Omega|\left((t-s)+\frac{4}{f}+\frac{1}{c}\right);\tag{3.6}$$

$$\|u(t)\|_{L^2}^2\le\left(8+\frac{\nu^2d^2}{4f^2b^2}\right)|\Omega|;\tag{3.7}$$

$$\begin{aligned}2a\int_s^t\|\nabla u(\tau)\|_{L^2}^2\,d\tau+f\int_s^t\|u(\tau)\|_{L^4}^4\,d\tau\\ \le 2f\left(1+\frac{\nu^2d^2}{8f^2b^2}\right)|\Omega|(t-s)+\left\{20+\frac{\nu^2d^2}{4f^2b^2}\left(5+\frac{f}{c}\right)\right\}|\Omega|;\end{aligned}\tag{3.8}$$

$$\|\Lambda^2\rho(t)\|_{L^2}^2\le\frac{2d^2f}{abc}\left(1+\frac{ab}{fc}+\frac{\nu^2d^2}{8f^2b^2}\right)|\Omega|;\tag{3.9}$$

$$\begin{aligned}\int_s^t\|\nabla\Lambda^2\rho(\tau)\|_{L^2}^2\,d\tau\le\frac{d^2f}{ab^2}\left(1+\frac{ab}{fc}+\frac{\nu^2d^2}{8f^2b^2}\right)|\Omega|\left((t-s)+\frac{2}{c}\right)\\ +\frac{d^2}{ab^2}\left\{10+\frac{4ab}{fc}+\frac{\nu^2d^2}{8f^2b^2}\left(5+\frac{f}{c}\right)\right\}|\Omega|;\end{aligned}\tag{3.10}$$

$$\|\nabla u(t)\|_{L^2}^2\le\frac{12a}{f}|\Omega|K_1;\tag{3.11}$$

$$\int_s^t\|\Lambda^2u(\tau)\|_{L^2}^2\,d\tau\le 4|\Omega|K_1\left((t-s)+\frac{10}{f}\right);\tag{3.12}$$

$$\|\Lambda^3\rho(t)\|_{L^2}^2 \le \frac{8d^2(b+c)}{bc^2}|\Omega|K_1; \tag{3.13}$$

$$\|\Lambda^2 u(t)\|_{L^2}^2 \le 16|\Omega|K_1K_2; \tag{3.14}$$

where $\Lambda^2 = -\Delta + 1$, $\varepsilon \in (0, 1/2)$ is an arbitrarily fixed exponent,

$$K_1 = 1 + \frac{f^2}{2a^2} + \frac{B_1''}{8}\left\{\frac{16\nu^2d^2f}{a^3bc}\left(1 + \frac{ab}{fc} + \frac{\nu^2d^2}{8f^2b^2}\right)|\Omega|\right\}^{\frac{2}{1-\varepsilon}}, \tag{3.15}$$

$$K_2 = 1 + \frac{4B_2'\nu^2d^2(b+c)}{a^2bc^2}|\Omega|K_1$$
$$+ \frac{B_3'f^2}{4a^2}\left\{1 + \left(8 + \frac{\nu^2d^2}{4f^2b^2}\right)|\Omega| + \frac{24a}{f}|\Omega|K_1\right\}\left(8 + \frac{\nu^2d^2}{4f^2b^2}\right)|\Omega|, \tag{3.16}$$

and the constants B_1'', B_2' and B_3' are determined only by ε and $B_1 = C_\varepsilon$ in (2.14), by $B_2 = C$ in (2.15) and by $B_3 = (2C_{3,2} + 3C_{6,2})C_{6,2}$ in (2.9), respectively.

The inequalities (3.1), (3.2) and (3.3) are easily verified by integrating the first equation of (CG) on Ω and utilizing the non-negativity of u,

$$\frac{d}{dt}\|u(t,x)\|_{L^1} = \int_\Omega \frac{\partial u}{\partial t}(x,t)dx \le f\|u\|_{L^2}^2 - f\|u\|_{L^3}^3$$
$$\le f\|u\|_{L^1} - f\|u\|_{L^2}^2 \le f|\Omega| - f\|u\|_{L^1}. \tag{3.17}$$

(3.4) and (3.5) follow from the energy equalities for ρ and $\nabla\rho$, that is, L^2-inner product of the second equation of (CG) with ρ and $-\triangle\rho$, respectively: for example, as for (3.5), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho(t)\|_{L^2}^2 = -b\|\triangle\rho\|_{L^2}^2 - c\|\nabla\rho\|_{L^2}^2 + d\langle u, -\triangle\rho\rangle_{L^2}$$
$$\le -\frac{b}{2}\|\triangle\rho\|_{L^2}^2 - c\|\nabla\rho\|_{L^2}^2 + \frac{d^2}{2b}\|u\|_{L^2}^2, \tag{3.18}$$

then, thanks to (3.17), utilize the formula

$$\int_0^t e^{-\alpha(t-s)}\|u(s)\|_{L^2}^2\,ds$$
$$\le \left(1 - \frac{\alpha}{f}\right)\int_0^t e^{-\alpha(t-s)}\|u(s)\|_{L^1}\,ds + \frac{e^{-\alpha t}}{f}\|u_0\|_{L^1} \le \frac{|\Omega|}{\alpha} + \frac{2e^{-\alpha t}}{f}\|u_0\|_{L^1}. \tag{3.19}$$

(3.6) is obtained by integrating (3.18) in t. To verify (3.7) we have from (CG) and (3.18) that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \le -a\|\nabla u\|_{L^2}^2 + \frac{\nu}{2}\|\triangle\rho\|_{L^2}\|u\|_{L^4}^2 + f\|u\|_{L^3}^3 - f\|u\|_{L^4}^4$$
$$\le -a\|\nabla u\|_{L^2}^2 + \frac{\nu^2}{8f}\|\triangle\rho\|_{L^2}^2 + f\|u\|_{L^3}^3 - \frac{f}{2}\|u\|_{L^4}^4$$

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$$\leq -a\|\nabla u\|_{L^2}^2 + \frac{\nu^2}{8f}\|\triangle\rho\|_{L^2}^2 + 4f|\Omega| - f\|u\|_{L^2}^2$$
$$\leq -a\|\nabla u\|_{L^2}^2 + 4f|\Omega| - f\|u\|_{L^2}^2 + \frac{\nu^2}{8f}\left(\frac{d^2}{b^2}\|u\|_{L^2}^2 - \frac{1}{b}\frac{d}{dt}\|\nabla\rho(t)\|_{L^2}^2\right), \tag{3.20}$$

and utilize (3.19). (3.8) is obtained by integrating (3.20) in t. (3.9) follows from the energy equality for $\Lambda^2\rho$,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^2\rho(t)\|_{L^2}^2 = -b\|\nabla\Lambda^2\rho\|_{L^2}^2 - c\|\Lambda^2\rho\|_{L^2}^2 + d\langle\Lambda u,\ \Lambda^3\rho\rangle_{L^2}$$
$$\leq -\frac{b}{2}\|\nabla\Lambda^2\rho\|_{L^2}^2 - \frac{c}{2}\|\Lambda^2\rho\|_{L^2}^2 + \frac{d^2}{2b}\|\nabla u\|_{L^2}^2 + \frac{d^2}{2c}\|u\|_{L^2}^2, \tag{3.21}$$

and utilize (3.19) and (3.20). (3.10) is obtained by integrating (3.21) in t. To show (3.11), from (2.14) with $B_1 = C_\varepsilon$ and (2.6) with $B_1' = C$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 \leq -a\|\triangle u\|_{L^2}^2 + B_1'B_1\nu\|u\|_{L^2}^{\frac{1-\varepsilon}{2}}\|\Lambda^2 u\|_{L^2}^{\frac{3+\varepsilon}{2}}\|\Lambda^2\rho\|_{L^2} + \frac{f}{3}\|\nabla u\|_{L^2}^2$$
$$\leq -\frac{a}{4}\|\Lambda^2 u\|_{L^2}^2 - \frac{f}{6}\|\nabla u\|_{L^2}^2 + \left[a + \frac{f^2}{2a} + \frac{B_1''a}{8}\left(\frac{4\nu^2}{a^2}\|\Lambda^2\rho(s)\|_{L^2}^2\right)^{\frac{2}{1-\varepsilon}}\right]\|u\|_{L^2}^2, \tag{3.22}$$

where $B_1'' = (1-\varepsilon)(3+\varepsilon)^{\frac{3+\varepsilon}{1-\varepsilon}}(B_1'B_1)^{\frac{4}{1-\varepsilon}}$, then utilize (3.9). (3.12) is obtained by integration of (3.22) in t. (3.13) follows from the energy equality for $\Lambda^3\rho$,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^3\rho(t)\|_{L^2}^2 = -b\|\nabla\Lambda^3\rho\|_{L^2}^2 - c\|\Lambda^3\rho\|_{L^2}^2 + d\langle\Lambda^2 u,\ \Lambda^4\rho\rangle_{L^2}$$
$$\leq -\frac{b}{2}\|\nabla\Lambda^3\rho\|_{L^2}^2 - \frac{c}{2}\|\Lambda^3\rho\|_{L^2}^2 + \frac{d^2(b+c)}{2bc}\|\Lambda^2 u\|_{L^2}^2, \tag{3.23}$$

and utilize (3.9), (3.19) and (3.22). (3.14) is verified from the following estimate: from (2.15) with $B_2 = C$ and (2.9) with $B_3 = (2C_{3,2} + 3C_{6,2})C_{6,2}$,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^2 u\|_{L^2}^2 = -a\|\nabla\Lambda^2 u\|_{L^2}^2 - \nu\langle\Lambda\nabla\cdot(u\nabla\rho),\Lambda^3 u\rangle_{L^2} + f\langle\Lambda(u^2-u^3),\Lambda^3 u\rangle_{L^2}$$
$$\leq -a\|\Lambda^3 u\|_{L^2}^2 + a\|\Lambda^2 u\|_{L^2}^2 + \nu\cdot B_2\|\Lambda^2 u\|_{L^2}\|\Lambda^3\rho\|_{L^2}\cdot\|\Lambda^3 u\|_{L^2}$$
$$+ f\cdot B_3(1+\|\Lambda u\|_{L^2})\|u\|_{L^2}\|\Lambda^2 u\|_{L^2}\cdot\|\Lambda^3 u\|_{L^2}$$
$$\leq -a\|\Lambda^2 u\|_{L^2}^2 + \left[2a + \frac{B_2^2\nu^2}{2a}\|\Lambda^3\rho\|_{L^2}^2 + \frac{B_3^2 f^2}{2a}(1+\|\Lambda u\|_{L^2}^2)\|u\|_{L^2}^2\right]\|\Lambda^2 u\|_{L^2}^2. \tag{3.24}$$

Then, $B_2' = B_2^2$ and $B_3' = B_3^2$ in (3.16).

4. Upper estimate of attractor dimension. Now we will apply Theorem 2.1 for the system (CG) to obtain an upper bound for $\dim\mathfrak{M}$ given by (2.4). As already

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seen, the asymptotic behaviour of solutions of (CG) is described by the dynamical system $(S_t, \mathfrak{X}, X)$ in the universal space $X = L^2(\Omega) \times H^1(\Omega)$. The phase space $\mathfrak{X}$ is a bounded set of $H^2_N(\Omega) \times H^3_N(\Omega)$ and, hence, a compact subset of $H^1(\Omega) \times H^2_N(\Omega) \subset X$. Therefore we can consider the non-linear semi-group S_t acting on $\mathfrak{X}$ as the one acting on $H^1(\Omega) \times H^2_N(\Omega)$.

In Section 4.1 we will show the estimate of the constant N^* appearing in Theorem 2.1. Section 4.2 is devoted to show the estimate of the Lipschitz constant L^* of the semi-group S_{t^*}. Then, in Section 4.3, we will show the upper estimate of $\dim \mathfrak{M}$.

4.1. Estimate of squeezing constant N^*. Let λ_n be the nth eigenvalue of $-\triangle + 1 = \Lambda^2$,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty, \tag{4.1}$$

and ϕ_n the corresponding eigenvector in $L^2(\Omega)$. Then we set

$$H_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}, \tag{4.2}$$

and P_N be the orthogonal projection onto H_N.

Take $U_0 = \left[\begin{smallmatrix} u_0 \\ \rho_0 \end{smallmatrix}\right]$ and $V_0 = \left[\begin{smallmatrix} v_0 \\ \xi_0 \end{smallmatrix}\right]$ from $\mathfrak{M}$, then the solutions $U(t) = \left[\begin{smallmatrix} u(t) \\ \rho(t) \end{smallmatrix}\right] = S_t U_0$ and $V(t) = \left[\begin{smallmatrix} v(t) \\ \xi(t) \end{smallmatrix}\right] = S_t V_0$ remain in $\mathfrak{M}$, and the difference $W(t) = \left[\begin{smallmatrix} w(t) \\ \eta(t) \end{smallmatrix}\right] = U(t) - V(t)$ satisfies the following equation:

$$\begin{cases} \dfrac{\partial w}{\partial t} = a\triangle w - \nu\nabla\cdot(v\nabla\eta + w\nabla\rho) & \\ \qquad\qquad + f(u + v - u^2 - uv - v^2)w & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial \eta}{\partial t} = b\triangle\eta - c\eta + dw & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial \eta}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ w(x, 0) = u_0(x) - v_0(x), \quad \eta(x, 0) = \rho_0(x) - \xi_0(x) & \text{in } \Omega. \end{cases} \tag{4.3}$$

Let us assume that U_0 and V_0 satisfy

$$\|P_N W(t^*)\| < \|(I - P_N)W(t^*)\| \tag{4.4}$$

with t^* fixed in Theorem 2.1. Then, thanks to the squeezing property, it holds that

$$\|W(t^*)\| \leq \delta^* \|W(0)\|. \tag{4.5}$$

We define

$$W_0(t) = \|w(t)\|^2_{L^2} + \frac{\nu^2}{bf}\|\Lambda\eta(t)\|^2_{L^2}, \tag{4.6}$$

$$W_1(t) = a\|\Lambda w(t)\|^2_{L^2} + \frac{\nu^2}{f}\|\Lambda^2\eta(t)\|^2_{L^2}, \tag{4.7}$$

$$W_2(t) = a^2\|\Lambda^2 w(t)\|^2_{L^2} + \frac{\nu^2 b}{f}\|\Lambda^3\eta(t)\|^2_{L^2}, \tag{4.8}$$

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and

$$\lambda(t) = \frac{W_1(t)}{W_0(t)}. \tag{4.9}$$

Then we see

$$\lambda(t^*) \geq \frac{a\|(1-P_N)\Lambda w(t^*)\|_{L^2}^2 + \frac{\nu^2}{f}\|(1-P_N)\Lambda^2\eta(t^*)\|_{L^2}^2}{2\left(\|(1-P_N)w(t^*)\|_{L^2}^2 + \frac{\nu^2}{bf}\|(1-P_N)\Lambda\eta(t^*)\|_{L^2}^2\right)} \geq \frac{\lambda_{N+1}}{2(a^{-1}+b^{-1})}. \tag{4.10}$$

Here we have, from (2.9) with $B_4 = C_{4,2}$,

$$\begin{aligned}
\frac{1}{2}\frac{dW_0}{dt} &\leq -a\|\nabla w\|_{L^2}^2 + \nu\|v\|_{L^4}\|\nabla\eta\|_{L^4}\|\nabla w\|_{L^2} + \frac{\nu}{2}\|w\|_{L^4}^2\|\triangle\rho\|_{L^2} \\
&\quad + \frac{f}{3}\|w\|_{L^2}^2 - \frac{\nu^2}{f}\|\nabla\Lambda\eta\|_{L^2}^2 - \frac{\nu^2 c}{bf}\|\Lambda\eta\|_{L^2}^2 + \frac{\nu^2 d}{bf}\|w\|_{L^2}\|\Lambda^2\eta\|_{L^2} \\
&\leq -\frac{1}{2}W_1 + \left(\frac{3a}{4} + \frac{b}{2} + \frac{f}{3} + \frac{\nu^2 d^2}{fb^2} + \frac{B_4^4\nu^2}{4a}\|\triangle\rho\|_{L^2}^2 + \frac{B_4^4 f^2 b}{a^2}\|v\|_{L^4}^4\right)W_0 \\
&\equiv \frac{1}{2}\left(-\lambda(t) + M_0(t)\right)W_0,
\end{aligned} \tag{4.11}$$

and hence

$$W_0(t) \leq \exp\left(\int_0^t (-\lambda(s) + M_0(s))\,ds\right)W_0(0), \tag{4.12}$$

where

$$M_0(s) = \frac{3a}{2} + b + \frac{2f}{3} + \frac{2\nu^2 d^2}{fb^2} + \frac{B_4^4\nu^2}{2a}\|\triangle\rho\|_{L^2}^2 + \frac{2B_4^4 f^2 b}{a^2}\|v\|_{L^4}^4. \tag{4.13}$$

Thus we can set

$$\delta^* \equiv \exp\left(\frac{1}{2}\int_0^{t^*}(-\lambda(s) + M_0(s))\,ds\right). \tag{4.14}$$

Applying (3.6) and (3.8), we easily see

$$\begin{aligned}
\int_0^{t^*} M_0(s)\,ds &\leq \left(\frac{3a}{2} + b + \frac{2f}{3} + \frac{2\nu^2 d^2}{fb^2}\right)t^* + \frac{B_4^4\nu^2 d^2}{2ab^2}|\Omega|\left(t^* + \frac{4}{f} + \frac{1}{c}\right) \\
&\quad + \frac{2B_4^4 fb}{a^2}|\Omega|\left[2f\left(1 + \frac{\nu^2 d^2}{8f^2b^2}\right)t^* + \left\{20 + \frac{\nu^2 d^2}{4f^2b^2}\left(5 + \frac{f}{c}\right)\right\}\right] \\
&= O((\nu d)^2) \quad \text{as } \nu d \to \infty.
\end{aligned} \tag{4.15}$$

Next let us estimate $\int_0^{t^*}\lambda(s)\,ds$. We have

$$\begin{aligned}
\frac{W_0^2}{2}\frac{d\lambda}{dt} &= \langle w_t, W_0 a\Lambda^2 w - W_1 w\rangle + \frac{\nu^2}{bf}\langle\Lambda\eta_t, W_0 b\Lambda^3\eta - W_1\Lambda\eta\rangle \\
&\leq -(W_0W_2 - W_1^2) + (a+b+c)W_0W_1
\end{aligned}$$

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$$+\left[\left\|-\nu\nabla\cdot(v\nabla\eta+w\nabla\rho)+f(u+v-u^2-uv-v^2)w\right\|_{L^2}^2+\frac{\nu^2d^2}{bf}\|\Lambda w\|_{L^2}^2\right]^{\frac{1}{2}}$$
$$\times W_0^{\frac{1}{2}}\left(W_0W_2-W_1^2\right)^{\frac{1}{2}}$$
$$\leq\left[a+b+c+\frac{\nu^2d^2}{4abf}+B_1^2f\|v\|_{H^{1+\varepsilon}}^2+B_1^2\frac{\nu^2}{a}\|\rho\|_{H^{2+\varepsilon}}^2\right.$$
$$\left.+B_4^2\frac{f^2}{2a}\|u+v-u^2-uv-v^2\|_{L^4}^2\right]W_0W_1\equiv\frac{1}{2}\mu(t)W_0W_1. \tag{4.16}$$

Hence,

$$\frac{d\lambda}{dt}\leq\mu(t)\lambda, \tag{4.17}$$

and then we have

$$\lambda(t^*)\leq\exp\left(\int_s^{t^*}\mu(\tau)\,d\tau\right)\lambda(s),\qquad 0\leq s\leq t^*. \tag{4.18}$$

Integrating in s and using (4.10), we have

$$\int_0^{t^*}\lambda(s)\,ds\geq\lambda(t^*)\int_0^{t^*}\exp\left(-\int_s^{t^*}\mu(\tau)\,d\tau\right)ds$$
$$\geq\frac{\lambda_{N+1}}{2(a^{-1}+b^{-1})}\int_0^{t^*}\exp\left(-\int_s^{t^*}\mu(\tau)\,d\tau\right)ds. \tag{4.19}$$

Here we recall (3.7), (3.9), (3.11), (3.13) and (3.14) to obtain

$$\mu(\tau)=2(a+b+c)+\frac{\nu^2d^2}{2abf}+2B_1^2f\|v\|_{H^{1+\varepsilon}}^2+2B_1^2\frac{\nu^2}{a}\|\rho\|_{H^{2+\varepsilon}}^2$$
$$+B_4^2\frac{f^2}{a}\|u+v-u^2-uv-v^2\|_{L^4}^2$$
$$\leq 2(a+b+c)+\frac{\nu^2d^2}{2abf}+2B_1^2fC_\varepsilon\|v\|_{H^1}^{2(1-\varepsilon)}\|v\|_{H^2}^{2\varepsilon}+2B_1^2\frac{\nu^2}{a}C_\varepsilon\|\rho\|_{H^2}^{2(1-\varepsilon)}\|\rho\|_{H^3}^{2\varepsilon}$$
$$+B_4^2\frac{f^2}{a}C\left\{\|u\|_{L^2}\|u\|_{H^1}\left(1+\|u\|_{H^1}^2\right)+\|v\|_{L^2}\|v\|_{H^1}\left(1+\|v\|_{H^1}^2\right)\right\}$$
$$\leq 2(a+b+c)+\frac{\nu^2d^2}{2abf}+2B_1^2fC_\varepsilon\left\{\left(8+\frac{\nu^2d^2}{4f^2b^2}\right)|\Omega|+\frac{12a}{f}|\Omega|K_1\right\}^{1-\varepsilon}\{16|\Omega|K_1K_2\}^\varepsilon$$
$$+2B_1^2\frac{\nu^2}{a}C_\varepsilon\left\{\frac{2d^2f}{abc}\left(1+\frac{ab}{fc}+\frac{\nu^2d^2}{8f^2b^2}\right)|\Omega|\right\}^{1-\varepsilon}\left\{\frac{8d^2(b+c)}{bc^2}|\Omega|K_1\right\}^\varepsilon$$
$$+2B_4^2\frac{f^2}{a}C\left\{\left(8+\frac{\nu^2d^2}{4f^2b^2}\right)|\Omega|\right\}^{\frac{1}{2}}\left\{1+\left(8+\frac{\nu^2d^2}{4f^2b^2}\right)|\Omega|+\frac{12a}{f}|\Omega|K_1\right\}^{\frac{3}{2}}$$
$$\equiv\mu_1=O\left((\nu d)^{\frac{13}{1-\varepsilon}}\right). \tag{4.20}$$

Then we have

$$\int_0^{t^*}\lambda(s)\,ds\geq\frac{\lambda_{N+1}}{2(a^{-1}+b^{-1})}\int_0^{t^*}e^{-\mu_1(t^*-s)}\,ds=\frac{\lambda_{N+1}}{2(a^{-1}+b^{-1})}\frac{1-e^{-\mu_1t^*}}{\mu_1}. \tag{4.21}$$

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Thus δ^* given in (4.14) is estimated by

$$\delta^* \leq \exp\left[-\frac{\lambda_{N+1}}{4(a^{-1}+b^{-1})}\frac{1-e^{-\mu_1 t^*}}{\mu_1} + \frac{1}{2}\int_0^{t^*} M_0(s)\,ds\right]. \tag{4.22}$$

If we take N^* large enough such that

$$-\frac{\lambda_{N^*+1}}{4(a^{-1}+b^{-1})}\frac{1-e^{-\mu_1 t^*}}{\mu_1} + \frac{1}{2}\int_0^{t^*} M_0(s)\,ds < \log\frac{1}{8} = -3\log 2 \tag{4.23}$$

as well as

$$-\frac{\lambda_{N^*}}{4(a^{-1}+b^{-1})}\frac{1-e^{-\mu_1 t^*}}{\mu_1} + \frac{1}{2}\int_0^{t^*} M_0(s)\,ds > \log\frac{1}{8} = -3\log 2, \tag{4.24}$$

that is,

$$\lambda_{N^*+1} > \frac{2(a^{-1}+b^{-1})}{1-e^{-\mu_1 t^*}}\mu_1\left(6\log 2 + \int_0^{t^*} M_0(s)\,ds\right) > \lambda_{N^*} \tag{4.25}$$

holds, then, we see that $\delta^* < 1/8$. By Theorem 5.6.2 of [**29**], we have

$$\lambda_N + 1 = O(N+1), \tag{4.26}$$

for integer N. Hence, we obtain

$$N^* \leq C\lambda_{N^*} \leq C\mu_1\left(1 + \int_0^{t^*} M_0(s)\,ds\right) = O\left((\nu d)^{\frac{15-2\varepsilon}{1-\varepsilon}}\right). \tag{4.27}$$

4.2. Estimate of Lipschitz constant L^*. The Lipschitz constant L^* of the operator S_{t^*} on $H^1(\Omega) \times H^2_N(\Omega)$ satisfies

$$W_1(t^*) \leq L^* W_1(0), \tag{4.28}$$

for $W_1(t)$ introduced in the preceding paragraph. But here we note that

$$\frac{dW_1}{dt} = W_0\frac{d\lambda}{dt} + \frac{W_1}{W_0}\frac{dW_0}{dt} \leq (\mu(t) + M_0(t))\,W_1, \tag{4.29}$$

and, hence, we can set

$$L^* = \exp\left(\int_0^{t^*} M_1(s)\,ds\right), \tag{4.30}$$

where

$$\begin{aligned} M_1(s) &= \mu(s) + M_0(s) \\ &= \frac{7a}{2} + 3b + 2c + \frac{2f}{3} + \frac{\nu^2 d^2(4a+b)}{2fab^2} + \frac{B_4^4\nu^2}{2a}\|\triangle\rho\|^2_{L^2} + \frac{2B_4^4 f^2 b}{a^2}\|v\|^4_{L^4} \\ &\quad + 2B_1^2 f\|v\|^2_{H^{1+\varepsilon}} + 2B_1^2\frac{\nu^2}{a}\|\rho\|^2_{H^{2+\varepsilon}} + B_4^2\frac{f^2}{a}\|u+v-u^2-uv-v^2\|^2_{L^4}. \end{aligned} \tag{4.31}$$

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We recall (3.6), (3.8), (3.10) and (3.12) to obtain

$$\log L^* = \int_0^{t^*} M_1(s)\, ds$$
$$= O\left((\nu d)^2 + (\nu d)^{2(1-\varepsilon)} K_1^{\varepsilon} + (\nu d)^3 K_1^{1/2}\right) = O\left((\nu d)^{\frac{7-3\varepsilon}{1-\varepsilon}}\right). \quad (4.32)$$

4.3. Upper estimate of $\dim \mathfrak{M}$**.** From (2.4), $\dim \mathfrak{M}$ is estimated from above by

$$\dim \mathfrak{M} \leq C^* N^* \log L^* = O\left((\nu d)^{\frac{22-5\varepsilon}{1-\varepsilon}}\right). \quad (4.33)$$

This is the desired estimate.

Appendix. Here we show that smoothing property leads to the squeezing property [**7**] correcting the proof of [**8**] on the convergence of the projections. Let H and H_1 be two Hilbert spaces such that the embedding $H_1 \subset H$ is compact, and let us assume that the map $S : X \to X$, where X is a bounded subset of H, enjoys the smoothing property

$$\|Su_1 - Su_2\|_{H_1} \leq L\|u_1 - u_2\|_H, \quad u_1, u_2 \in X, \quad (A.1)$$

with some Lipschitz constant L.

Let then $M : H \to H_1$ be a self-adjoint linear onto mapping such that $\|Mu\|_{H_1} = \|u\|_H$, $u \in H$. The mapping M is obviously compact in H. Therefore, we can consider the projections P_n based on the spectrum of M, that is, $P_n : H \to H_n := \operatorname{span}\{e_1, \dots, e_n\}$ is an orthogonal projection, $Me_n = \lambda_n e_n$, $\lambda_n \to 0$ as $n \to \infty$. Obviously, we have $P_n u \to u$ as $n \to \infty$ for each $u \in H$.

Let us now check the squeezing property. We have, for u_1 and u_2 in X,

$$\|(I - P_n)(Su_1 - Su_2)\|_H = \|M(I - P_n)(Su_1 - Su_2)\|_{H_1}$$
$$\leq \|M(I - P_n)\|_{\mathcal{L}(H_1, H_1)} \cdot L\|u_1 - u_2\|_H. \quad (A.2)$$

Let now δ belong to $(0, 1/4)$. If $\|Su_1 - Su_2\|_H \leq \delta\|u_1 - u_2\|_H$, then the squeezing property is satisfied. So, let us assume that

$$\|Su_1 - Su_2\|_H > \delta\|u_1 - u_2\|_H, \quad \text{that is, } \|u_1 - u_2\|_H < \frac{1}{\delta}\|Su_1 - Su_2\|_H. \quad (A.3)$$

We have to show that there exists $n_0 = n_0(\delta) \in \mathbb{N}$ such that

$$\|(I - P_{n_0})(Su_1 - Su_2)\|_H \leq \|P_{n_0}(Su_1 - Su_2)\|_H. \quad (A.4)$$

Indeed, it follows from (A.2) and (A.3) that

$$\|(I - P_n)(Su_1 - Su_2)\|_H < \frac{L}{\delta}\|M(I - P_n)\|_{\mathcal{L}(H_1, H_1)} \cdot \|Su_1 - Su_2\|_H$$
$$\leq \epsilon_n \left(\|P_n(Su_1 - Su_2)\|_H + \|(I - P_n)(Su_1 - Su_2)\|_H\right), \quad (A.5)$$

where $\epsilon_n = (L/\delta)\|M(I-P_n)\|_{\mathcal{L}(H_1,H_1)}$ is a constant independent of the choice of u_1 and u_2, which yields

$$(1-\epsilon_n)\|(I-P_n)(Su_1 - Su_2)\|_H \le \epsilon_n \|P_n(Su_1 - Su_2)\|_H. \tag{A.6}$$

So, we have proved the squeezing property if $\epsilon_n \to 0$ as $n \to \infty$ (we note that ϵ_n does not depend on the choice of u_1 and u_2). Indeed, we have

$$\begin{aligned}\|M(I-P_n)\|_{\mathcal{L}(H_1,H_1)} &= \sup_{0\neq u\in H_1} \frac{\|M(I-P_n)u\|_{H_1}}{\|u\|_{H_1}} \\ &= \sup_{0\neq v\in H} \frac{\|M(I-P_n)Mv\|_{H_1}}{\|Mv\|_{H_1}} = \sup_{0\neq v\in H} \frac{\|(I-P_n)Mv\|_H}{\|v\|_H},\end{aligned} \tag{A.7}$$

and $P_nMv \to Mv$ as $n \to \infty$ for each $v \in H$, hence $\epsilon_n \to 0$ as $n \to \infty$, the squeezing property. Now, since the smoothing property implies the squeezing property in a Hilbert setting, we could also have estimated the dimension of $\mathfrak{M}$ by using the classical method of [7].

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REFERENCES

1. M. Aida, M. Efendiev and A. Yagi, Quasilinear abstract parabolic evolution equations and exponential attractors, *Osaka J. Math.* **42** (2005), 101–132.

2. M. Aida, T. Tsujikawa, M. Efendiev, A. Yagi and M. Mimura, Lower estimate of attractor dimension for chemotaxis growth system, *J. London Math. Soc.* **74** (2006), 453–474.

3. W. Alt and D. A. Lauffenburger, Transient behavior of a chemotaxis system modelling certain types of tissue inflammation, *J. Math. Biol.* **24** (1985), 691–722.

4. A. V. Babin and M. I. Vishik, *Attraktory Evolutsionnyky Uravnenii* (Nauka, Moskow, 1989). *Attractors of evolution equations*, English translation, North-Holland, Amsterdam, 1992

5. E. O. Budrene and H. C. Berg, Complex patterns formed by motile cells of *Escherichia coli*, *Nature* **349** (1991), 630–633.

6. L. Dung and B. Nicolaenko, Exponential attractors in Banach spaces, *J. Dyn. Diff. Eq.* **13** (2001), 791–806.

7. A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential attractors for dissipative evolution equations* (Masson, Paris, 1994).

8. M. Efendiev and A. Miranville, The dimension of the global attractor for dissipative reaction-diffusion systems, *Appl. Math. Lett.* **16** (2003), 351–355.

9. M. Efendiev, A. Miranville and S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, *C. R. Acad. Sci. Paris* **330** (2000), 713–718.

10. M. Efendiev and E. Nakaguchi, Upper and lower estimate of dimension of the global attractor for the chemotaxis–growth system: Part I, *Adv. Math. Sci. Appl.* **16** (2006), 569–579.

11. M. Efendiev and E. Nakaguchi, Upper and lower estimate of dimension of the global attractor for the chemotaxis–growth system II: Two-dimensional case, *Adv. Math. Sci. Appl.* **16** (2006), 581–590.

12. E. Nakaguchi and M. Efendiev, On a new dimension estimate of the global attractor for chemotaxis–growth systems, *Osaka J. Math.* **45** (2008), 273–281.

13. M. Efendiev, E. Nakaguchi and W. L. Wendland, Uniform estimate of dimension of the global attractor for a semi-discretized chemotaxis–growth system, *Discrete Conti. Dyn. Syst.* **2007** (Suppl.) (2007), 334–343.

14. M. Efendiev and A. Yagi, Continuous dependence on a parameter of exponential attractors for chemotaxis–growth system, *J. Math. Soc. Japan* **57** (2005), 167–181.

15. R. M. Ford and D. A. Lauffenburger, Analysis of chemotactic bacterial distributions in population migration assays using a mathematical model applicable to steep or shallow attractant gradients, *Bull. Math. Biol.* **53** (1991), 721–749.

16. H. Haken, *Synergetics—An introduction*, 3rd ed. (Springer-Verlag, New York, 1983).

17. E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* **26** (1970), 399–415.

18. I. N. Kostin, Rate of attraction to a non-hyperbolic attractor, *Asymptot. Anal.* **16** (1998), 203–222.

19. K. Kuto, N. Kurata, K. Osaki, T. Tsujikawa and T. Sakurai, Hexagonal pattern formatioin in a chemotaxis-diffusion-growth model, preprint (2006).

20. O. Ladyzhenskaya, *Attractors for semigroups and evolution equations* (Cambridge University Press, Cambridge, UK, 1991).

21. D. A. Lauffenburger and C. R. Kennedy, Localized bacterial infection in a distributed model for tissue inflammation, *J. Math. Biol.* **16** (1983), 141–163.

22. M. Mimura and T. Tsujikawa, Aggregating pattern dynamics in a chemotaxis model including growth, *Phys. A* **230** (1996), 499–543.

23. J. D. Murray, *Mathematical biology*, 3rd ed. (Springer-Verlag, Berlin, 2002).

24. M. R. Myerscough and J. D. Murray, Analysis of propagating pattern in a chemotaxis system, *Bull. Math. Biol.* **54** (1992), 77–94.

25. E. Nakaguchi and A. Yagi, Fully discrete approximations by Galerkin Runge-Kutta method for quasilinear parabolic systems, *Hokkaido Math. J.* **31** (2002), 385–429.

26. G. Nicolis and I. Prigogine, *Self-organization in nonequilibrium system—From dissipation structure to order through fluctuations* (John Wiley & Sons, Chichester, 1977).

27. K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractor for a chemotaxis–growth system of equations, *Nonlinear Anal.* **51** (2002), 119–144.

28. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, 2nd ed. (Springer-Verlag, Berlin, 1997).

29. H. Triebel. *Interpolation theory, function spaces, differential operators* (North-Holland, Amsterdam, 1978).

30. D. E. Woodward, R. Tyson, M. R. Myerscough, J. D. Murray, E. O. Budrene and H. C. Berg, Spatio-temporal patterns generated by Salmonella typhimurium, *Biophys. J.* **68** (1995), 2181–2189.