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WIDTH-DIAMETER RELATIONS FOR PLANAR CONVEX SETS WITH LATTICE POINT CONSTRAINTS

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We obtain an inequality concerning the width and diameter of a planar convex set with interior containing no point of the rectangular lattice. We then use the result to obtain a corresponding inequality for a planar convex set with interior containing exactly two points of the integral lattice.

1. INTRODUCTION

Let K be a compact, non-empty convex set in E^2 with minimal width $w(K) = w$ and diameter $d(K) = \delta$. Let K^o denote the interior of K and let Γ denote the integral lattice. A number of results are known concerning the relationship between the width and the diameter of a convex set. The following elegant result was obtained by Scott [**3**].

THEOREM 1. *If K^o contains no point of Γ, then $(w-1)(\delta-1) \leqslant 1$ with equality when and only when K is a triangle of diameter δ and width $w = \delta/(\delta-1)$* (Figure 1).

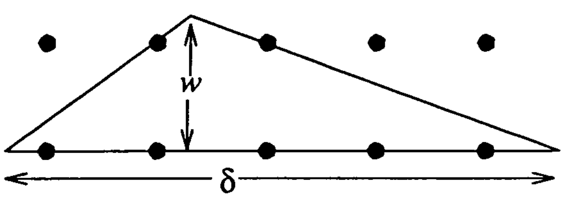

Figure 1.

Theorem 1 has been extended to sets containing exactly one point of Γ in the interior [4]. The analogous result is:

THEOREM 2. *If K^o contains one point of Γ, then $\left(w-\sqrt{2}\right)\left(\delta-\sqrt{2}\right) \leqslant 2$; the inequality is best possible.*

The purpose of this paper is to generalise Theorem 1 to rectangular lattices and to use the result to obtain analogous inequalities for convex sets containing exactly two points of Γ in the interior. Let $\Lambda_R(u,v)$ be a rectangular lattice generated by the vectors $(u,0)$ and $(0,v)$. We prove the following two pretty results:

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THEOREM 3. *Suppose that $u \leqslant v$ and that K^o contains no point of $\Lambda_R(u,v)$. Then $(w-v)(\delta-u) \leqslant uv$; equality is attained when and only when K is a triangle with diameter δ and width $w = \delta v/(\delta - u)$* (Figure 2).

THEOREM 4. *If K^o contains exactly two points of Γ then $(w-2)(\delta-1) \leqslant 2$; equality is attained when and only when K is a triangle with diameter δ and width $w = 2\delta/(\delta-1)$* (Figure 3).

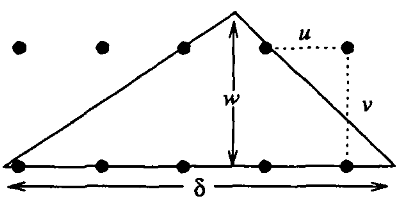

Figure 2.

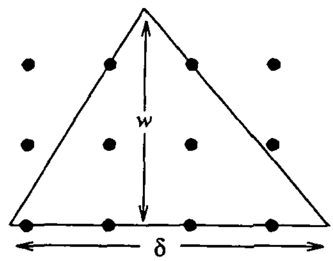

Figure 3.

2. Three useful lemmas

We shall denote lines by lower case letters: thus x is a line containing the point X of $\Lambda_R(u,v)$. Let the slope of x be m_x and let $d(Y,x)$ denote the perpendicular distance from the point Y to the line x.

Let K be a set containing no point of $\Lambda_R(u,v)$ in its interior. A set for which $(w-v)(\delta-u)$ is as large as possible is called a *maximal set.* Clearly we may assume that $\delta \geqslant w > v \geqslant u$. We first establish three lemmas which will help us narrow down the possibilities for a maximal set.

We say that a triangle *circumscribes* a rectangle (or equivalently, a rectangle is *inscribed* in a triangle) if all vertices of the rectangle lie on the sides of the triangle. Lemma 1 establishes the maximal value of $(w-v)(d-u)$ where K is a triangle circumscribing a fundamental rectangular cell of $\Lambda_R(u,v)$. Lemmas 2 and 3 will help us eliminate those cases for which K is not maximal.

LEMMA 1. *Let K be a triangle circumscribing a fundamental rectangular cell of $\Lambda_R(u,v)$. Then $(w-v)(\delta-u) \leqslant uv$ with equality when and only when the side of the rectangular cell having length u lies on the edge of K with length δ* (Figure 4).

PROOF: Let the vertices of K be X, Y and Z and let C denote the fundamental rectangular cell inscribed in K. Without loss of generality, let XY be the side of K containing two vertices of C. Let XY have length b and let the altitude from Z to XY be h.

We first let the side of C with length u lie on the edge XY. Then the area of K is $(1/2)bh(= (1/2)w\delta)$. The edges of C partition K into four regions. The area of K

may therefore be calculated as the sum of the areas of the four component parts (Figure 4).

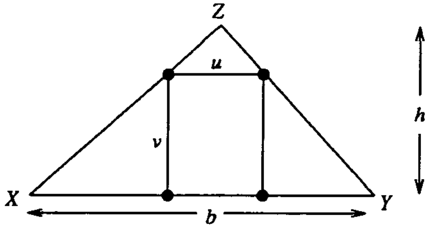

Figure 4.

Hence

$$\frac{1}{2}w\delta = \frac{1}{2}bh = \frac{1}{2}(b-u)v + \frac{1}{2}(h-v)u + uv$$
$$= \frac{1}{2}(bv + hu),$$

that is,

$$w\delta = bh = bv + hu.$$

From the identity $(\alpha+\beta)^2 = (\alpha-\beta)^2 + 4\alpha\beta$, we note that the sum of two numbers with a given product is smallest when the difference between them is least. Applying this first to the pair (bv, hu) and then to the pair $(\delta v, wu)$, and noting that $bv - hu \leqslant \delta v - wu$, we have

$$bv + hu \leqslant \delta v + wu.$$

We thus have

$$w\delta \leqslant \delta v + wu.$$

Adding uv to both sides of the inequality gives

$$(w-v)(\delta-u) \leqslant uv.$$

Equality is attained here when $XY = b = \delta$ and $h = w$.

If, on the other hand, the side of length v of C lies on XY, then by the same argument we obtain $(w-u)(\delta-v) \leqslant uv$. In this case we write

(1) $$(w-v)(\delta-u) = (w-u)(\delta-v) + (w-\delta)(v-u).$$

Since $u \leqslant v$ and $w < \delta$ for triangles, we have

$$(w-v)(\delta-u) < (w-u)(\delta-v) \leqslant uv.$$

Hence for circumscribed triangles K, $(w-v)(\delta-u) \leqslant uv$ with equality when and only when the side of C of length u lies on the edge of K with length δ. ▯

From Lemma 1, we deduce that if K is a maximal set, then $(w-v)(\delta-u) \geqslant uv$.

Lemma 2. *Let $ABCD$ be a fundamental rectangular cell of $\Lambda_R(u,v)$ labelled in an anticlockwise direction. Let $\triangle$ be a triangle determined by the lines a, b and c with points A, B and C interior to the edges of $\triangle$ and point D exterior to $\triangle$. Further, let line c containing an edge of $\triangle$ intercept the closed line segment AD. Then $(w(\triangle)-v)(d(\triangle)-u) < uv$.*

Proof: Let $b.c = P$, $a.c = Q$ and $a.b = R$. By a suitable rotation of the plane together with a reflection of the set $\triangle$ in the mediator of the segment AB, if necessary, we may assume that $m_b > m_c \geqslant 0$ (see Figure 5).

Suppose first that $\angle Q \leqslant \pi/2$. Let c make an acute angle $\theta(\neq 0)$ with the line CD. Let V be a point on QR with BV parallel to PQ. Then $BV < AB$ and BV is distant $BC\cos\theta < BC$ from PQ. We rotate $\triangle$ about B until PQ is parallel to CD. Let the rotated triangle be $\triangle'$. Clearly $\triangle'$ contains no lattice point in its interior and B is the only lattice point on the boundary of $\triangle'$. Hence $\triangle'$ may be enlarged to a triangle $\triangle^*$ inscribing the rectangle $ABCD$. Using Lemma 1,

$$(w(\triangle)-v)(d(\triangle)-u) < (w(\triangle^*)-v)(d(\triangle^*)-u) \leqslant uv. \tag{2}$$

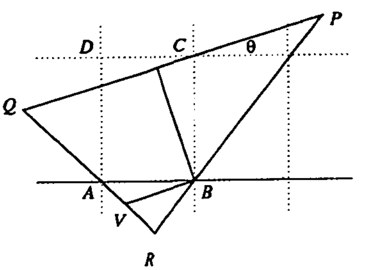

Figure 5.

Now suppose that $\angle Q > \pi/2$. We consider the following two cases:

Case (i): Q lies in the closed rectangle $ABCD$. We show that

$$(w(\triangle)-v)(d(\triangle)-u) < uv.$$

We first inscribe a rectangle $R_\triangle$ in $\triangle$ with side lengths $u' < u$ and $v' = v$ as follows: Let b' be a line parallel to b and distant v from b. Since $w > v$, b' intersects $\triangle$ in a line segment $M'N'$ of length $s > 0$ (see Figure 6).

Let M and N be the feet of the perpendiculars from M' and N' to the line b and let $R_\triangle$ be the rectangle with vertices M, N, N' and M'. We shall show that $s < u$. Let b' intersect the lines CD and AD in the points Z and Y respectively. Clearly $s < YZ$.

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We now consider the following two subcases:

(a) If AB has length u and BC has length v, we take the coordinates of B, Z and Y to be $(u,0)$, (x,v) and $(0,y)$ respectively. Hence

$$\text{Area of } \triangle BZY = \frac{1}{2}v.ZY = \frac{1}{2}\begin{vmatrix} u & 0 & 1 \\ x & v & 1 \\ 0 & y & 1 \end{vmatrix},$$

that is,

$$ZY = u + (x-u)\frac{y}{v}.$$

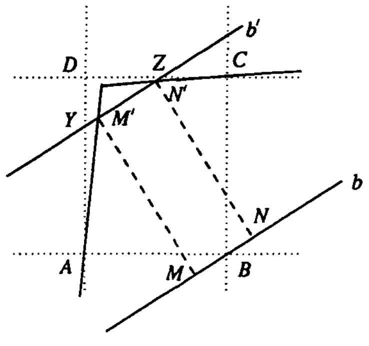

Figure 6.

Now since $x < u$, we have $ZY < u$. We now rotate $R_\triangle$ so that the edge of $R_\triangle$ of length s lies on the edge of $ABCD$ of length u and $R_\triangle$ is contained in the closed rectangle $ABCD$. The same rotation transforms $\triangle$ to $\triangle'$ say. Clearly $\triangle'$ contains no interior lattice points and since $s < u$, at least one of C and D lies in the exterior of $\triangle'$. Hence $\triangle'$ may be enlarged to a triangle $\triangle^*$ inscribing the rectangle $ABCD$, and (2) applies immediately.

(b) If now AB has length v and BC has length u, we inscribe a rectangle in $\triangle$ with side lengths $u' = s$ and $v' = v$ as described above. We now let the coordinates of B, Z and Y be $(v,0)$, (x,u) and $(0,y)$ respectively. Noting that $x < v$, we obtain

$$ZY = u + (x-v)\frac{y}{v} < u.$$

By the rotation argument above, we again obtain (2).

Case (ii): Q lies exterior to the closed rectangle $ABCD$. Let a make an acute angle $\varphi(\neq 0)$ with the line AD. Let T be the point on PQ with BT parallel to QR. Now $BT < BC$ and BT is distant $AB\cos\varphi < AB$ from QR. We rotate $\triangle$ clockwise about B until BT lies on the edge BC. Let the transformed triangle $\triangle'$ have vertices

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P', Q' and R' corresponding to points P, Q and R respectively. Then clearly $Q'R'$ is parallel to AD. We note also that points A and C are exterior to $\triangle P'Q'R'$. We can now construct a triangle Δ'' with vertices P'', Q'', R'' such that line $P''Q''$ is parallel to $P'Q'$ and contains the point C, line $Q''R''$ is coincident with line AD and line $R''P''$ is coincident with $R'P'$. Clearly $\triangle P''Q''R''$ is a triangle of the type described in Case (i). Hence

$$\begin{aligned}(w(\Delta) - v)(d(\Delta) - u) &= (w(\Delta') - v)(d(\Delta') - u) \\ &< (w(\Delta'') - v)(d(\Delta'') - u) \\ &< uv.\end{aligned}$$

This completes the proof of Lemma 2. □

Suppose now that K is contained in a triangle satisfying the conditions of Lemma 2. Since $K \subseteq \Delta$, $w(K) \leqslant w(\Delta)$ and $d(K) \leqslant d(\Delta)$. From Lemma 2, it follows that K is not maximal.

Henceforth we shall use the shorthand notation $L2(a, b, c)$ to mean:

> K is contained in a triangle determined by the lines a, b, c satisfying the conditions of Lemma 2. Hence K is not maximal.

Lemma 3. *Let $ABCD$ be a rectangular cell of $\Lambda_R(u, v)$ labelled anticlockwise and let Q be a proper convex quadrilateral determined by lines a, b, c, d, with A, B, C and D interior to the edges of Q on a, b, c and d respectively. Then amongst all convex sets containing no interior lattice points, a set K contained in Q can not be maximal.*

Proof: Since $K \subseteq Q$, it suffices to show that Q is not maximal. Let $a.b = X$, $b.c = Y$, $c.d = Z$ and $d.a = W$ (Figure 7).

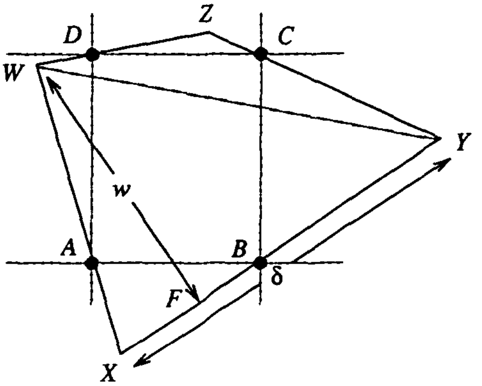

Figure 7.

We now recall that the diameter of a polygonal set is the maximum distance between a pair of vertices of the polygon. Suppose first that δ is the length of an edge, XY say, of Q. Without loss of generality, suppose that W is the vertex of Q furthest from b. Then $w \leqslant d(W,b)$. Let $\triangle$ be the triangle XYW. Clearly $d(\triangle) = XY$ and so $w(\triangle) = d(W,b)$ and $w \leqslant w(\triangle)$. But since $\triangle \subset Q$, $w(\triangle) \leqslant w$. Hence $w = w(\triangle) = d(W,b)$. Since $\triangle$ and Q have the same width and diameter, it suffices to show that $\triangle$ is not maximal. Noting that the edge WY contains no lattice points, $\triangle$ may be enlarged about the point X to $\triangle' = \triangle W'XY'$ where $W'Y'$ contains the point D. By a simple variant of Lemma 2,

$$(w(\triangle) - v)(d(\triangle) - v) < (w(\triangle') - v)(d(\triangle') - u) < uv.$$

Hence $\triangle$ (and so Q) is not maximal.

We now suppose that δ is the length of a diagonal of Q, WY say. Let t be the width of Q in a direction perpendicular to WY (see Figure 8). Since the (minimal) width of Q occurs in a direction perpendicular to an edge of Q (see for example [1]), we have $w < t$. Let WY make an acute angle θ with CD and let XZ intersect WY in the point O. Now the area of Q is $(1/2)t\delta$. This area is also obtained by adding the areas of the quadrilaterals $ODWA$, $OBYC$ to $OCZD$, $OAXB$.

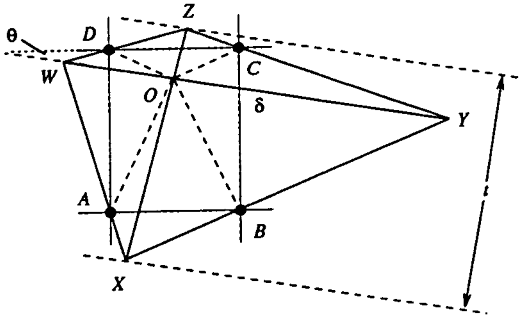

Figure 8.

Suppose first that AB has length u and BC has length v. Then we have

$$\frac{1}{2}t\delta = \frac{1}{2}v\delta\cos\theta + \frac{1}{2}ut\cos\theta.$$

Hence

$$t\delta = (tu + \delta v)\cos\theta \leqslant tu + \delta v.$$

Adding uv to both sides of the inequality and factorising, we have

$$(t - v)(\delta - u) \leqslant uv.$$

Since $w < t$, we have

$$(w-v)(\delta-u) < uv.$$

Hence Q is not maximal.

Now suppose that AB has length v and BC has length u. Repeating the above argument, we obtain the corresponding inequality

$$(w-u)(\delta-v) < uv.$$

By (1), $(w-v)(\delta-u) < uv$. So again, Q is not maximal. $\square$

3. Proof of Theorem 3

We now assume that K is a maximal set. We may assume that $\delta \geqslant w > v \geqslant u$. Let the radius of the largest circle inscribed in K be r. It is shown in [**2**] that for any convex set K,

$$(w-2r)\delta \leqslant 2\sqrt{3}r^2.$$

If $r \leqslant u/2 \leqslant v/2$, then

$$(w-v)(\delta-u) < (w-v)\delta \leqslant (w-2r)\delta \leqslant 2\sqrt{3}r^2 \leqslant 2\sqrt{3}.\frac{u}{2}.\frac{v}{2} = \frac{\sqrt{3}}{2}uv < uv.$$

Hence K is not maximal. We may therefore assume that K contains a disk $\mathcal{D}$ of radius $r > u/2$.

By translating K through a suitable lattice vector, we may bring the centre of $\mathcal{D}$ to lie in $0 < y < v$. For easier reference, we list the properties of $\mathcal{D}$ as follows:

D1. $r > u/2$.

D2. The centre of $\mathcal{D}$ lies in $0 < y < v$.

Since $w > v$, K^o intercepts one of $y = 0$ and $y = v$. Without loss of generality, we may assume that K^o intercepts $y = 0$. Since K^o contains no point of $\Lambda_R(u,v)$, we may assume that K^o intercepts $y = 0$ between two adjacent lattice points. By translating through a suitable lattice vector we may take these points to be $E(0,0)$ and $F(u,0)$. Let G and H be the points (u,v) and $(0,v)$ respectively. We shall show that K is a triangle with diameter δ and width $w = \delta v/(\delta-u)$ (see for example Figure 2).

From D1 and D2, K^o must intercept one of the edges EH and FG. Without losing generality, we may assume that K^o intercepts FG. Hence K lies above a line f with $m_f > 0$. We now consider the following two cases:

Case 1: K is bounded by $y = v$. By D1 and D2, lines e and f intersect in the halfplane $y < 0$ and K is contained in the triangle $\triangle$ determined by the lines e, f and $y = v$. Since K^o intercepts EF, $m_e \neq 0$. If $m_e > 0$, then H is exterior to $\triangle$

and $L2(e,f,g)$. We may now assume that $m_e < 0$ (possibly infinite). In this case, $\triangle$ circumscribes the rectangular cell $EFGH$. By Lemma 1, K is maximal when K is the triangle bounded by $y = v$ and the lines e and f with $m_e < 0$ (possibly infinite) and $m_f > 0$, and having diameter on the line $y = v$.

CASE 2: K crosses the line $y = v$. We again show that K is not maximal. Suppose that K crosses the line $y = v$ between the adjacent lattice points X and Y on the line $y = v$. Without losing generality, we may assume that X and Y are the points (ku, v) and $((k+1)u, v)$ respectively where $k \geqslant 0$. If $k = 0$, then $X = H$ and $Y = G$ and we have $m_g < 0$ and $m_h \neq 0$. If $m_h > 0$ and $m_e < 0$, then K is contained in a proper convex quadrilateral Q, and by Lemma 3, K is not maximal. If $m_h < 0$ then $L2(f,g,h)$ or if $m_e > 0$ then $L2(f,g,e)$. Finally, if h has infinite slope, K is contained in a triangle circumscribing the rectangle $EFGH$ with the edge EH of length v on $x = 0$. By Lemma 1, K is not maximal.

We may therefore assume that $XY \neq GH$. The set K is therefore bounded by lines x and y with $m_x > 0$. By D1 and D2, e and f intersect in the halfplane $y < 0$ and x and y intersect in the halfplane $y > v$. If $m_f > m_x > 0$, K is contained in a triangle $\triangle$ determined by lines e, f and x. Let g_f denote the line containing G and parallel to f and let π_H be the open half plane bounded by g_f and containing the point H. Since $w(\triangle) > v > d(G,f)$, e and x intersect in a point Q lying in the intersection of the half planes $y \leqslant v$ and π_H. It follows that K is also contained in a triangle $\triangle'$ determined by lines e, f and g_x where g_x is a line containing G and parallel to x. Hence $L2(e,f,g_x)$. If, on the other hand, $m_x > m_f > 0$, then by a similar argument, K is contained in a triangle determined by the lines x, y and w_f where w_f is the line containing the point $W(ku, 0)$ and parallel to f. Hence $L2(y,x,w_f)$.

This completes the proof of Theorem 3. $\square$

4. Proof of Theorem 4

Let K now be a set satisfying the conditions of Theorem 4. We may assume that the origin O is one of the lattice points. Let $L(z_1, z_2)$ denote the other lattice point contained in K^o. Without loss of generality, we may assume that $z_1 \geqslant 0$ and $z_2 \geqslant 0$. By a reflection about the line $y = x$ if necessary, it suffices to consider the cases for which $z_1 \geqslant z_2$. Since K^o contains no other lattice points, the open line segment OL contains no lattice point. Hence we may assume that z_1 and z_2 are relatively prime.

If z_1 and z_2 are both odd, we consider the sublattice

$$\Gamma' = \{(x,y) : x + y \equiv 1 \pmod 2\}.$$

Clearly $O \notin \Gamma'$, $L \notin \Gamma'$ and K^o contains no point of Γ'. By Theorem 3, we have

$$\left(w - \sqrt{2}\right)\left(\delta - \sqrt{2}\right) \leqslant 2.$$

However,

$$\begin{aligned}(w-2)(\delta-1)-\left(w-\sqrt{2}\right)\left(\delta-\sqrt{2}\right)&=w\left(\sqrt{2}-1\right)+\delta\left(\sqrt{2}-2\right)\\&\leqslant\delta\left(\sqrt{2}-1\right)+\delta\left(\sqrt{2}-2\right)\\&=\delta\left(2\sqrt{2}-3\right)<0.\end{aligned}$$

It follows that $(w-2)(\delta-1)<\left(w-\sqrt{2}\right)\left(\delta-\sqrt{2}\right)\leqslant 2$. Hence K is not maximal.

If say, z_1 is odd and z_2 is even, we consider the sublattice

$$\Gamma'=\{(x,y): x=n,\ y=2m+1,\ m,n\in Z\}.$$

Clearly $O\notin\Gamma'$, $L\notin\Gamma'$ and K^o contains no point of Γ'. By Theorem 3, we have

$$(w-2)(\delta-1)\leqslant 2.$$

Equality occurs when and only when K is a triangle with diameter δ and width $w=2\delta/(\delta-1)$ as shown in Figure 3. ∎

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