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A SPECIAL CASE OF THE VANISHING OF A (G, σ)-PRODUCT IN A (G, σ)-SPACE

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In [1] we constructed a (G, σ)-space and determined a condition which is both necessary and sufficient for the (G, σ)-product of the vectors $v_1, v_2, \ldots, v_n$ to be zero. The purpose of the present paper is to give a criterion for $v_1 \triangle v_2 \triangle \cdots \triangle v_n$ to be zero, in the particular case when V is a unitary space and the group G belongs to a special class of groups which we shall define below. As a result, we get a criterion which is very simple to determine whether $v_1 \triangle v_2 \triangle \cdots \triangle v_n$ is zero or not. We repeat some definitions and results of [1] in order to make this paper self-contained.

1. Let G be a permutation group on the set $I=\{1, 2, 3, \ldots, n\}$, F an arbitrary field, σ a linear character of G into F^*, the multiplicative group of the field F. Consider the Cartesian product $W=V\times V\times \cdots \times V$ (n copies), where V is an m-dimensional vector space over F.

1.1 Definition. A mapping $f\colon W \to U$, where U is any vector space over F, is called (G, σ) iff

$$(w_1, w_2, \ldots, w_n)f = \sigma(g)(w_{g(1)}, w_{g(2)}, \ldots, w_{g(n)})f$$

for all $g \in G$, $w_i \in V$, and $i \in I$.

1.2 Definition. An element $(w_1, w_2, \ldots, w_n) \in W$ is called a (G, σ) element iff $\exists\, g \in G$ such that $\sigma(g) \neq 1$ and w_i, $w_{g(i)}$ are linearly dependent for all $i \in I$.

1.3 Definition. A vector space T over F is called a (G, σ)-space of W, iff $\exists$ a mapping τ on W into T such that

(i) τ is multilinear and (G, σ).

(ii) T has a "Universal mapping property", i.e. if U is any vector space over F and f is any multilinear and (G, σ) mapping of W into U, then $\exists$ a unique linear transformation $\bar{f}$ of T into U, such that $\tau\bar{f}=f$.

In [1] we have shown that, given G, σ, and W, there exists a (G, σ)-space which is unique up to isomorphism.

1.4 *Notation.* If $(w_1, w_2, \ldots, w_n) \in W$, we denote its image $(w_1, w_2, \ldots, w_n)\tau$ under τ by $w_1 \triangle w_2 \triangle \cdots \triangle w_n$ and call it the (G, σ)-product of the vectors $w_1, w_2, \ldots, w_n$.

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1.5 Definition. An element $(w_1, w_2, \ldots, w_n) \in W$ is called a trivial element iff $w_i = 0$ for some i. Otherwise it is called nontrivial.

1.6 Remark. If $(w_1, w_2, \ldots, w_n) \in W$ is a trivial element, then since τ is multilinear, we have $w_1 \triangle w_2 \triangle \cdots \triangle w_n = 0$. Thus we shall assume from here on that $(w_1, w_2, \ldots, w_n)$ is a nontrivial element of W.

We have the following sufficient condition for $v_1 \triangle v_2 \triangle \cdots \triangle v_n = 0$.

1.7 Theorem. *If $(v_1, v_2, \ldots, v_n) \in W$ is nontrivial and a (G, σ) element, then $v_1 \triangle v_2 \triangle \cdots \triangle v_n = 0$.*

Proof. $(v_1, v_2, \ldots, v_n)$ is a (G, σ) element implies there exists $g \in G$ such that $\sigma(g) \neq 1$ and v_i, $v_{g(i)}$ are linearly dependent for all $i \in I$. Let $v_{g(i)} = \lambda_{g(i)} v_i$, where $\lambda_{g(i)} \in F$ for all $i \in I$. We shall first show that $\lambda_{g(1)}\lambda_{g(2)} \ldots \lambda_{g(n)} = 1$. Let $g = C_1 C_2 \ldots C_k$ be the cyclic decomposition of g, which also includes the cycles of length one, if any. Let $D_i = \operatorname{dom} C_i$, $i = 1, 2, \ldots, k$. Then $I = \bigcup_{i=1}^{k} D_i$ and if $i, j \in I$, $i \neq j$, then $D_i \cap D_j = \varnothing$. Let $C_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i})$, where $n_i \geq 1$ is the length of the cycle C_i, $i = 1, 2, \ldots, k$. Then $n_1 + n_2 + \cdots + n_k = n$.

$$\begin{aligned} v_{\alpha_{i,n_i}} &= v_{g(\alpha_{i,n_i-1})} = \lambda_{g(\alpha_{i,n_i-1})} v_{\alpha_{i,n_i-1}} = \cdots = \cdots \\ &= \lambda_{g(\alpha_{i,n_i-1})} \cdots \lambda_{g(\alpha_{i,1})} \lambda_{g(\alpha_{i,n_i})} v_{\alpha_{i,n_i}}. \end{aligned}$$

Hence $\prod_{\alpha \in D_i} \lambda_{g(\alpha)} = 1$, and since i is arbitrary, we have

$$\prod_{\alpha \in D} \lambda_{g(\alpha)} = \prod_{i=1}^{k} \prod_{\alpha \in D_i} \lambda_{g(\alpha)} = 1.$$

Thus,

$$\begin{aligned} v_1 \triangle v_2 \triangle \cdots \triangle v_n &= (v_1, v_2, \ldots, v_n)\tau \\ &= \sigma(g)(v_{g(1)}, v_{g(2)}, \ldots, v_{g(n)})\tau \\ &= \sigma(g)(\lambda_{g(1)} v_1, \lambda_{g(2)} v_2, \ldots, \lambda_{g(n)} v_{g(n)})\tau \\ &= \sigma(g) \prod_{\alpha \in D} \lambda_{g(\alpha)} (v_1, v_2, \ldots, v_n)\tau \\ &= \sigma(g) v_1 \triangle v_2 \triangle \cdots \triangle v_n \end{aligned}$$

and since $\sigma(g) \neq 1$, we have $v_1 \triangle v_2 \triangle \cdots \triangle v_n = 0$.

1.8 Remark. The converse of Theorem 1.7 is false; for take $G = S_3$, and $\sigma : G \to F^*$, defined by

$$\sigma(g) = \begin{cases} 1 & \text{if } g \text{ is an even permutation,} \\ -1 & \text{if } g \text{ is an odd permutation.} \end{cases}$$

Then the (G, σ) space in this case is the Grassman space $\bigwedge^3 V$. Clearly $(v_1, v_2, v_1 + v_2) \in W$ is not a (G, σ) element, but it is well known in the theory of Grassman space that since $v_1, v_2, v_1 + v_2$ are linearly dependent, we have

$$v_1 \triangle v_2 \triangle (v_1 + v_2) = v_1 \wedge v_2 \wedge (v_1 + v_2) = 0.$$

However if V is a unitary space and G belongs to a certain class of groups $\mathbf{G}$, which we shall define below, then the converse of the Theorem 1.7 is also true.

2. Particularizing V and G. Let G be a subgroup of S_n, the symmetric group of degree n. If T is an orbit of G, let g^T denote the restriction of g to T. Let $G^T=\{g^T \mid g \in G\}$. Then G^T is a subgroup of S_T, the symmetric group on T. Let $\mathbf{G}=\{G \mid G$ is a subgroup of S_n and if T is any orbit of G, then G^T is cyclic$\}$. Clearly $\mathbf{G}$ contains every cyclic group. As to the other members of $\mathbf{G}$, they are all abelian.

Let $G \in \mathbf{G}$ and $W=V\times V\times \cdots \times V$ (n copies), where V is a unitary space of dimension m. Let σ be any linear character of G and consider the (G, σ)-space of W.

2.1 Definition. If $v=(v_1, v_2, \ldots, v_n) \in W$, then a mapping $\gamma\colon I \to I$ is called an indicator of v iff $\gamma_i=\gamma_j$, where $\gamma_i=\gamma(i)$, when and only when v_i and v_j are linearly dependent.

2.2 Definition. If γ is an indicator of v, then we define $G_\gamma=\{g \mid g \in G, \gamma_i=\gamma_{g(i)}$ for all $i \in I\}$. It is proved in [2, Theorem 5, p. 4], that $v_1 \triangle v_2 \triangle \cdots \triangle v_n=0$ iff $\sum_{g \in G_\gamma} \sigma(g)=0$, for any indicator γ of v.

2.3 Theorem. *With G and W as defined in* §2, $v_1 \triangle v_2 \triangle \cdots \triangle v_n=0$ *iff* $(v_1, v_2, \ldots, v_n)$ *is a* (G, σ) *element.*

Proof. ($\Leftarrow$) It is a particular case of Theorem 1.7. ($\Rightarrow$) Let γ be any indicator of $v=(v_1, v_2, \ldots, v_n)$. Then $\sum_{g \in G} \sigma(g)=0$. This implies that there exists $g \in G$ such that $\sigma(g)\neq 1$. Also $g \in G_\gamma$ implies $\gamma_i=g(i)$ for all $i \in I$ and this implies that v_i and $v_{g(i)}$ are linearly dependent for all $i \in I$. Thus there exists $g \in G$ such that $\sigma(g)\neq 1$ and v_i and $v_{g(i)}$ are linearly dependent for all $i \in I$, i.e. $(v_1, v_2, \ldots, v_n)$ is a (G, σ)-element.

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