Proceedings of the Edinburgh Mathematical Society (2010) **53**, 393–413 ©
DOI:10.1017/S0013091508001004 Printed in the United Kingdom

HOMOGENEOUS KÄHLER AND SASAKIAN STRUCTURES RELATED TO COMPLEX HYPERBOLIC SPACES

P. M. GADEA[1] AND J. A. OUBIÑA[2]

[1]*Instituto de Física Fundamental, CSIC, Serrano 113-bis, 28006 Madrid, Spain* (pmgadea@iec.csic.es)

[2]*Departamento de Xeometría e Topoloxía, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain* (ja.oubina@usc.es)

(Received 7 October 2008)

Abstract We study homogeneous Kähler structures on a non-compact Hermitian symmetric space and their lifts to homogeneous Sasakian structures on the total space of a principal line bundle over it, and we analyse the case of the complex hyperbolic space.

Keywords: homogeneous Riemannian structures; homogeneous Kähler structures; homogeneous almost-contact Riemannian manifolds; non-compact Hermitian symmetric spaces; Sasakian spaces

2010 *Mathematics subject classification:* Primary 53C30
Secondary 53C25; 53C35; 53C55

1. Introduction

The general theory of homogeneous Kähler manifolds is well known, as is the relation between homogeneous symplectic and homogeneous contact manifolds (see, for example, [**6**, **10**, **11**]).

As is also widely known, a connected, simply connected and complete Riemannian manifold is a symmetric space if and only if its curvature tensor field is parallel. Ambrose and Singer [**2**] extended this result to obtain a characterization of homogeneous Riemannian manifolds in terms of the existence of a tensor field S of type $(1,2)$ on the manifold, called a homogeneous Riemannian structure (see [**28**], where a classification of such structures is also given), satisfying certain properties (see (2.1); if $S = 0$, one has the symmetric case). Moreover, Sekigawa [**26**] obtained the corresponding result for almost-Hermitian manifolds, defining homogeneous almost-Hermitian structures (among them the homogeneous Kähler structures), which were classified in [**1**]. Its odd-dimensional version, the almost-contact-metric case, has also been studied (see, for example, [**8**, **12**, **15**, **21**]).

In § 2, we give basic results about homogeneous Riemannian and homogeneous Kähler structures. In particular, we consider these structures on Hermitian symmetric spaces of non-compact type. Besides the trivial homogeneous structure $S = 0$ associated to the description of one such space as a symmetric space, other structures can be obtained associated to other descriptions as a homogeneous space and, in particular, to its description as a solvable Lie group given by an Iwasawa decomposition (see § 2.2). We also give a construction of homogeneous Sasakian structures on the bundle space of a principal line bundle over a Hermitian symmetric space of non-compact type, endowed with a connection 1-form that is the contact form of a Sasakian structure on the total space (Proposition 2.5).

The complex hyperbolic space $\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ with the Bergman metric is an irreducible Hermitian symmetric space of non-compact type, and, up to homotheties, is the simply connected complete complex space form of negative curvature. It has been characterized in [**14**] in terms of the existence of certain type of homogeneous Kähler structure on it, and in [**7**] a Lie-theoretical description of its homogeneous structure of linear type is found. From an alternate point of view, in § 3 we study the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)$, which, in particular, provide an infinite number of descriptions of $\mathbb{C}\mathrm{H}(n)$ as non-isomorphic solvable Lie groups. Moreover, we consider the principal line bundle over $\mathbb{C}\mathrm{H}(n)$, with its Sasakian structure given in a natural way from a connection form on the bundle, and we obtain the families of homogeneous Sasakian structures on its bundle space following our previous general construction. In summary, we obtain the following.

(a) All the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n) \equiv AN$: these are given in terms of some 1-forms related by a system of differential equations on the solvable Lie group AN (Theorem 3.1).

(b) The explicit description of a multi-parametric family of homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)$, given by using the generators of $\mathfrak{a} + \mathfrak{n}$ (Proposition 3.6), and the corresponding subgroups of the full isometry group $\mathrm{SU}(n,1)$ of AN (Theorem 3.7).

(c) The explicit description of a one-parametric family of homogeneous Sasakian structures on the bundle space of the line bundle $\bar{M} \to \mathbb{C}\mathrm{H}(n)$, given in terms of the horizontal lifts of the generators of $\mathfrak{a} + \mathfrak{n}$ and the fundamental vector field ξ on $\bar{M}$ (Proposition 3.9), and their associated reductive decompositions (Propositions 3.11 and 3.12). One of them describes $\bar{M}$ as the complete simply connected φ-symmetric Sasakian space $\widetilde{\mathrm{SU}}(n,1)/\,\mathrm{SU}(n)$, which is also a Sasakian space form.

On the other hand, complex hyperbolic space was the first target space-time where Nishino's [**22**] alternative (i.e. neither necessarily hyper-Kähler nor quaternion-Kähler) $N = (4,0)$ superstring theory proved to work. This model has some interesting features, among them not having the incompatibility (which is a trait common to heterotic σ-models) between the torsion tensor and quaternion-Kähler manifolds found by de Wit and van Nieuwenhuizen [**9**]. Another peculiarity is that, in this case, one of the two scalars of the relevant global multiplet is promoted to coordinates on $\mathbb{C}\mathrm{H}(n)$, while the other plays the role of a tangent vector under the holonomy group $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$.

2. Homogeneous Riemannian structures

Ambrose and Singer [**2**] proved that a connected, simply connected and complete Riemannian manifold is homogeneous if and only if there exists a tensor field S of type $(1,2)$ on M such that the connection $\tilde{\nabla} = \nabla - S$ satisfies the following equations:

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} S = 0, \tag{2.1}$$

where ∇ is the Levi-Cività connection of g and R is its curvature tensor field, for which we adopt the conventions

$$R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z, \qquad R_{XYZW} = g(R_{XY}Z, W).$$

Such a tensor field S is called a homogeneous Riemannian structure [**28**]. We also denote by S the associated tensor field of type $(0,3)$ on M defined by $S_{XYZ} = g(S_X Y, Z)$.

2.1. Homogeneous Kähler structures

An almost-Hermitian manifold (M, g, J) is said to be a homogeneous almost-Hermitian manifold if there exists a Lie group of holomorphic isometries which acts transitively and effectively on M. Sekigawa proved the following theorem.

Theorem 2.1 (Sekigawa [26]). *A connected, simply connected and complete almost-Hermitian manifold (M, g, J) is homogeneous if and only if there is a tensor field S of type $(1,2)$ on M which satisfies Equations (2.1) and $\tilde{\nabla} J = 0$.*

A tensor S satisfying the Equations (2.1) and $\tilde{\nabla} J = 0$ is called a homogeneous almost-Hermitian structure. The almost-Hermitian manifold (M, g, J) is Kähler if and only if J is integrable and the fundamental 2-form Ω on M, given by $\Omega(X, Y) = g(X, JY)$, is closed, or equivalently $\nabla J = 0$. In this case, a homogeneous almost-Hermitian structure is also called a homogeneous Kähler structure, and we have the following proposition.

Proposition 2.2. *A homogeneous Riemannian structure S on a Kähler manifold (M, g, J) is a homogeneous Kähler structure if and only if $S \cdot J = 0$ or, equivalently, $S_{XYZ} = S_{XJYJZ}$ for all the vector fields X, Y, Z on M.*

Corollary 2.3. *A connected, simply connected and complete Kähler manifold (M, g, J) is a homogeneous Kähler manifold if and only if there exists a homogeneous Kähler structure on M.*

If $(M = G/H, g)$ is a homogeneous Riemannian manifold, where G is a connected Lie group acting transitively and effectively on M as a group of isometries and H is the isotropy group at a point $o \in M$, then the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector-space direct sum $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. If G is connected and M is simply connected, then H is connected, and the condition $\mathrm{Ad}(H)\,\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The vector space $\mathfrak{m}$ is identified with $T_o(M)$ by the isomorphism $X \in \mathfrak{m} \to X^*_o \in T_o(M)$, where X^* is the Killing vector field on M generated by the one-parameter subgroup $\{\exp tX\}$ of G

acting on M. If $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we write $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$, $X_{\mathfrak{h}} \in \mathfrak{h}$, $X_{\mathfrak{m}} \in \mathfrak{m}$. The canonical connection $\tilde{\nabla}$ of $M = G/H$ (with regard to the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$) is determined by

$$(\tilde{\nabla}_{X^*} Y^*)_o = [X^*, Y^*]_o = -[X, Y]^*_o = -([X, Y]_{\mathfrak{m}})^*_o, \quad X, Y \in \mathfrak{m}. \tag{2.2}$$

Then $S = \nabla - \tilde{\nabla}$ satisfies the Ambrose–Singer Equations (2.1), and it is the homogeneous Riemannian structure associated to the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. If (M, g) is endowed with a compatible almost-complex structure J invariant by G (so that $(M = G/H, g, J)$ is a homogeneous almost-Hermitian manifold), restricting J to $T_o(M) \equiv \mathfrak{m}$, we obtain a linear endomorphism J_o of $\mathfrak{m}$ such that $J_o^2 = -1$, and $J_o \operatorname{ad}_{\mathfrak{h}} = \operatorname{ad}_{\mathfrak{h}} J_o$. Moreover, J is integrable if and only if

$$[J_o X, J_o Y]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - J_o[X, J_o Y]_{\mathfrak{m}} - J_o[J_o X, Y]_{\mathfrak{m}} = 0$$

for all $X, Y \in \mathfrak{m}$ (see [**20**, Chapter 10, Proposition 6.5]).

Conversely, suppose that (M, g) is a connected, simply connected and complete Riemannian manifold, and let S be a homogeneous Riemannian structure on (M, g). We set $\mathfrak{m} = T_o(M)$, where $o \in M$. If $\tilde{R}$ is the curvature tensor of the connection $\tilde{\nabla} = \nabla - S$, the holonomy algebra $\tilde{\mathfrak{h}}$ of $\tilde{\nabla}$ is the Lie subalgebra of the Lie algebra of antisymmetric endomorphisms $\mathfrak{so}(\mathfrak{m})$ of $(\mathfrak{m}, g_o)$ generated by the operators $\tilde{R}_{XY}$, where $X, Y \in \mathfrak{m}$. A Lie bracket is defined [**23**] in the vector-space direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \mathfrak{m}$ by

$$\left.\begin{aligned} &[U, V] = UV - VU, && U, V \in \tilde{\mathfrak{h}}, \\ &[U, X] = U(X), && U \in \tilde{\mathfrak{h}}, \quad X \in \mathfrak{m}, \\ &[X, Y] = \tilde{R}_{XY} + S_X Y - S_Y X, && X, Y \in \mathfrak{m}, \end{aligned}\right\} \tag{2.3}$$

and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \mathfrak{m}$ is the reductive decomposition corresponding to the homogeneous Riemannian structure S. Let $\tilde{G}$ be the connected, simply connected Lie group whose Lie algebra is $\tilde{\mathfrak{g}}$ and let $\tilde{H}$ be the connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. Then $\tilde{G}$ acts transitively on M as a group of isometries and M is diffeomorphic to $\tilde{G}/\tilde{H}$. If Γ is the set of the elements of $\tilde{G}$ which act trivially on M, then Γ is a discrete normal subgroup of $\tilde{G}$, and the Lie group $G = \tilde{G}/\Gamma$ acts transitively and effectively on M as a group of isometries, with isotropy group $H = \tilde{H}/\Gamma$. Then M is diffeomorphic to G/H. Now, if J is a compatible almost-complex structure on (M, g) and S is a homogeneous almost-Hermitian structure, then the holonomy algebra $\tilde{\mathfrak{h}}$ is a subalgebra of the Lie algebra $\mathfrak{u}(\mathfrak{m}) = \{A \in \mathfrak{so}(\mathfrak{m}) : A \cdot J = 0\}$ of the unitary group, and $M \approx \tilde{G}/\tilde{H} \approx G/H$ is a homogeneous almost-Hermitian manifold.

2.2. Hermitian symmetric spaces of non-compact type

Suppose that $(M = G/K, g, J)$ is a connected Hermitian symmetric space of non-compact type, where $G = I_0(M)$ is the identity component of the group of (holomorphic) isometries and K is a maximal compact subgroup of G. Then M is simply connected and the Hermitian structure is Kähler. We consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{a} \subset \mathfrak{p}$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is a nilpotent subalgebra. Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M, so M is isometric to AN equipped with the left-invariant Riemannian metric defined by the scalar product $\langle \cdot , \cdot \rangle$, induced on $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{g} / \mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B|_{\mathfrak{p} \times \mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$.

Now, let $\hat{G}$ be a connected closed Lie subgroup of G which acts transitively on M. The isotropy group of this action at $o = K \in M$ is $H = \hat{G} \cap K$. Then $M = G/K$ has also the description $M \equiv \hat{G}/H$, and $o \equiv H \in \hat{G}/H$. Let $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ corresponding to $M \equiv \hat{G}/H$.

We have the isomorphisms of vector spaces

$$\mathfrak{p} \cong \mathfrak{g} / \mathfrak{k} \cong \hat{\mathfrak{g}} / \mathfrak{h} \cong \mathfrak{m} \cong T_o(M) \cong \mathfrak{a} + \mathfrak{n},$$

with

$$\xi : \mathfrak{p} \xrightarrow{\cong} \mathfrak{m}, \qquad \mu : \mathfrak{m} \xrightarrow{\cong} T_o(M), \qquad \zeta : T_o(M) \xrightarrow{\cong} \mathfrak{a} + \mathfrak{n},$$

given by

$$\xi^{-1}(Z) = Z_{\mathfrak{p}}, \quad \mu(Z) = Z^*_o, \quad \zeta^{-1}(X) = X^*_o, \quad Z \in \mathfrak{m},\ X \in \mathfrak{a} + \mathfrak{n}.$$

For each $X \in \mathfrak{g}$, we have $(X_{\mathfrak{k}})^*_o = 0$ and $(\nabla (X_{\mathfrak{p}})^*)_o = 0$, and since the Levi-Cività connection ∇ has no torsion, for each $X, Y \in \mathfrak{g}$, we have

$$(\nabla_{X^*} Y^*)_o = (\nabla_{(X_{\mathfrak{p}})^*} (Y_{\mathfrak{k}})^*)_o = [(X_{\mathfrak{p}})^*, (Y_{\mathfrak{k}})^*]_o = -[X_{\mathfrak{p}}, Y_{\mathfrak{k}}]^*_o. \tag{2.4}$$

The reductive decomposition $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ defines the homogeneous Riemannian structure $S = \nabla - \tilde{\nabla}$, where $\tilde{\nabla}$ is the canonical connection of $M \equiv \hat{G}/H$ with respect to $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$, which is $\hat{G}$-invariant and uniquely determined by $(\tilde{\nabla}_{X^*} Y^*)_o = -[X, Y]^*_o$, for $X, Y \in \mathfrak{m}$ (see (2.2)). The tensor field S is also uniquely determined by its value at o because $M \equiv \hat{G}/H$ and S is $\hat{G}$-invariant. Since J is $\hat{G}$-invariant, from [**20**, Chapter 10, Proposition 2.7], it follows that $\tilde{\nabla} J = 0$ and, by Theorem 2.1, S is a homogeneous Kähler structure.

We have

$$(S_{X^*} Y^*)_o = (\nabla_{X^*} Y^*)_o + [X, Y]^*_o = \nabla_{Y^*_o} X^*, \quad X, Y \in \mathfrak{m}. \tag{2.5}$$

By (2.4) and (2.5), S is given by

$$S_{X^*_o} Y^*_o = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}]^*_o, \quad X, Y \in \mathfrak{m}.$$

Then, for each $X, Y \in \mathfrak{a} + \mathfrak{n}$, we have

$$S_{X^*_o} Y^*_o = S_{\xi(X_{\mathfrak{p}})^*_o} \xi(Y_{\mathfrak{p}})^*_o = [(\xi(X_{\mathfrak{p}}))_{\mathfrak{k}}, Y_{\mathfrak{p}}]^*_o.$$

The complex structure J on $M = G/K$ is defined by an element E_J in the centre of $\mathfrak{k}$, and it defines the complex structure $J \in \operatorname{End}(\mathfrak{a} + \mathfrak{n})$ such that the following diagram

https://doi.org/10.1017/S0013091508001004 Published online by Cambridge University Press

is commutative, and $(\mathfrak{a}+\mathfrak{n}, \langle\cdot\,,\cdot\rangle, J)$ becomes a Hermitian vector space isomorphic to $(T_o(M), g_o, J_o)$:

$$\begin{array}{ccccccc} \mathfrak{p} & \xrightarrow{\xi} & \mathfrak{m} & \xrightarrow{\mu} & T_o(M) & \xrightarrow{\zeta} & \mathfrak{a}+\mathfrak{n} \\ \downarrow{\scriptstyle \mathrm{ad}_{E_J}} & & \downarrow{\scriptstyle J_o} & & \downarrow{\scriptstyle J_o} & & \downarrow{\scriptstyle J} \\ \mathfrak{p} & \xrightarrow{\xi} & \mathfrak{m} & \xrightarrow{\mu} & T_o(M) & \xrightarrow{\zeta} & \mathfrak{a}+\mathfrak{n} \end{array}$$

Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M. Then M is isometric to AN equipped with the left-invariant Riemannian metric defined by the scalar product induced on $\mathfrak{a}+\mathfrak{n} \cong \mathfrak{g}\,/\,\mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B|_{\mathfrak{p}\times\mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$, so that AN equipped with the left-invariant almost-complex structure defined by J is a Kähler manifold.

2.3. Homogeneous almost-contact Riemannian manifolds

An almost-contact structure on a $(2n+1)$-dimensional manifold $\bar{M}$ is a triple (φ, ξ, η), where φ is a tensor field of type $(1,1)$, ξ is a vector field (called the characteristic vector field) and η is a differential 1-form on $\bar{M}$ such that

$$\varphi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Then $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and φ has rank $2n$. If $\bar{g}$ is a Riemannian metric on $\bar{M}$ such that $\bar{g}(\varphi\tilde{X}, \varphi\tilde{Y}) = \bar{g}(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y})$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ on $\bar{M}$, then $(\varphi, \xi, \eta, \bar{g})$ is said to be an almost-contact-metric structure on $\bar{M}$. In this case, $\bar{g}(\tilde{X}, \xi) = \eta(\tilde{X})$. The 2-form Φ on M defined by $\Phi(\tilde{X}, \tilde{Y}) = \bar{g}(\tilde{X}, \varphi\tilde{Y})$ is called the fundamental 2-form of the almost-contact-metric structure $(\varphi, \xi, \eta, \bar{g})$. If $\mathrm{d}\eta(\tilde{X}, \tilde{Y}) = \tilde{X}\eta(\tilde{Y}) - \tilde{Y}\eta(\tilde{X}) - \eta([\tilde{X}, \tilde{Y}]) = 2\Phi(\tilde{X}\tilde{Y})$, then $(\phi, \xi, \eta, \bar{g})$ is called a contact metric (or contact Riemannian) structure; in particular, $\eta \wedge (\mathrm{d}\eta)^n \neq 0$, that is, η is a contact form on $\bar{M}$. If

$$(D_{\tilde{X}}\varphi)\tilde{Y} = \bar{g}(\tilde{X}, \tilde{Y})\xi - \eta(\tilde{Y})\tilde{X}, \tag{2.6}$$

where D is the Levi-Cività connection of $\bar{g}$, then $(\varphi, \xi, \eta, \bar{g})$ is called a Sasakian structure, and the manifold $\bar{M}$ with such a structure is a Sasakian manifold. Sasakian manifolds can also be characterized as normal contact metric manifolds and they are in some sense odd-dimensional analogues of Kähler manifolds [**3**, **4**].

If $(\varphi, \xi, \eta, \bar{g})$ is an almost-contact-metric structure on $\bar{M}$ and $(\bar{M} = \bar{G}/H, \bar{g})$ is a homogeneous Riemannian manifold such that φ is invariant under the action of the connected Lie group $\bar{G}$ (and hence so are ξ and η), then $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ is called a homogeneous almost-contact Riemannian manifold [**8**, **15**, **21**]. Let $\bar{R}$ be the curvature tensor field of the Levi-Cività connection D of $\bar{g}$. Let S be a homogeneous Riemannian structure on $\bar{M}$, that is $\tilde{D}\bar{g} = 0$, $\tilde{D}\bar{R} = 0$ and $\tilde{D}S = 0$, where $\tilde{D} = D - S$. If S satisfies the additional condition $\tilde{D}\varphi = 0$ (and hence $\tilde{D}\xi = 0$ and $\tilde{D}\eta = 0$), then S is called a homogeneous almost-contact-metric structure on $(\bar{M}, \varphi, \xi, \eta, \bar{g})$. From the results of Kiričenko [**18**] on homogeneous Riemannian spaces with invariant tensor structure, we have the following.

https://doi.org/10.1017/S0013091508001004 Published online by Cambridge University Press

Theorem 2.4. *A connected, simply connected and complete almost-contact-metric manifold* $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ *is a homogeneous almost-contact Riemannian manifold if and only if there exists a homogeneous almost-contact-metric structure on* $\bar{M}$*.*

A homogeneous almost-contact-metric structure on a Sasakian manifold will also be called a homogeneous Sasakian structure.

2.4. Principal 1-bundles over almost-Hermitian manifolds

Let (M, g, J) be an almost-Hermitian manifold and let $\bar{M}$ be the bundle space of a principal 1-bundle over M. Let η be a connection (form) on the principal bundle $\pi : \bar{M} \to M$, and let ξ be the fundamental vector field on $\bar{M}$ defined by the element 1 of the Lie algebra $\mathbb{R}$ of the structure group of the bundle. Then $\eta(\xi) = 1$. For each vector field X on M, we denote by X^{H} the horizontal lift of X with respect to η. If $\bar{X}$ is a vector field on $\bar{M}$, its vertical part is $\eta(\bar{X})\xi$. Then, for any vector fields X and Y on M, we have

$$[X^{\mathrm{H}}, Y^{\mathrm{H}}] = [X, Y]^{\mathrm{H}} + \eta([X^{\mathrm{H}}, Y^{\mathrm{H}}])\xi.$$

Moreover, $[X^{\mathrm{H}}, \xi] = 0$, because X^{H} is invariant under the action of the structural group. We define a tensor field φ of type $(1, 1)$ and a Riemannian metric $\bar{g}$ on $\bar{M}$ by

$$\varphi X^{\mathrm{H}} = (JX)^{\mathrm{H}}, \qquad \varphi\xi = 0, \qquad \bar{g} = \pi^* g + \eta \otimes \eta, \tag{2.7}$$

where X and Y are vector fields on M. Clearly, $(\varphi, \xi, \eta, \bar{g})$ is an almost-contact-metric structure on $\bar{M}$, and we have $\bar{g}(X^{\mathrm{H}}, Y^{\mathrm{H}}) = g(X, Y) \circ \pi$ and $\bar{g}(X^{\mathrm{H}}, \xi) = 0$. Let Φ be its 2-fundamental form. If Ω is the fundamental 2-form of the almost-Hermitian manifold (M, g, J), then $\pi^*\Omega = \Phi$.

If ∇ and D are the Levi-Cività connections of g and $\bar{g}$, respectively, then [**24**]

$$D_{X^{\mathrm{H}}} Y^{\mathrm{H}} = (\nabla_X Y)^{\mathrm{H}} + \tfrac{1}{2}\eta([X^{\mathrm{H}}, Y^{\mathrm{H}}])\xi = (\nabla_X Y)^{\mathrm{H}} - \tfrac{1}{2}\mathrm{d}\eta(X^{\mathrm{H}}, Y^{\mathrm{H}})\xi,$$

and $D_{X^{\mathrm{H}}}\xi = D_\xi X^{\mathrm{H}} = -\varphi X^{\mathrm{H}}$. Now, if $2\Phi = \mathrm{d}\eta$, Equation (2.6) is satisfied, as one can easily see by replacing $(\tilde{X}, \tilde{Y})$ by $(X^{\mathrm{H}}, Y^{\mathrm{H}})$, (X^{H}, ξ) and (ξ, Y^{H}), respectively. Then, if the almost-contact-metric structure $(\varphi, \xi, \eta, \bar{g})$ is a contact structure, it is also Sasakian.

Suppose now that the structural group of the principal 1-bundle $\pi : \bar{M} \to M$ is $\mathbb{R}$ and that the base manifold is a $2n$-dimensional connected Hermitian symmetric space of non-compact type $(M = G/K, g, J)$, so that M is isometric to the solvable Lie group AN as in § 2.2. Then M is holomorphically diffeomorphic to a bounded symmetric domain, i.e. to a simply connected open subset of $\mathbb{C}^n$ such that each point is an isolated fixed point of an involutive holomorphic diffeomorphism of itself [**16**, Chapter VIII, Theorem 7.1]. Since $\pi : \bar{M} \to M$ is a principal line bundle over the paracompact manifold M, it admits a global section [**19**, Chapter I, Theorem 5.7], so there exists a diffeomorphism $\bar{M} \to M \times \mathbb{R}$, and the bundle space $\bar{M}$ may be identified with $AN \times \mathbb{R}$, with π being the projection on AN. On the other hand, since the fundamental 2-form Ω associated to the Kähler structure (g, J) is closed, $\Omega = \mathrm{d}\zeta$ for some real analytic 1-form ζ on AN. We consider the connection form $\eta = 2\pi^*\zeta + \mathrm{d}t$ on $\bar{M}$, where t is the coordinate of $\mathbb{R}$. The vertical vector field ξ with $\eta(\xi)=1$ can be identified with $\mathrm{d}/\mathrm{d}t$, and we consider φ and $\bar g$ given by (2.7). Then $2\Phi = 2\pi^*\Omega = 2\pi^*\mathrm{d}\zeta = \mathrm{d}\eta$, and hence $(\varphi,\xi,\eta,\bar g)$ is a Sasakian structure on $\bar M$.

If $\bar S$ is a homogeneous almost-contact-metric structure on $\bar M$, and $\tilde D = D - \bar S$, then $\tilde D\xi = 0$, and hence $\bar S_{X^{\mathrm{H}}}\xi = D_{X^{\mathrm{H}}}\xi = -\varphi X^{\mathrm{H}}$. We have the following proposition.

Proposition 2.5. *Let $(M = G/K, g, J)$ be a connected Hermitian symmetric space of non-compact type. Let $\pi : \bar M \to M$ be a principal line bundle with connection form η such that the almost-contact-metric structure $(\varphi,\xi,\eta,\bar g)$ on $\bar M$ defined by (2.7) is Sasakian.*

(a) *If S is a homogeneous Kähler structure on M, then the tensor field $\bar S$ on $\bar M$ defined by*

$$\bar S_{X^{\mathrm{H}}}Y^{\mathrm{H}} = (S_XY)^{\mathrm{H}} - \bar g(X^{\mathrm{H}},\varphi Y^{\mathrm{H}})\xi, \quad \bar S_{X^{\mathrm{H}}}\xi = -\varphi X^{\mathrm{H}} = \bar S_\xi X^{\mathrm{H}}, \quad \bar S_\xi\xi = 0,$$

for all vector fields X and Y on M, is a homogeneous Sasakian structure on $\bar M$.

(b) *$\{S^t : t\in\mathbb{R}\}$, defined by*

$$\begin{aligned} S^t_{X^{\mathrm{H}}}Y^{\mathrm{H}} &= -\bar g(X^{\mathrm{H}},\varphi Y^{\mathrm{H}})\xi, \qquad & S^t_{X^{\mathrm{H}}}\xi &= -\varphi X^{\mathrm{H}}, \\ S^t_\xi X^{\mathrm{H}} &= -t\varphi X^{\mathrm{H}}, & S^t_\xi\xi &= 0, \end{aligned}$$

is a family of homogeneous Sasakian structures on $\bar M$.

Proof. (a) If $\tilde D = D - \bar S$, then since $\bar S_{X^{\mathrm{H}}Y^{\mathrm{H}}Z^{\mathrm{H}}} = \bar g((S_XY)^{\mathrm{H}}, Z^{\mathrm{H}}) = g(S_XY,Z)\circ\pi = -g(Y,S_XZ)\circ\pi = -\bar g(Y^{\mathrm{H}},(S_XZ)^{\mathrm{H}}) = -\bar S_{X^{\mathrm{H}}Z^{\mathrm{H}}Y^{\mathrm{H}}}$ and $\bar S_{X^{\mathrm{H}}Y^{\mathrm{H}}\xi} = -\bar S_{X^{\mathrm{H}}\xi Y^{\mathrm{H}}}$, the condition $\tilde D\bar g = 0$ is satisfied. On the other hand, if $\tilde\nabla = \nabla - S$, we have

$$\tilde D_{X^{\mathrm{H}}}Y^{\mathrm{H}} = (\tilde\nabla_XY)^{\mathrm{H}}, \qquad \tilde D_{X^{\mathrm{H}}}\xi = \tilde D_\xi X^{\mathrm{H}} = 0. \tag{2.8}$$

We can identify $M = G/K$ with the solvable Lie group AN in an Iwasawa decomposition $G = KAN$ and consider the Lie algebra $\mathfrak{a}+\mathfrak{n}$ of AN. If $\tilde U$, $\tilde V$, $\tilde X$, $\tilde Y$, $\tilde Z$ are horizontal lifts of elements of $\mathfrak{a}+\mathfrak{n}$ or some of them are the vertical vector field ξ, then

$$(\tilde D_{\tilde U}\bar R)_{\tilde X\tilde Y\tilde Z\tilde V} = -\bar R_{\tilde X\tilde Y\tilde Z\tilde D_{\tilde U}\tilde V} + \bar R_{\tilde X\tilde Y\tilde V\tilde D_{\tilde U}\tilde Z} - \bar R_{\tilde Z\tilde V\tilde X\tilde D_{\tilde U}\tilde Y} + \bar R_{\tilde Z\tilde V\tilde Y\tilde D_{\tilde U}\tilde X}, \tag{2.9}$$

since $\tilde U(\bar R_{\tilde X\tilde Y\tilde Z\tilde V}) = 0$. Now, if $X,Y,Z,V\in\mathfrak{a}+\mathfrak{n}$, then

$$\left.\begin{aligned} \bar R_{X^{\mathrm{H}}Y^{\mathrm{H}}Z^{\mathrm{H}}V^{\mathrm{H}}} &= (R_{XYZV} - 2g(X,JY)g(Z,JV) \\ &\qquad + g(X,JV)g(Y,JZ) - g(X,JZ)g(Y,JV))\circ\pi, \\ \bar R_{X^{\mathrm{H}}Y^{\mathrm{H}}Z^{\mathrm{H}}\xi} &= -\bar g([X,Y]^{\mathrm{H}},\varphi Z^{\mathrm{H}}) \\ &\qquad + \bar g((\nabla_XZ)^{\mathrm{H}},\varphi Y^{\mathrm{H}}) - \bar g((\nabla_YZ)^{\mathrm{H}},\varphi X^{\mathrm{H}}), \\ \bar R_{X^{\mathrm{H}}\xi Z^{\mathrm{H}}\xi} &= \bar g(D_{X^{\mathrm{H}}}\xi, D_{Z^{\mathrm{H}}}\xi). \end{aligned}\right\} \tag{2.10}$$

By using (2.8) and (2.10), together with the conditions $\tilde\nabla R = 0$ and $\tilde\nabla J = 0$ for the homogeneous Kähler structure S on M, and the formula

$$\bar R_{\tilde X\tilde Y}\xi = \eta(\tilde X)\tilde Y - \eta(\tilde Y)\tilde X$$

https://doi.org/10.1017/S0013091508001004 Published online by Cambridge University Press

for the Sasakian manifold $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ [**4**, Proposition 7.3], one obtains from (2.9) that $\tilde{D}\bar{R} = 0$. Now,

$$(\tilde{D}_{U^{\mathrm{H}}}\bar{S})_{X^{\mathrm{H}}}Y^{\mathrm{H}} = ((\tilde{\nabla}_U S)_X Y)^{\mathrm{H}}, \quad (\tilde{D}_{U^{\mathrm{H}}}\bar{S})_{X^{\mathrm{H}}}\xi = -((\tilde{\nabla}_U J)X)^{\mathrm{H}} \quad \text{and} \quad \tilde{D}_\xi S = 0;$$

thus $\tilde{D}S = 0$. Moreover, $(\tilde{D}_{X^{\mathrm{H}}}\varphi)Y^{\mathrm{H}} = ((\tilde{\nabla}_X J)Y)^{\mathrm{H}}$ and $(\tilde{D}_{X^{\mathrm{H}}}\varphi)\xi = 0$. Then $\tilde{D}\varphi = 0$, and $\bar{S}$ is a homogeneous Sasakian structure on $\bar{M}$.

(b) If $t = 1$, the corresponding tensor S^1 coincides with $\bar{S}$ in (a) for $S = 0$. For arbitrary t, if $\tilde{D}^t = D - S^t$ we have $\tilde{D}^t_\xi X^{\mathrm{H}} = (t-1)(JX)^{\mathrm{H}}$, and we get $\tilde{D}^t\bar{g} = 0$, $\tilde{D}^t\bar{R} = 0$, $\tilde{D}^t\bar{S}^t = 0$, $\tilde{D}^t\varphi = 0$. □

3. The complex hyperbolic space $\mathbb{C}\mathrm{H}(n)$

3.1. $\mathbb{C}\mathrm{H}(n)$ as a solvable Lie group

The complex hyperbolic space $\mathbb{C}\mathrm{H}(n)$, which may be identified with the unit ball in $\mathbb{C}^n$ endowed with the hyperbolic metric of constant holomorphic sectional curvature -4, may also be viewed as the irreducible Hermitian symmetric space of non-compact type $\mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$.

The Lie algebra $\mathfrak{su}(n,1)$ of $\mathrm{SU}(n,1)$ can be described as the subalgebra of $\mathfrak{sl}(n+1,\mathbb{C})$ of all matrices of the form

$$X = \begin{pmatrix} Z & P^{\mathrm{T}} \\ \bar{P} & \mathrm{i}c \end{pmatrix}, \tag{3.1}$$

where $Z \in \mathfrak{u}(n)$, $c \in \mathbb{R}$ and $P = (p_1, \dots, p_n) \in \mathbb{C}^n$. The involution τ of $\mathfrak{su}(n,1)$ given by $\tau(X) = -\bar{X}^{\mathrm{T}}$ defines the Cartan decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & \mathrm{i}c \end{pmatrix} : \operatorname{tr} Z + \mathrm{i}c = 0 \right\} \cong \mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1)), \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & P^{\mathrm{T}} \\ \bar{P} & 0 \end{pmatrix} \right\}.$$

The element A_0 of $\mathfrak{p}$ defined by $P = (0, \dots, 0, 1)$ generates a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of $\mathfrak{su}(n,1)$. Let f_0 be the linear functional on $\mathfrak{a}$ given by $f_0(A_0) = 1$. If $n > 1$, the set of roots of $(\mathfrak{su}(n,1), \mathfrak{a})$ is $\Sigma = \{\pm f_0, \pm 2f_0\}$, the set $\Pi = \{f_0\}$ is a system of simple roots and the corresponding positive root system is $\Sigma^+ = \{f_0, 2f_0\}$. If $n = 1$, then $\Sigma = \{\pm 2f_0\}$ and $\Pi = \Sigma^+ = \{2f_0\}$.

Let E_{ij} be the matrix in $\mathfrak{gl}(n,\mathbb{C})$ such that the entry at the ith row and the jth column is 1 and the other entries are all 0. The root vector spaces are

$$\begin{aligned} \mathfrak{g}_{f_0} &= \langle Z_j, Z'_j : 1 \leqslant j \leqslant n-1 \rangle \text{ (if } n > 1), & \mathfrak{g}_{2f_0} &= \langle U \rangle, \\ \mathfrak{g}_{-f_0} &= \langle W_j, W'_j : 1 \leqslant j \leqslant n-1 \rangle \text{ (if } n > 1), & \mathfrak{g}_{-2f_0} &= \langle V \rangle, \end{aligned}$$

https://doi.org/10.1017/S0013091508001004 Published online by Cambridge University Press

where

$$\begin{aligned} Z_j &= E_{jn} - E_{j,n+1} - E_{nj} - E_{n+1,j}, \\ Z'_j &= \mathrm{i}(E_{jn} - E_{j,n+1} + E_{nj} + E_{n+1,j}), \\ W_j &= E_{jn} + E_{j,n+1} - E_{nj} + E_{n+1,j}, \\ W'_j &= \mathrm{i}(E_{jn} + E_{j,n+1} + E_{nj} - E_{n+1,j}), \\ U &= \mathrm{i}(E_{nn} - E_{n,n+1} + E_{n+1,n} - E_{n+1,n+1}), \\ V &= \mathrm{i}(E_{nn} + E_{n,n+1} - E_{n+1,n} - E_{n+1,n+1}). \end{aligned}$$

If $n > 2$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is $Z_{\mathfrak{k}}(\mathfrak{a}) = \langle C_r, F_{jk}, H_{jk} : r, j, k = 1, \dots, n-1, j < k\rangle \cong \mathfrak{u}(n-1)$, where

$$C_r = 2\mathrm{i}E_{rr} - \mathrm{i}E_{nn} - \mathrm{i}E_{n+1,n+1}, \quad F_{jk} = E_{jk} - E_{kj}, \quad H_{jk} = \mathrm{i}(E_{jk} + E_{kj})$$

and $\mathfrak{su}(n,1) = (Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}) + \sum_{f \in \Sigma} \mathfrak{g}_f$ is the restricted-root space decomposition. We also have the Iwasawa decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_{f_0} + \mathfrak{g}_{2f_0} = \langle U, Z_j, Z'_j : 1 \leqslant j \leqslant n-1\rangle$.

If $n = 2$, we set $C = C_1 = \mathrm{diag}(2\mathrm{i}, -\mathrm{i}, -\mathrm{i})$, $Z = Z_1$, $Z' = Z'_1$, and in this case C generates $Z_{\mathfrak{k}}(\mathfrak{a})$, and $\mathfrak{a} + \mathfrak{n} = \langle A_0, U, Z, Z'\rangle$. If $n = 1$, $Z_{\mathfrak{k}}(\mathfrak{a}) = 0$, we have the restricted-root space decomposition $\mathfrak{su}(1,1) = \mathfrak{a} + (\mathfrak{g}_{2f_0} + \mathfrak{g}_{-2f_0}) = \langle A_0\rangle + \langle U, V\rangle$, and the solvable part in the Iwasawa decomposition is $\mathfrak{a} + \mathfrak{n} = \langle A_0, U\rangle$.

By using the Cartan decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p}$, we express each element $X \in \mathfrak{su}(n,1)$ as the sum $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ ($X_{\mathfrak{k}} \in \mathfrak{k}$, $X_{\mathfrak{p}} \in \mathfrak{p}$). In particular, we have

$$\begin{aligned} U_{\mathfrak{k}} &= \mathrm{i}(E_{nn} - E_{n+1,n+1}), & U_{\mathfrak{p}} &= \mathrm{i}(E_{n+1,n} - E_{n,n+1}), \\ (Z_j)_{\mathfrak{k}} &= E_{jn} - E_{nj}, & (Z_j)_{\mathfrak{p}} &= -(E_{n+1,j} + E_{j,n+1}), \\ (Z'_j)_{\mathfrak{k}} &= \mathrm{i}(E_{jn} + E_{nj}), & (Z'_j)_{\mathfrak{p}} &= \mathrm{i}(E_{n+1,j} - E_{j,n+1}). \end{aligned}$$

From the basis $\{A_0, U, Z_j, Z'_j : 1 \leqslant j \leqslant n-1\}$ of $\mathfrak{a} + \mathfrak{n}$ and the generators of $Z_{\mathfrak{k}}(\mathfrak{a})$ above, we get the basis $\{C_r, F_{jk}, H_{jk}, U_{\mathfrak{k}}, (Z_r)_{\mathfrak{k}}, (Z'_r)_{\mathfrak{k}} : r, j, k = 1, \dots, n-1, j < k\}$ of $\mathfrak{k}$, and the basis $\{A_0, U_{\mathfrak{p}}, (Z_j)_{\mathfrak{p}}, (Z'_j)_{\mathfrak{p}} : 1 \leqslant j \leqslant n-1\}$ of $\mathfrak{p}$. Notice that if $n = 1$, $\mathfrak{k} = \langle U_{\mathfrak{k}}\rangle$ and $\mathfrak{p} = \langle A_0, U_{\mathfrak{p}}\rangle$, and if $n = 2$, we have $\mathfrak{k} = \langle C, U_{\mathfrak{k}}, Z_{\mathfrak{k}}, Z'_{\mathfrak{k}}\rangle$ and $\mathfrak{p} = \langle A, U_{\mathfrak{p}}, Z_{\mathfrak{p}}, Z'_{\mathfrak{p}}\rangle$. We also decompose $\mathfrak{k} = \mathfrak{k}' + \mathfrak{c}$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = \langle C_r - U_{\mathfrak{k}}, F_{jk}, H_{jk}, (Z_r)_{\mathfrak{k}}, (Z'_r)_{\mathfrak{k}} : r, j, k = 1, \dots, n-1, j < k\rangle \cong \mathfrak{su}(n)$, and $\mathfrak{c}$ is the centre of $\mathfrak{k}$, which is generated by the element

$$E_J = \frac{1}{2n+1}(C_1 + \cdots + C_{n-1} + (n+1)U_{\mathfrak{k}})$$

such that $\mathrm{ad}_{E_J} : \mathfrak{p} \to \mathfrak{p}$ defines the complex structure on $\mathbb{C}\mathrm{H}(n)$. By the isomorphisms $\mathfrak{p} \cong \mathfrak{su}(n,1)/\mathfrak{k} \cong \mathfrak{a} + \mathfrak{n}$, we obtain the complex structure J acting on $\mathfrak{a} + \mathfrak{n}$ as follows:

$$JA_0 = -U, \qquad JU = A_0, \qquad JZ_r = Z'_r, \qquad JZ'_r = -Z_r. \tag{3.2}$$

We consider the scalar product $\langle \cdot\,, \cdot\rangle$ on $\mathfrak{a} + \mathfrak{n}$ defined by the isomorphism $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{p}$ and

$$\frac{1}{4(n+1)} B\bigg|_{\mathfrak{p} \times \mathfrak{p}}.$$

Then $(\mathfrak{a}+\mathfrak{n}, \langle\cdot\,,\cdot\rangle, J)$ is a Hermitian vector space, and the basis $\{A_0, U, Z_r, Z'_r : 1 \leqslant r \leqslant n-1\}$ of $\mathfrak{a}+\mathfrak{n}$ is orthonormal. We consider the solvable factor AN (with Lie algebra $\mathfrak{a}+\mathfrak{n}$) of the Iwasawa decomposition of $\mathrm{SU}(n,1)$ with the invariant metric g and almost-complex structure J defined by $\langle\cdot\,,\cdot\rangle$ and J, respectively.

The Lie brackets of the elements of the basis of $\mathfrak{a}+\mathfrak{n}$ are given by

$$[A_0, U] = 2U, \qquad [A_0, Z_j] = Z_j, \qquad [A_0, Z'_j] = Z'_j, \qquad [Z_j, Z'_r] = -\delta_{jr}2U,$$
$$[U, Z_j] = [U, Z'_j] = [Z_j, Z_r] = [Z'_j, Z'_r] = 0.$$

The Levi-Civitá connection ∇ is given by $2g(\nabla_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$ for all $X, Y, Z \in \mathfrak{a}+\mathfrak{n}$. So, the covariant derivatives between generators of $\mathfrak{a}+\mathfrak{n}$ are given by

$$\left.\begin{array}{cccc} & \nabla_{A_0}A_0 = \nabla_{A_0}U = \nabla_{A_0}Z_r = \nabla_{A_0}Z'_r = 0, & & \\ \nabla_U A_0 = -2U, & \nabla_U U = 2A_0, & \nabla_U Z_r = Z'_r, & \nabla_U Z'_r = -Z_r, \\ \nabla_{Z_j} A_0 = -Z_j, & \nabla_{Z_j} U = Z'_j, & \nabla_{Z_j} Z_r = \delta_{jr}A_0, & \nabla_{Z_j} Z'_r = -\delta_{jr}U, \\ \nabla_{Z'_j} A_0 = -Z'_j, & \nabla_{Z'_j} U = -Z_j, & \nabla_{Z'_j} Z_r = \delta_{jr}U, & \nabla_{Z'_j} Z'_r = \delta_{jr}A_0. \end{array}\right\} \tag{3.3}$$

The components of the curvature tensor field R are given by

$$\begin{array}{llll} R_{A_0U}A_0 = -4U, & R_{A_0U}U = 4A_0, & R_{A_0U}Z_r = 2Z'_r, & R_{A_0U}Z'_r = -2Z_r, \\ R_{A_0Z_j}A_0 = -Z_j, & R_{A_0Z_j}U = Z'_j, & R_{A_0Z_j}Z_r = \delta_{jr}A_0, & R_{A_0Z_j}Z'_r = -\delta_{jr}U, \\ R_{A_0Z'_j}A_0 = -Z'_j, & R_{A_0Z'_j}U = -Z_j, & R_{A_0Z'_j}Z_r = \delta_{jr}U, & R_{A_0Z'_j}Z'_r = \delta_{jr}A_0, \\ R_{UZ_j}A_0 = -Z'_j, & R_{UZ_j}A_0 = -Z_j, & R_{UZ_j}Z_r = \delta_{jr}U, & R_{UZ_j}Z'_r = \delta_{jr}A_0, \\ R_{UZ'_j}A_0 = Z_j, & R_{UZ'_j}U = -Z'_j, & R_{UZ'_j}Z_r = -\delta_{jr}A_0, & R_{UZ'_j}Z'_r = \delta_{jr}U, \\ R_{Z_kZ_j}A_0 = R_{Z_kZ_j}U = 0, & R_{Z_jZ'_r}A_0 = 2\delta_{jr}U, & & R_{Z_jZ'_r}U = -2\delta_{jr}A_0, \\ R_{Z_kZ_j}Z_r = \delta_{jr}Z_k - \delta_{kr}Z_j, & R_{Z_kZ_j}Z'_r = \delta_{jr}Z'_k - \delta_{kr}Z'_j, & & R_{Z'_kZ'_j} = R_{Z_kZ_j}, \\ R_{Z_jZ'_j}Z_r = -2(1+\delta_{jr}Z'_r), & R_{Z_jZ'_j}Z'_r = 2(1+\delta_{jr})Z_r, & & \end{array}$$

and

$$R_{Z_kZ'_j}Z_r = -\delta_{jr}Z'_k - \delta_{kr}Z'_j, \quad R_{Z_kZ'_j}Z_r = \delta_{jr}Z_k - \delta_{kr}Z_j, \quad \text{where } k \neq j.$$

In particular, we see that the invariant metric on AN has constant holomorphic sectional curvature -4.

3.2. Homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n) \equiv AN$

We will determine the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n) \equiv AN$ in terms of the basis of left-invariant forms α, β, γ^j, γ'^j, $1 \leqslant j \leqslant n-1$, dual to A_0, U, Z_j, Z'_j. If S is a homogeneous Riemannian structure on AN and $\tilde{\nabla} = \nabla - S$, the condition $\tilde{\nabla} g = 0$

https://doi.org/10.1017/S0013091508001004 Published online by Cambridge University Press

in (2.1) is equivalent to $S_{XYZ} + S_{XZY} = 0$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$. Moreover, $\tilde{\nabla} R = 0$ is equivalent to the condition

$$(\nabla_X R)_{Y_1 Y_2 Y_3 Y_4} = -R_{S_X Y_1 Y_2 Y_3 Y_4} - R_{Y_1 S_X Y_2 Y_3 Y_4} - R_{Y_1 Y_2 S_X Y_3 Y_4} - R_{Y_1 Y_2 Y_3 S_X Y_4}$$

for all $Y_1, Y_1, Y_3, Y_4 \in \mathfrak{a} + \mathfrak{n}$. Replacing (Y_1, Y_2, Y_3, Y_4) by (A_0, U, A_0, Z_j), (A_0, U, A_0, Z'_j), (A_0, U, Z_k, Z_j) and (A_0, U, Z_k, Z'_j), one obtains that $S_{XUZ_j} = S_{XA_0Z'_j}$, $S_{XUZ'_j} = -S_{XA_0Z_j}$, $S_{XZ_kZ'_j} = -S_{XZ'_kZ_j}$ and $S_{XZ_kZ_j} = S_{XZ'_kZ'_j}$, respectively. It is easy to see that the condition $\tilde{\nabla} R = 0$ holds if and only if the last four equations are satisfied for all $X \in \mathfrak{a} + \mathfrak{n}$. These equations also show (see (3.2)) that the condition $S \cdot J = 0$ of homogeneous Kähler structures (see Proposition 2.2) is fulfilled. We set

$$\omega(X) = S_{XA_0U}, \qquad \sigma^j(X) = S_{XA_0Z_j} = -S_{XUZ'_j}, \qquad \tau^j(X) = S_{XA_0Z'_j} = S_{XUZ_j}, \tag{3.4}$$

$$\theta^{kj}(X) = S_{XZ_kZ'_j} = S_{XZ_jZ'_k}, \qquad \psi^{kj}(X) = S_{XZ_kZ_j} = S_{XZ'_kZ'_j}. \tag{3.5}$$

We have $\theta^{kj} = \theta^{jk}$ and $\psi^{kj} = -\psi^{jk}$. Now, we must determine the conditions for the 1-forms ω, σ^j, τ^j, θ^{kj} and σ^{kj} under which the condition $\tilde{\nabla} S = 0$ in (2.1) is satisfied.

By (3.3)–(3.5), the connection $\tilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned}
\tilde{\nabla}_X A_0 &= -(2\beta + \omega)(X)U - \sum_j (\gamma^j + \sigma^j)(X) Z_j - \sum_j (\gamma'^j + \tau^j)(X) Z'_j, \\
\tilde{\nabla}_X U &= (2\beta + \omega)(X) A_0 - \sum_j (\gamma'^j + \tau^j)(X) Z_j + \sum_j (\gamma^j + \sigma^j)(X) Z'_j, \\
\tilde{\nabla}_X Z_j &= (\gamma^j + \sigma^j)(X) A_0 + (\gamma'^j + \tau^j)(X) U + (\beta - \theta^j)(X) Z'_j \\
&\qquad + \sum_{k \neq j} (\psi^{kj}(X) Z_k - \theta^{kj}(X) Z'_k), \\
\tilde{\nabla}_X Z'_j &= (\gamma'^j + \tau^j)(X) A_0 - (\gamma^j + \sigma^j)(X) U + (\theta^j - \beta)(X) Z_j \\
&\qquad + \sum_{k \neq j} (\theta^{kj}(X) Z_k - \psi^{kj}(X) Z'_k).
\end{aligned}$$

Now, replacing (V_1, V_2) in the equation $(\tilde{\nabla}_X S)(W, V_1, V_2) = 0$ by (A_0, U), (A_0, Z_j), (A_0, Z'_j), (Z_k, Z_j) and (Z_k, Z'_j), respectively, we obtain that the condition $\tilde{\nabla} S = 0$ is equivalent to the following conditions:

$$\left.\begin{aligned}
\tilde{\nabla} \omega &= 2 \sum_j ((\gamma^j + \sigma^j) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \sigma^j), \\
\tilde{\nabla} \sigma^j &= -(\beta + \omega + \theta^j) \otimes \tau^j + (\gamma'^j + \tau^j) \otimes (\omega + \theta^j) \\
&\quad + \sum_{k \neq j} (\psi^{kj} \otimes \sigma^k - \theta^{kj} \otimes \tau^k + (\gamma'^k + \tau^k) \otimes \theta^{kj} - (\gamma^k + \sigma^k) \otimes \psi^{kj}), \\
\tilde{\nabla} \tau^j &= (\beta + \omega + \theta^j) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes (\omega + \theta^j) \\
&\quad + \sum_{k \neq j} (\theta^{kj} \otimes \sigma^k + \psi^{kj} \otimes \tau^k - (\gamma^k + \sigma^k) \otimes \theta^{kj} - (\gamma'^k + \tau^k) \otimes \psi^{kj}),
\end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned}\tilde\nabla\theta^{kj} &= (\gamma^j+\sigma^j)\otimes\tau^k+(\gamma^k+\tau^k)\otimes\tau^j-(\gamma'^j+\tau^j)\otimes\sigma^k-(\gamma'^k+\tau^k)\otimes\sigma^j\\ &\quad+\sum_l\psi^{lk}\wedge\theta^{jl}+\sum_l\theta^{lk}\wedge\psi^{jl},\\ \tilde\nabla\psi^{kj} &= (\gamma^k+\sigma^k)\otimes\sigma^j-(\gamma^j+\sigma^j)\otimes\sigma^k-(\gamma'^k+\tau^k)\otimes\tau^j-(\gamma'^j+\tau^j)\otimes\tau^k\\ &\quad+\sum_l\theta^{lk}\wedge\theta^{jl}-\sum_l\psi^{lk}\wedge\psi^{jl},\end{aligned}\right\}\qquad(3.6\ cont.)$$

where $\theta^j=\theta^{jj}$. Thus, from (3.4) and (3.5), we have the following.

Theorem 3.1. *All the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)\equiv AN$ are given by*

$$\begin{aligned}S &= \omega\otimes(\alpha\wedge\beta)\\ &\quad+\sum_{j=1}^{n-1}(\sigma^j\otimes(\alpha\wedge\gamma^j-\beta\wedge\gamma'^j)+\tau^j\otimes(\alpha\wedge\gamma'^j+\beta\wedge\gamma^j)+\theta^{jj}\otimes(\gamma^j\wedge\gamma'^j))\\ &\qquad+\sum_{1\leqslant k<j\leqslant n-1}(\psi^{kj}\otimes(\gamma^k\wedge\gamma^j+\gamma'^k\wedge\gamma'^j)+\theta^{kj}\otimes(\gamma^k\wedge\gamma'^j+\gamma^j\wedge\gamma'^k)),\end{aligned}$$

where ω, σ^j, τ^j, θ^{kj}, ψ^{kj} $(1\leqslant k,j\leqslant n-1)$, are 1-forms on AN satisfying $\theta^{jk}=\theta^{kj}$, $\psi^{jk}=-\psi^{kj}$ and Equations (3.6).

If $n=2$, we set $\gamma=\gamma^1$, $\gamma'=\gamma'^1$, so that $\{\alpha,\beta,\gamma,\gamma'\}$ is the basis of left-invariant forms on $AN=\mathbb{C}\mathrm{H}(2)$ dual to $\{A_0,U,Z,Z'\}$, and we have the following.

Corollary 3.2. *All the homogeneous Kähler structures on the complex hyperbolic plane $\mathbb{C}\mathrm{H}(2)\equiv AN$ are given by*

$$S=\omega\otimes(\alpha\wedge\beta)+\sigma\otimes(\alpha\wedge\gamma-\beta\wedge\gamma')+\tau\otimes(\alpha\wedge\gamma'+\beta\wedge\gamma)+\theta\otimes(\gamma\wedge\gamma'),$$

where ω, σ, τ and θ are 1-forms on AN satisfying

$$\begin{aligned}\tilde\nabla\omega &= 2(\gamma+\sigma)\otimes\tau-2(\gamma'+\tau)\otimes\sigma=\tilde\nabla\theta,\\ \tilde\nabla\sigma &= -(\beta+\omega+\theta)\otimes\gamma+(\gamma'+\tau)\otimes(\omega+\theta),\\ \tilde\nabla\tau &= (\beta+\omega+\theta)\otimes\sigma-(\gamma+\sigma)\otimes(\omega+\theta).\end{aligned}$$

If $n=1$, $\{\alpha,\beta\}$ is the basis of 1-invariant forms on the two-dimensional solvable Lie group $AN=\mathbb{C}\mathrm{H}(1)$ dual to the basis $\{A_0,U\}$ of $\mathfrak{a}+\mathfrak{n}$, and we have the following.

Corollary 3.3. *All the homogeneous Kähler structures on the complex hyperbolic line (or real hyperbolic plane) $\mathbb{C}\mathrm{H}(1)\equiv AN$ are given by $S=\omega\otimes(\alpha\wedge\beta)$, where ω is a 1-form on AN satisfying $\tilde\nabla\omega=0$.*

Remark 3.4. If $S=\omega\otimes(\alpha\wedge\beta)$ is a homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(1)$, and $\omega=\lambda\alpha+\mu\beta$, where λ and μ are functions on $\mathbb{C}\mathrm{H}(1)$, the condition $\tilde\nabla\omega=0$ together with the structure equation $[A_0,U]=2U$ gives $\lambda=\mu=0$ or $\lambda^2+\mu^2=4$, and we have that there are infinite homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(1)$. However, up to isomorphism [**28**, Theorem 4.4], there are only two homogeneous structures on the real hyperbolic plane: one of them is $S = 0$ ($\lambda = \mu = 0$), and the other, which is given by $S_X Y = g(X,Y)\xi_0 - g(\xi_0, Y)X$, with $\xi_0 = 2A_0$ (for $X, Y \in \mathfrak{a} + \mathfrak{n} = \langle A_0, U\rangle$), corresponds to $S = \omega \otimes (\alpha \wedge \beta)$, with $\omega = -2\beta$ ($\lambda = 0$, $\mu = -2$).

Remark 3.5. For each $n > 0$, $S = 0$ is a homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(n) \equiv AN$; the corresponding canonical connection is $\tilde{\nabla} = \nabla$, its holonomy algebra is $\mathfrak{k} \cong \mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1))$, the associated reductive decomposition is the Cartan decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p}$ and it gives the description of $\mathbb{C}\mathrm{H}(n)$ as symmetric space $\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$.

Now, our purpose is to obtain non-trivial homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)$, $n \geqslant 2$, their associated reductive decompositions, and the corresponding descriptions as homogeneous Kähler spaces.

We will seek for solutions for which $\sigma^j = -\gamma^j$, $\tau^j = -\gamma'^j$. In this case, we have

$$\tilde{\nabla}\gamma^j = (\beta - \theta^j) \otimes \gamma'^j + \sum_{k \neq j} (\psi^{kj} \otimes \gamma^k - \theta^{kj} \otimes \gamma'^k),$$
$$\tilde{\nabla}\gamma'^j = (\theta^j - \beta) \otimes \gamma^j + \sum_{k \neq j} (\theta^{kj} \otimes \gamma^k + \psi^{kj} \otimes \gamma'^k).$$

(Obviously, the last summands on the right hand-side in each of the two equations above do not appear if $n = 2$.) By the second and third equations in (3.6), we must have $\omega = -2\beta$, which also satisfies the first equation in (3.6), because

$$\tilde{\nabla}\beta = (2\beta + \omega) \otimes \alpha - \sum_j (\gamma'^j + \tau^j) \otimes \gamma^j + \sum_j (\gamma^j + \sigma^j) \otimes \gamma'^j = 0.$$

If $n = 2$, by Corollary 3.2, we have only to determine θ such that $\tilde{\nabla}\theta = 0$. If we set $\theta = a\alpha + b\beta + c\gamma + c'\gamma'$, by also using the structure equations of $\mathfrak{a} + \mathfrak{n} = \langle A_0, U, Z, Z'\rangle$, we obtain that $c = c' = 0$ and a and b are constant. For $n > 2$ we set $\theta^j = \theta^{jj} = a_j\alpha + b_j\beta$, $\theta^{kj} = c_{kj}\alpha$, $\psi^{kj} = p_{kj}\alpha$, $k \neq j$, with $a_j, b_j, c_{kj}, p_{kj} \in \mathbb{R}$. Then, if $\sigma^j = -\gamma^j$, $\tau^j = -\gamma'^j$ and $\omega = -2\beta$, Equations (3.6) are satisfied if and only if one has

$$p_{kj}(b_k - b_j) = c_{kj}(b_k - b_j) = 0.$$

Consequently, we get the following.

Proposition 3.6. *For $n > 2$, the space $\mathbb{C}\mathrm{H}(n)$ admits the multi-parametric family of homogeneous Kähler structures $S = S^{a_j, b_j, c_{kj}, p_{kj}}$ given in terms of the generators of $\mathfrak{a} + \mathfrak{n}$ by Table 1.*

The complex hyperbolic plane $\mathbb{C}\mathrm{H}(2)$ admits the two-parametric family of homogeneous Kähler structures $S = S^{a,b}$ given in terms of the generators of $\mathfrak{a} + \mathfrak{n}$ by Table 2.

Table 1. *Homogeneous Kähler structure* $S = S^{a_j,b_j,c_{kj},p_{kj}}$.

	A_0	U	Z_j	Z'_j
S_{A_0}	0	0	$a_j Z'_j + \sum_{l\neq j}(p_{jl}Z_l + c_{jl}Z'_l)$	$-a_j Z_j + \sum_{l\neq j}(p_{jl}Z'_l - c_{jl}Z_l)$
S_U	$-2U$	$2A_0$	$b_j Z'_j$	$-b_j Z_j$
S_{Z_k}	$-Z_k$	Z'_k	$\delta_{kj}A_0$	$-\delta_{kj}U$
$S_{Z'_k}$	$-Z'_k$	$-Z_k$	$\delta_{kj}U$	$\delta_{kj}A_0$

Table 2. *Homogeneous Kähler structure* $S = S^{a,b}$.

	A_0	U	Z	Z'
S_{A_0}	0	0	aZ'	$-aZ$
S_U	$-2U$	$2A_0$	bZ'	$-bZ$
S_Z	$-Z$	Z'	A_0	$-U$
$S_{Z'}$	$-Z'$	$-Z$	U	A_0

If $S = S^{a_j,b_j,c_{kj},p_{kj}}$, with respect to the basis $\{A_0, U, Z_j, Z'_j\}$ of $\mathfrak{a}+\mathfrak{n}$, the connection $\tilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned} \tilde{\nabla}_{A_0} Z_j &= -a_j Z'_j - \sum_{l\neq j}(p_{jl}Z_l + c_{jl}Z'_l), \quad & \tilde{\nabla}_U Z_j &= (1-b_j)Z'_j, \\ \tilde{\nabla}_{A_0} Z'_j &= a_j Z_j - \sum_{l\neq j}(p_{jl}Z'_l - c_{jl}Z_l), \quad & \tilde{\nabla}_U Z'_j &= (b_j-1)Z_j, \end{aligned}$$

with the rest vanishing. Hence, the components of the curvature tensor field are

$$\tilde{R}_{A_0 U} = -\tilde{R}_{Z_k Z'_k} = 2\sum_j (1-b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma'^j),$$

and the rest are zero.

If $b_j = 1$ for all $j = 1,\dots,n-1$, the holonomy algebra of $\tilde{\nabla}$ is trivial and the reductive decompositions associated to the homogeneous Kähler structures given in Proposition 3.6 are given by $\tilde{\mathfrak{g}}^{a_j,c_{kj},p_{kj}} = \{0\} + (\mathfrak{a}+\mathfrak{n})$. From (2.3), the non-vanishing brackets are given by

$$\left.\begin{aligned} [A_0, Z_j] &= Z_j + a_j Z'_j + \sum_{l\neq j}(p_{jl}Z_l + c_{jl}Z'_l), \quad & [A_0, U] &= 2U, \\ [A_0, Z'_j] &= -a_j Z_j + Z'_j + \sum_{l\neq j}(p_{jl}Z'_l + c_{jl}Z_l), \quad & [Z_j, Z'_j] &= -2U. \end{aligned}\right\} \tag{3.7}$$

On the other hand, the element

$$\hat{A}_0 = \lambda_1 C_1 + \cdots + \lambda_{n-1}C_{n-1} + \sum_{j<l}(c_{jl}H_{jl} - p_{jl}F_{jl}) + A_0$$

of $\mathfrak{su}(n,1)$ generates a subspace $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}$ of $Z_{\mathfrak{k}}(\mathfrak{a})+\mathfrak{a}$, and the structure equations of the Lie subalgebra $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}+\mathfrak{n}$ of $\mathfrak{su}(n,1)$ are

$$\left.\begin{aligned}
[\hat{A}_0,Z_j]&=Z_j+\Big(3\lambda_j+\sum_{l\neq j}\lambda_l\Big)Z'_j+\sum_{l\neq j}(p_{jl}Z_l+c_{jl}Z'_l), \qquad [\hat{A}_0,U]=2U,\\
[\hat{A}_0,Z'_j]&=-\Big(3\lambda_j+\sum_{l\neq j}\lambda_l\Big)Z_j+Z'_j+\sum_{l\neq j}(p_{jl}Z'_l+c_{jl}Z_l), \quad [Z_j,Z'_j]=-2U,
\end{aligned}\right\} \tag{3.8}$$

with the rest vanishing. From (3.7) and (3.8), it follows that $\tilde{\mathfrak{g}}^{a_j,c_{kj},p_{kj}}$ is isomorphic to $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}+\mathfrak{n}$.

Now, for the structure $S=S^{a_j,b_j,c_{kj},p_{kj}}$ in Table 1, suppose that $b_j\neq 1$ for some $j=1,\dots,n-1$. Then,

$$\rho=\tilde{R}_{A_0U}=-\tilde{R}_{Z_kZ'_k}=2\sum_j(1-b_j)(Z'_j\otimes\gamma^j-Z_j\otimes\gamma'^j)$$

generates the holonomy algebra $\tilde{\mathfrak{h}}^{a_j,b_j,c_{kj},p_{kj}}$ of $\tilde{\nabla}=\nabla-S$, and the reductive decomposition associated to S is

$$\tilde{\mathfrak{g}}^{a_j,b_j,c_{kj},p_{kj}}=\tilde{\mathfrak{h}}^{a_j,b_j,c_{kj},p_{kj}}+(\mathfrak{a}+\mathfrak{n})=\langle\rho,A_0,U,Z_j,Z'_j\rangle.$$

From (2.3), the structure equations are given by

$$\left.\begin{aligned}
&[\rho,A_0]=[\rho,U]=0, \qquad [\rho,Z_j]=2(1-b_j)Z'_j, \quad [\rho,Z'_j]=2(b_j-1)Z_j,\\
&[A_0,U]=\rho+2U, \qquad [A_0,Z_j]=Z_j+a_jZ'_j+\sum_{l\neq j}(p_{jl}Z_l+c_{jl}Z'_l),\\
&[A_0,Z'_j]=-a_jZ_j+Z'_j+\sum_{l\neq j}(p_{jl}Z'_l+c_{jl}Z_l),\\
&[U,Z_j]=(b_j-1)Z'_j, \qquad [U,Z'_j]=(1-b_j)Z_j, \quad [Z_k,Z'_j]=-\delta_{kj}(\rho+2U).
\end{aligned}\right\} \tag{3.9}$$

If $\mathfrak{u}\cong\mathfrak{u}(1)$ is the subspace of $Z_{\mathfrak{k}}(\mathfrak{a})$ generated by $C=C_1+\dots+C_{n-1}$, it is easy to see that the Lie algebra $\tilde{\mathfrak{g}}^{a_j,b_j,c_{kj},p_{kj}}$ is isomorphic to the Lie subalgebra

$$\mathfrak{u}+\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}+\mathfrak{n}=\langle C,\hat{A}_0,U,Z_j,Z'_j\rangle$$

of $\mathfrak{su}(n,1)$. We deduce the following.

Theorem 3.7. *Let $S=S^{a_j,b_j,c_{kj},p_{kj}}$ be the homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(n)$, $n>2$, given by Table 1, and let $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}$ be the subspace of $Z_{\mathfrak{k}}(\mathfrak{a})+\mathfrak{a}$ generated by*

$$\hat{A}_0=\sum_j\lambda_jC_j+\sum_{1\leqslant j<l\leqslant n-1}(c_{jl}H_{jl}-p_{jl}F_{jl})+A_0 \quad \left(\lambda_j=\frac{na_j-\sum_{l\neq j}a_l}{2n+2}\right),$$

and $\mathfrak{u}=\langle C_1+\dots+C_{n-1}\rangle$. If $b_j=1$ for all $j=1,\dots,n-1$, the corresponding group of isometries is the connected subgroup $E^{\lambda_j,c_{kj},p_{kj}}N$ of $\mathrm{SU}(n,1)$ whose lie algebra is $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}+\mathfrak{n}$. If $b_j\neq 1$ for some $j=1,\dots,n-1$, the corresponding group of

Table 3. *Homogeneous Sasakian structure S^t.*

	A_0^{H}	U^{H}	Z_j^{H}	$Z_j^{\prime\mathrm{H}}$	ξ
$S^t_{A_0^{\mathrm{H}}}$	0	$-\xi$	0	0	U^{H}
$S^t_{U^{\mathrm{H}}}$	ξ	0	0	0	$-A^{\mathrm{H}}$
$S^t_{Z_k^{\mathrm{H}}}$	0	0	0	$\delta_{kj}\xi$	$-Z_k^{\prime\mathrm{H}}$
$S^t_{Z_k^{\prime\mathrm{H}}}$	0	0	$-\delta_{kj}\xi$	0	Z_k^{H}
S^t_{ξ}	tU^{H}	$-tA^{\mathrm{H}}$	$-tZ_j^{\prime\mathrm{H}}$	tZ_j^{H}	0

isometries is the connected subgroup $\mathrm{U}(1)E^{\lambda_j,c_{kj},p_{kj}}N$ *of* $\mathrm{SU}(n,1)$ *whose Lie algebra is* $\mathfrak{u}+\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}+\mathfrak{n}$.

If $S^{a,b}$ *is the homogeneous Kähler structure on the complex hyperbolic plane* $\mathbb{C}\mathrm{H}(2)$ *given by Table 2,* $\mathfrak{e}^{\lambda}=\langle\hat{A}_0\rangle$, *where* $\hat{A}_0=\lambda C+A_0$ $(\lambda=a/3)$, *and* $\mathfrak{u}=\langle C\rangle$, *then the corresponding group of isometries is*

(i) *the subgroup* $E^{\lambda}N$ *of* $\mathrm{SU}(2,1)$ *generated by the Lie subalgebra* $\mathfrak{e}^{\lambda}+\mathfrak{n}$ *of* $\mathfrak{su}(2,1)$, *if* $b=1$,

(ii) *the subgroup* $\mathrm{U}(1)E^{\lambda}N$ *of* $\mathrm{SU}(2,1)$ *generated by* $\mathfrak{u}+\mathfrak{e}^{\lambda}+\mathfrak{n}$, *if* $b\neq 1$.

Remark 3.8. Each structure $S^{a_j,b_j,c_{kj},p_{kj}}$, with $b_j=1$ for all j, is also characterized by the fact that $\tilde{\nabla}=\nabla-S^{a_j,b_j,c_{kj},p_{kj}}$ is the canonical connection for the Lie group $E^{\lambda_j,c_{kj},p_{kj}}N$, which is the connection for which every left-invariant vector field on $E^{\lambda_j,c_{kj},p_{kj}}N$ is parallel. Each one of these groups acts simply transitively on $\mathbb{C}\mathrm{H}(n)$ and it provides a description of $\mathbb{C}\mathrm{H}(n)$ as a homogeneous space. If all the parameters a_j, c_{kj}, p_{kj} are zero, then $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}}=\mathfrak{a}$, and we get the usual description as a solvable Lie group $\mathbb{C}\mathrm{H}(n)=AN$. In this case, the corresponding homogeneous structure is given by $S_XY=\nabla_XY$ for all $X,Y\in\mathfrak{a}+\mathfrak{n}$. If $b_j\neq 1$ for some $j=1,\dots,n-1$, we get the descriptions as homogeneous space $\mathbb{C}\mathrm{H}(n)=\mathrm{U}(1)E^{\lambda_j,c_{kj},p_{kj}}N/\mathrm{U}(1)$.

3.3. Principal line bundle over $\mathbb{C}\mathrm{H}(n)$

By (3.2), the fundamental 2-form of the Kähler structure (J,g) of $\mathbb{C}\mathrm{H}(n)\equiv AN$ is given by

$$\Omega=\alpha\wedge\beta-\sum_{j=1}^{n-1}\gamma^j\wedge\gamma'^j=-\tfrac{1}{2}\mathrm{d}\beta,$$

where $\{\alpha,\beta,\gamma^j,\gamma'^j:1\leqslant j\leqslant n-1\}$ is the basis of left-invariant 1-forms on AN dual to the basis $\{A_0,U,Z_j,Z'_j\}$ of $\mathfrak{a}+\mathfrak{n}$. We consider the principal line bundle $\pi:\bar{M}\to\mathbb{C}\mathrm{H}(n)$, and identify the bundle space $\bar{M}$ with $AN\times\mathbb{R}$ and π with the projection on AN. The fundamental vector field ξ is identified with $\mathrm{d}/\mathrm{d}t$, and the 1-form $\eta=\mathrm{d}t-\pi^*\beta$ is also regarded as a connection form on the bundle. If φ and $\bar{g}$ are given by (2.7), then $(\varphi,\xi,\eta,\bar{g})$ is a Sasakian structure on $\bar{M}$.

https://doi.org/10.1017/S0013091508001004 Published online by Cambridge University Press

By Proposition 2.5 (a), each homogeneous Kähler structure $S^{a_j,b_j,c_{kj},p_{kj}}$ on $\mathbb{C}\mathrm{H}(n)$ given in Theorem 3.7 defines a homogeneous Sasakian structure $\bar{S}^{a_j,b_j,c_{kj},p_{kj}}$ on $\bar{M}$ which gives a description of $\bar{M}$ as either the connected subgroup $E^{\lambda_j,c_{kj},p_{kj}}N\times\mathbb{R}$ of $\mathrm{SU}(n,1)\times\mathbb{R}$ (if $b_j=1$ for all $j=1,\dots,n-1$), or as the homogeneous space $(\mathrm{U}(1)E^{\lambda_j,c_{kj},p_{kj}}N\times\mathbb{R})/\mathrm{U}(1)$.

On the other hand, from (b) of Proposition 2.5, we get the following.

Proposition 3.9. *The bundle space $\bar{M}$ of the line bundle $\pi:\bar{M}\to\mathbb{C}\mathrm{H}(n)$ admits the family of homogeneous Sasakian structures $\{S^t : t\in\mathbb{R}\}$ given, in terms of the horizontal lifts of the generators of $\mathfrak{a}+\mathfrak{n}$ and the fundamental vector field ξ, by Table 3.*

Remark 3.10. For each $p\in\bar{M}$, if $c_{12}(S^t)_p$ is the map from the tangent space $T_p(\bar{M})$ to its dual given by

$$c_{12}(S^t)_p(\tilde{X})=\sum_{i=1}^{2n+1}S_{e_ie_i\tilde{X}},$$

where $\{e_i\}$ is an orthonormal basis of $T_p(\bar{M})$, then $c_{12}(S^t)_p$ vanishes for every $t\in\mathbb{R}$. According to Tricerri and Vanhecke's classification of homogeneous Riemannian structures in [**28**], each S^t is of type $\mathcal{T}_2\oplus\mathcal{T}_3$. Moreover, if $t=-1$, we have $S_{\tilde{X}}\tilde{Y}+S_{\tilde{Y}}\tilde{X}=0$. Then S^{-1} is of type $\mathcal{T}_3$, which means that $\bar{M}$ is a naturally reductive Riemannian space. If $t=2$, then each cyclic sum $\mathfrak{S}_{\tilde{X}\tilde{Y}\tilde{Z}}S_{\tilde{X}\tilde{Y}\tilde{Z}}$ vanishes, and hence $\bar{M}$ is of type $\mathcal{T}_2$, which may also be expressed by saying that $\bar{M}$ is a cotorsionless manifold [**13**].

We will construct the reductive decomposition $\tilde{\mathfrak{g}}_t=\tilde{\mathfrak{h}}_t+\bar{\mathfrak{m}}$ associated to each homogeneous Sasakian structure S^t, where $\bar{\mathfrak{m}}=T_o(\bar{M})$, with $o\in\bar{M}$, is generated by $\tilde{A}=(A_0^{\mathrm{H}})_o$, $\tilde{U}=(U^{\mathrm{H}})_o$, $\tilde{Z}_j=(Z_j^{\mathrm{H}})_o$, $\tilde{Z}'_j=(Z'_j)_o^{\mathrm{H}}$, $\bar{\xi}=\xi_o$, $1\leqslant j\leqslant n-1$, and $\tilde{\mathfrak{h}}_t$ is the holonomy algebra of the connection $\tilde{D}^t=D-S^t$. Each connection $\tilde{D}^t$ is given by Table 4.

Let $\tilde{R}^t$ be the curvature of $\tilde{D}^t$, and let $\{\bar{\alpha},\bar{\beta},\bar{\gamma}^j,\bar{\gamma}'^j,\bar{\eta}\}$ be the basis dual to the basis $\{\tilde{A},\tilde{U},\tilde{Z}_j,\tilde{Z}'_j,\bar{\xi}\}$ of $\bar{\mathfrak{m}}$. The holonomy algebra $\tilde{\mathfrak{h}}_t$ of $\tilde{D}^t$ is generated by the curvature operators ρ_0, ρ_r, φ_r, ψ_r, σ_{jk}, τ_{jk} $(r,j,k=1,\dots,n-1,\ j<k)$, given by

$$\rho_0=\tilde{R}^t_{\tilde{A}\tilde{U}}=2(t-3)(\bar{\alpha}\otimes\tilde{U}-\bar{\beta}\otimes\tilde{A})+2(2-t)\sum_{j=1}^{n-1}(\bar{\gamma}^j\otimes\tilde{Z}'_j-\bar{\gamma}'^j\otimes\tilde{Z}_j),$$

$$\begin{aligned}\rho_r&=\tilde{R}^t_{\tilde{Z}_r\tilde{Z}'_r}\\&=2(2-t)(\bar{\alpha}\otimes\tilde{U}-\bar{\beta}\otimes\tilde{A})+2(t-3)(\bar{\gamma}^r\otimes\tilde{Z}'_r-\bar{\gamma}'^r\otimes\tilde{Z}_r)\\&\qquad+2(t-2)\sum_{j\neq r}(\bar{\gamma}^j\otimes\tilde{Z}'_j-\bar{\gamma}'^j\otimes\tilde{Z}_j),\end{aligned}$$

$$\varphi_r=\tilde{R}^t_{\tilde{A}\tilde{Z}_r}=-\tilde{R}^t_{\tilde{U}\tilde{Z}'_r}=-\bar{\alpha}\otimes\tilde{Z}_r+\bar{\beta}\otimes\tilde{Z}'_r+\bar{\gamma}^r\otimes\tilde{A}-\bar{\gamma}'^r\otimes\tilde{U},$$

$$\psi_r=\tilde{R}^t_{\tilde{U}\tilde{Z}_r}=\tilde{R}^t_{\tilde{A}\tilde{Z}'_r}=-\bar{\alpha}\otimes\tilde{Z}'_r-\bar{\beta}\otimes\tilde{Z}_r+\bar{\gamma}^r\otimes\tilde{U}+\bar{\gamma}'^r\otimes\tilde{A},$$

$$\sigma_{jk}=\tilde{R}^t_{\tilde{Z}_j\tilde{Z}_k}=\tilde{R}^t_{\tilde{Z}'_j\tilde{Z}'_k}=-\bar{\gamma}^j\otimes\tilde{Z}_k-\bar{\gamma}'^j\otimes\tilde{Z}'_k+\bar{\gamma}^k\otimes\tilde{Z}_j+\bar{\gamma}'^k\otimes\tilde{Z}'_j,$$

$$\tau_{jk}=\tilde{R}^t_{\tilde{Z}_j\tilde{Z}'_k}=\tilde{R}^t_{\tilde{Z}_k\tilde{Z}'_j}=-\bar{\gamma}^j\otimes\tilde{Z}'_k+\bar{\gamma}'^j\otimes\tilde{Z}_k-\bar{\gamma}^k\otimes\tilde{Z}'_j+\bar{\gamma}'^k\otimes\tilde{Z}_j.$$

Table 4. *Connection* $\tilde{D}^t = D - S^t$.

	A_0^{H}	U^{H}	Z_j^{H}	$Z_j'^{\mathrm{H}}$	ξ
$\tilde{D}^t_{A_0^{\mathrm{H}}}$	0	0	0	0	0
$\tilde{D}^t_{U^{\mathrm{H}}}$	$-2U^{\mathrm{H}}$	$2A_0^{\mathrm{H}}$	$Z_j'^{\mathrm{H}}$	$-Z_j^{\mathrm{H}}$	0
$\tilde{D}^t_{Z_k^{\mathrm{H}}}$	$-Z_k^{\mathrm{H}}$	$Z_k'^{\mathrm{H}}$	$\delta_{kj}A_0^{\mathrm{H}}$	$-\delta_{kj}U^{\mathrm{H}}$	0
$\tilde{D}^t_{Z_k'^{\mathrm{H}}}$	$-Z_k'^{\mathrm{H}}$	$-Z_k^{\mathrm{H}}$	$\delta_{kj}U^{\mathrm{H}}$	$\delta_{kj}A_0^{\mathrm{H}}$	0
$\tilde{D}^t_{\xi}$	$(1-t)U^{\mathrm{H}}$	$(t-1)A^{\mathrm{H}}$	$(t-1)Z_j'^{\mathrm{H}}$	$(1-t)Z_j^{\mathrm{H}}$	0

(If $n = 2$, the operators σ_{jk} and τ_{jk} do not appear, that is, $\tilde{\mathfrak{h}}_t = \langle \rho_0, \rho_1, \varphi_1, \psi_1 \rangle$, and if $n = 1$, then $\tilde{\mathfrak{h}}_t$ is generated by $\rho_0 = \tilde{R}^t_{\tilde{A}\tilde{U}} = 2(t-3)(\bar{\alpha} \otimes \tilde{U} - \bar{\beta} \otimes \tilde{A})$.) The Lie structure of $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \bar{\mathfrak{m}}$ is defined by Equations (2.3). If $t \neq (2n+1)/n$, the subalgebra $\tilde{\mathfrak{h}}_t$ is isomorphic to the Lie algebra $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) \cong \mathfrak{u}(n)$ in § 3.1, via the map $h : \tilde{\mathfrak{h}}_t \to \mathfrak{k}$ given by $h(\rho_0) = 2U_{\mathfrak{k}}$, $h(\rho_r) = -(C_r + U_{\mathfrak{k}})$, $h(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $h(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $h(\sigma_{jk}) = F_{jk}$, $h(\tau_{jk}) = -H_{jk}$. If we set $\hat{\rho}_0 = \frac{1}{2}(\rho_0 - 2\bar{\xi})$, $\hat{\rho}_r = -\frac{1}{2}\rho_0 - \rho_r - \bar{\xi}$, then

$$\widehat{\mathfrak{su}}(n,1) = \langle \hat{\rho}_0, \hat{\rho}_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}, \tilde{A}, \tilde{U}, \tilde{Z}_r, \tilde{Z}'_r : r, j, k = 1, \dots, n-1, j < k \rangle$$

is an ideal of $\tilde{\mathfrak{g}}_t$, and the map h extends to a Lie algebra isomorphism

$$\tilde{h} : \widehat{\mathfrak{su}}(n,1) \to \mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p},$$

given by $\tilde{h}(\hat{\rho}_0) = U_{\mathfrak{k}}$, $\tilde{h}(\hat{\rho}_r) = C_r$, $\tilde{h}(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $\tilde{h}(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $\tilde{h}(\sigma_{jk}) = F_{jk}$, $\tilde{h}(\tau_{jk}) = -H_{jk}$, $\tilde{h}(\tilde{A}) = A_0$, $\tilde{h}(\tilde{U}) = U_{\mathfrak{p}}$, $\tilde{h}(\tilde{Z}_r) = (Z_r)_{\mathfrak{p}}$, $\tilde{h}(\tilde{Z}'_r) = (Z'_r)_{\mathfrak{p}}$. Moreover, $\tilde{\mathfrak{g}}_t$ is the semidirect product of $\widehat{\mathfrak{su}}(n,1)$ and the line generated by $\bar{\xi}$ under the homomorphism

$$\delta_t : \langle \bar{\xi} \rangle \to \operatorname{Der}(\widehat{\mathfrak{su}}(n,1)),$$

given by $\delta_t(\bar{\xi})(\tilde{A}) = (t-1)\tilde{U}$, $\delta_t(\bar{\xi})(\tilde{U}) = (1-t)\tilde{A}$, $\delta_t(\bar{\xi})(\tilde{Z}_r) = (1-t)\tilde{Z}'_r$, $\delta_t(\bar{\xi})(\tilde{Z}'_r) = (t-1)\tilde{Z}_r$, and $\delta_t(\bar{\xi})(\langle \hat{\rho}_0, \hat{\rho}_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk} \rangle) = 0$. So, we have the following.

Proposition 3.11. *The reductive decomposition associated to the homogeneous Sasakian structure* S^t, $t \neq (2n+1)/n$, *on the total space of the line bundle* $\bar{M} \to \mathbb{C}\mathrm{H}(n)$ *is* $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \bar{\mathfrak{m}}$, *where* $\tilde{\mathfrak{h}}_t \cong \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) \cong \mathfrak{u}(n) \subset \mathfrak{su}(n,1)$, *and*

$$\bar{\mathfrak{m}} = \mathfrak{p} + \langle \bar{\xi} \rangle = \langle A_0, U_{\mathfrak{p}}, (Z_r)_{\mathfrak{p}}, (Z'_r)_{\mathfrak{p}}, \bar{\xi} : 1 \leqslant r \leqslant n-1 \rangle.$$

Moreover, $\tilde{\mathfrak{g}}_t$ *is the semidirect product* $\tilde{\mathfrak{g}}_t = \langle \bar{\xi} \rangle \ltimes_{\delta_t} \mathfrak{su}(n,1)$, *where* $\delta_t(\bar{\xi})(A_0) = (t-1)U_{\mathfrak{p}}$, $\delta_t(\bar{\xi})(U_{\mathfrak{p}}) = (1-t)A_0$, $\delta_t(\bar{\xi})((Z_r)_{\mathfrak{p}}) = (1-t)(Z'_r)_{\mathfrak{p}}$, $\delta_t(\bar{\xi})((Z'_r)_{\mathfrak{p}}) = (t-1)(Z_r)_{\mathfrak{p}}$, *and* $\delta_t(\bar{\xi})(\tilde{\mathfrak{h}}_t) = 0$.

If $n \geqslant 2$ and $t = (2n+1)/n$, then it is easy to see that $\rho_0 = \rho_1 + \cdots + \rho_{n-1}$, and we set $\tilde{\rho}_r = \frac{1}{2}(\rho_0 + \rho_r)$, $1 \leqslant r \leqslant n-1$. In this case, $\tilde{\mathfrak{g}}_{(2n+1)/n} = \tilde{\mathfrak{h}}_{(2n+1)/n} + \bar{\mathfrak{m}}$ coincides with the reductive decomposition $\mathfrak{su}(n,1) = \mathfrak{k}' + \mathfrak{m}'$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(n)$, and $\mathfrak{m}' = \mathfrak{p} + \langle \mathfrak{c} \rangle$,

$\mathfrak{c}$ being the centre of $\mathfrak{k}$, which is generated by the element E_J such that $\mathrm{ad}_{E_J} : \mathfrak{p} \to \mathfrak{p}$ defines the complex structure of $\mathbb{C}\mathrm{H}(n)$. In fact, we have the isomorphism

$$f : \tilde{\mathfrak{g}}_{(2n+1)/n} \to \mathfrak{su}(n,1)$$

given by $f(\tilde{\rho}_r) = \frac{1}{2}(U_{\mathfrak{k}} - C_r)$, $f(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $f(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $f(\sigma_{jk}) = F_{jk}$, $f(\tau_{jk}) = -H_{jk}$, $f(\tilde{A}) = A_0$, $f(\tilde{U}) = U_{\mathfrak{p}}$, $f(\tilde{Z}_r) = (Z_r)_{\mathfrak{p}}$, $f(\tilde{Z}'_r) = (Z'_r)_{\mathfrak{p}}$ and

$$f(\bar{\xi}) = -\frac{n+1}{n} E_J = -\frac{1}{2n}(C_1 + \cdots + C_{n-1} + (n+1)U_{\mathfrak{k}})$$

and, in particular, $f(\tilde{\mathfrak{h}}_{(2n+1)/n}) = \mathfrak{k}'$ and $f(\bar{\mathfrak{m}}) = \mathfrak{m}'$. If $n = 1$ and $t = 3$, then $\rho_0 = 0$. In this case, $\tilde{\mathfrak{h}}_3 = 0$, $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = 0$, $\mathfrak{c} = \langle E_J \rangle$, $E_J = \frac{1}{2}U_{\mathfrak{k}}$, $\tilde{\mathfrak{g}}_3 = \{0\} + \bar{\mathfrak{m}}$ is the reductive decomposition $\mathfrak{su}(1,1) = \{0\} + \mathfrak{m}'$, where $\bar{\mathfrak{m}} = \langle \tilde{A}, \tilde{U}, \bar{\xi} \rangle$, $\mathfrak{m}' = \langle A_0, U_{\mathfrak{p}}, U_{\mathfrak{k}} \rangle$, and $f : \tilde{\mathfrak{g}}_3 \to \mathfrak{su}(1,1)$ such that $f(\tilde{A}) = A_0$, $f(\tilde{U}) = U_{\mathfrak{p}}$, $f(\bar{\xi}) = -U_{\mathfrak{k}}$. Hence, we have obtained the following.

Proposition 3.12. *The reductive decomposition associated to the homogeneous Sasakian structure* S^t, *with* $t = (2n+1)/n$, *on the total space of the line bundle* $\bar{M} \to \mathbb{C}\mathrm{H}(n)$ *is* $\mathfrak{su}(n,1) = \mathfrak{k}' + \mathfrak{m}'$, *where* $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(n)$ *and* $\mathfrak{m}' = \mathfrak{p} + \mathfrak{c}$, $\mathfrak{c} = \langle E_J \rangle$ *being the centre of* $\mathfrak{k}$.

Remark 3.13. The reductive decomposition $\mathfrak{su}(n,1) = \mathfrak{k}' + \mathfrak{m}'$ associated to the homogeneous Sasakian structure S^t, with $t = (2n+1)/n$, provides the description of $\bar{M}$ as the homogeneous space $\widetilde{\mathrm{SU}}(n,1)/K'$, where $\widetilde{\mathrm{SU}}(n,1)$ is the universal covering of $\mathrm{SU}(n,1)$, and $K' \cong \mathrm{SU}(n)$ is the connected subgroup of $\widetilde{\mathrm{SU}}(n,1)$ whose Lie algebra is $\mathfrak{k}' \cong \mathfrak{su}(n)$. (In particular, if $n = 1$, $\bar{M}$ is the universal covering space of $Sl(2,\mathbb{R})$.) These spaces appear in the classification by Jiménez and Kowalski [**17**] of complete simply connected φ-symmetric Sasakian manifolds, and they are also Sasakian space forms (they have constant φ-sectional curvature -7). Notice that for a Sasakian manifold the condition of being a locally symmetric space is too strong, because in this case it is a space of constant curvature [**25**]. For this reason, Takahashi [**27**] introduced φ-symmetric spaces in Sasakian geometry as generalizations of Sasakian space forms. They are also analogues of Hermitian symmetric spaces. A φ-symmetric space is a complete connected regular Sasakian manifold $\bar{M}$ that fibres over a Hermitian symmetric space M so that the geodesic involutions of M lift to involutive automorphisms of the Sasakian structure on $\bar{M}$. Moreover, each complete simply connected φ-symmetric space is a naturally reductive homogeneous space [**5**].

Acknowledgements. This research was partly supported by the Ministry of Science and Innovation, Spain, under Project no. MTM2008-01386.

References

1. E. Abbena and S. Garbiero, Almost-Hermitian homogeneous structures, *Proc. Edinb. Math. Soc.* **31** (1988), 375–395.
2. W. Ambrose and I. M. Singer, On homogeneous Riemannian manifolds, *Duke Math. J.* **25** (1958), 647–669.

3. D. E. Blair, *Contact manifolds in Riemannian geometry* (Springer, 1976).
4. D. E. Blair, *Riemannian geometry of contact and symplectic manifolds* (Birkhäuser, 2002).
5. D. E. Blair and L. Vanhecke, New characterizations of φ-symmetric spaces, *Kōdai Math. J.* **10** (1987), 102–107.
6. W. M. Boothby and H. C. Wang, On contact manifolds, *Annals Math.* **68** (1958), 721–734.
7. M. Castrillón López, P. M. Gadea and A. F. Swann, Homogeneous structures on real and complex hyperbolic spaces, *Illinois J. Math.* **53**(2) (2009), 561–574.
8. D. Chinea and C. González, An example of an almost cosymplectic homogeneous manifold, in *Differential geometry*, Lecture Notes in Mathematics, Volume 1209, pp. 133–142 (Springer, 1986).
9. B. de Wit and P. van Nieuwenhuizen, Rigidly and locally supersymmetric two-dimensional nonlinear σ-models with torsion, *Nucl. Phys.* B **312** (1989), 58–94.
10. A. Díaz Miranda and A. Reventós, Homogeneous contact compact manifolds and homogeneous symplectic manifolds, *Bull. Sci. Math.* **106**(4) (1982), 337–350.
11. J. Dorfmeister and K. Nakajima, The fundamental conjecture for homogeneous Kähler manifolds, *Acta Math.* **161** (1988), 23–70.
12. A. Fino, Almost contact homogeneous structures, *Boll. UMI* A **9** (1995), 299–311.
13. P. M. Gadea and J. A. Oubiña, Reductive homogeneous pseudo-Riemannian manifolds, *Monatsh. Math.* **124** (1997), 17–34.
14. P. M. Gadea, A. Montesinos Amilibia and J. Muñoz Masqué, Characterizing the complex hyperbolic space by Kähler homogeneous structures, *Math. Proc. Camb. Phil. Soc.* **27** (2000), 87–94.
15. C. González and D. Chinea, Quasi-Sasakian homogeneous structures on the generalized Heisenberg group $H(p, 1)$, *Proc. Am. Math. Soc.* **105** (1989), 173–184.
16. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, New York, 1978).
17. J. Jiménez and O. Kowalski, The classification of φ-symmetric Sasakian manifolds, *Monatsh. Math.* **115** (1993), 83–98.
18. V. F. Kiričenko, On homogeneous Riemannian spaces with an invariant structure tensor, *Sov. Math. Dokl.* **21** (1980), 734–737.
19. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Volume I (Wiley, 1963).
20. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Volume II (Wiley, 1969).
21. T. Koda and Y. Watanabe, Homogeneous almost contact Riemannian manifolds and infinitesimal models, *Boll. UMI* B **11** (supplement) (1997), 11–24.
22. H. Nishino, Alternative $N = (4, 0)$ superstring and σ-models, *Phys. Lett.* B **355** (1995), 117–126.
23. K. Nomizu, Invariant affine connections on homogeneous spaces, *Am. J. Math.* **76** (1954), 33–65.
24. K. Ogiue, On fiberings of almost contact manifolds, *Kōdai Math. Sem. Rep.* **17** (1965), 53–62.
25. M. Okumura, Some remarks on spaces with a certain contact structure, *Tohoku Math. J.* **14** (1962), 135–145.
26. K. Sekigawa, Notes on homogeneous almost Hermitian manifolds, *Hokkaido Math. J.* **7** (1978), 206–213.
27. T. Takahashi, Sasakian φ-symmetric spaces, *Tohoku Math. J.* **29** (1997), 91–113.
28. F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds* (Cambridge University Press, 1983).