The Solutions of the Differential Equations

$$\left\{\cos\left(\lambda \, . \, \frac{d}{dx}\right)\right\} . \, y = f(x) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

$$\left\{\sin\left(\lambda \, . \, \frac{d}{dx}\right)\right\} . \, y = \phi(x) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

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These equations are solved by the ordinary methods applicable to linear equations with constant coefficients.

The complementary function of equation (1) is the primitive of

$$\left\{\cos\left(\lambda \, . \, \frac{d}{dx}\right)\right\} . \, \mathrm{Y} = 0 \qquad \ldots \qquad \ldots \qquad . \, . \qquad (3)$$

Now $\mathrm{Y} = e^{m \, . \, x}$ will be a particular solution of (3) if m be such as to make $\cos(\lambda \, . \, m) = 0$.

The required values of m are found by giving to r all positive integral values in succession, from zero to infinity in $\left(r + \frac{1}{2}\right)\frac{\pi}{\lambda}$ and $-\left(r + \frac{1}{2}\right)\frac{\pi}{\lambda}$. Thus $e^{(r+\frac{1}{2})\frac{\pi}{\lambda}}$ and $e^{-(r+\frac{1}{2})\frac{\pi}{\lambda}}$ are each particular integrals of equation (3) r having any integral value from 0 to $+\infty$.

Hence the solution of equation (3) is

$$\mathrm{Y} = \sum_{r=0}^{r=\infty} \left\{ \mathrm{A}_r e^{(r+\frac{1}{2})\frac{\pi x}{\lambda}} + \mathrm{B}_r e^{-(r+\frac{1}{2})\frac{\pi x}{\lambda}} \right\} \qquad \ldots \qquad \ldots \qquad (4)$$

The expression on the right side of the above equation is the complementary function of equation (1).

In order to find a particular integral of the equation we write it in the form

$$y = \left\{\frac{1}{\cos(\lambda \, . \, \mathrm{D})}\right\} . \, f(x) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5)$$

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Decomposing the operating function into partial fractions we obtain

$$\frac{1}{\cos\lambda\,.\,\mathrm{D}}=\left(\frac{1}{\lambda\,.\,\mathrm{D}+\frac{\pi}{2}}-\frac{1}{\lambda\,.\,\mathrm{D}-\frac{\pi}{2}}\right)-\left(\frac{1}{\lambda\,.\,\mathrm{D}+\frac{3\pi}{2}}-\frac{1}{\lambda\,.\,\mathrm{D}-\frac{3\pi}{2}}\right)+\ldots \text{ to infin.}$$

$$=\sum_{r=0}^{r=\infty}\frac{(-1)^r}{\lambda}\,.\left\{\frac{1}{\mathrm{D}+(r+\frac{1}{2})\frac{\pi}{\lambda}}-\frac{1}{\mathrm{D}-(r+\frac{1}{2})\frac{\pi}{\lambda}}\right\}$$

Equation (5) may now be written

$$y=\sum_{r=0}^{r=\infty}\frac{(-1)^r}{\lambda}\,.\left\{\frac{1}{\mathrm{D}+(r+\frac{1}{2})\,.\,\frac{\pi}{\lambda}}-\frac{1}{\mathrm{D}-(r+\frac{1}{2})\,.\,\frac{\pi}{\lambda}}\right\}.f(x) \qquad \ldots \quad (6)$$

Using the general theorem $\mathrm{D}^{-1}\{e^{ax}\mathrm{X}\}=e^{ax}\,.\,\frac{1}{\mathrm{D}+a}\,.\,\mathrm{X}$ equation (6) becomes

$$y=\sum_{r=0}^{r=\infty}\frac{(-1)^r}{\lambda}\left\{e^{-(r+\frac{1}{2})\,.\,\frac{\pi\,.\,x}{\lambda}}\int e^{(r+\frac{1}{2})\,.\,\frac{\pi\,.\,x}{\lambda}}\,.\,f(x)dx-e^{(r+\frac{1}{2})\frac{\pi\,.\,x}{\lambda}}\right.$$

$$\left.\int e^{-(r+\frac{1}{2})\frac{\pi\,.\,x}{\lambda}}\,.\,f(x)dx\right\}$$

This is the particular solution of equation (1), and the complete solution is found by adding the complementary function to the expression on the right side of the above equation.

In the equation $\left\{\sin\left(\lambda\,.\,\frac{d}{dx}\right)\right\}y=\phi(x)$, the solution takes a simpler form than in the first equation.

The complementary function is the primitive of

$$\left\{\sin\left(\lambda\,.\,\frac{d}{dx}\right)\right\}\mathrm{Y}=0\,;$$

$y=e^{m\,.\,x}$ will be a particular solution of this if m be such as to make $\sin(\lambda\,.\,m)=0$. The required values of m are found by giving to r all integral values (including zero) from $+\infty$ to $-\infty$ in $\frac{r\,.\,\pi\,.}{\lambda}$

Hence $Y = \sum_{r=-\infty}^{r=+\infty} \left\{ A_r e^{\frac{r\pi x}{\lambda}} \right\}$ the expression on the right side of this equation is the complementary function in the solution of the second equation.

To find the particular integral we write equation (2) in the form $y = \left\{ \frac{1}{\sin(\lambda \cdot D)} \right\} \phi(x)$ and decompose the operator into partial fractions.

$$\frac{1}{\sin\lambda \cdot D} = \frac{1}{\lambda \cdot D} - \frac{1}{\lambda . D - \pi} - \frac{1}{\lambda \cdot D + \pi} + \frac{1}{\lambda \cdot D - 2\pi} + \frac{1}{\lambda . D + 2\pi} - \text{etc. to infinity}$$

$$= \sum_{r=-\infty}^{r=+\infty} \frac{(-1)^r}{\lambda} \frac{1}{D + \frac{r\pi}{\lambda}}$$

$$\therefore \quad y = \sum_{r=-\infty}^{r=+\infty} \left\{ \frac{(-1)^r}{\lambda} \cdot \frac{1}{D + \frac{r\pi}{\lambda}} \right\} \cdot \phi(x)$$

$$= \sum_{r=-\infty}^{r=+\infty} \frac{(-1)^r}{\lambda} e^{\frac{-r\pi x}{\lambda}} \int e^{\frac{r\pi x}{\lambda}} \phi(x) dx$$

The complete solution is found by adding the complementary function to the expression on the right of the above equation.

The solution of the equation (2) is therefore

$$y = \sum_{r=-\infty}^{r=+\infty} \left\{ A_r e^{\frac{r\pi x}{\lambda}} + \frac{(-1)^r}{\lambda} e^{\frac{-r\pi x}{\lambda}} \int e^{\frac{r\pi x}{\lambda}} \phi(x) dx \right\} \quad \ldots \quad \ldots \quad (7)$$

and the complete solution of equation (1) is

$$y = \sum_{r=0}^{r=\infty} \left\{ A_r e^{(r+\frac{1}{2})\frac{\pi . x}{\lambda}} + B_r e^{-(r+\frac{1}{2})\frac{\pi . x}{\lambda}} + \frac{(-1)^r}{\lambda} e^{-(r+\frac{1}{2})\frac{\pi . x}{\lambda}} \cdot \right.$$
$$\left. \int e^{(r+\frac{1}{2}) \cdot \frac{\pi x}{\lambda}} \phi(x) dx - \frac{(-1)^r}{\lambda} e^{(r+\frac{1}{2}) \cdot \frac{\pi . x}{\lambda}} \int e^{-(r+\frac{1}{2}) \cdot \frac{\pi . x}{\lambda}} \phi(x) dx \right\} \quad \ldots \quad (8)$$