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ON THE CHARACTERISATION OF ALTERNATING GROUPS BY CODEGREES

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Abstract

Let G be a finite group and $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G. Define the codegree of $\chi \in \mathrm{Irr}(G)$ as $\mathrm{cod}(\chi) := |G : \ker(\chi)|/\chi(1)$ and let $\mathrm{cod}(G) := \{\mathrm{cod}(\chi) \mid \chi \in \mathrm{Irr}(G)\}$ be the codegree set of G. Let A_n be an alternating group of degree $n \geq 5$. We show that A_n is determined up to isomorphism by $\mathrm{cod}(\mathrm{A}_n)$.

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1. Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G. For any $\chi \in \mathrm{Irr}(G)$, define the codegree of χ as $\mathrm{cod}(\chi) := |G : \ker(\chi)|/\chi(1)$ and the codegree set of G as $\mathrm{cod}(G) := \{\mathrm{cod}(\chi) \mid \chi \in \mathrm{Irr}(G)\}$. We refer the reader to the authors' previous paper [8] for the current literature on codegrees.

The following conjecture appears in the *Kourovka Notebook of Unsolved Problems in Group Theory* [12, Question 20.79].

CODEGREE VERSION OF HUPPERT'S CONJECTURE. Let H be a finite nonabelian simple group and G a finite group such that $\mathrm{cod}(H) = \mathrm{cod}(G)$. Then $G \cong H$.

In [8], the authors verified the conjecture for all sporadic simple groups. In this paper, we provide a general proof verifying this conjecture for all alternating groups of degree greater than or equal to 5.

THEOREM 1.1. *Let* A_n *be an alternating group of degree* $n \geq 5$ *and* G *a finite group. If* $\mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n)$, *then* $G \cong \mathrm{A}_n$.

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Throughout the paper, we follow the notation used in Isaacs' book [10] and the ATLAS of Finite Groups [6].

2. Proof of Theorem 1.1

First, we note that the cases $n = 5, 6$ and 7 have already been proven in [1, 2], so in the following, we always assume that $n > 7$. Now, let G be a minimal counterexample and N be a maximal normal subgroup of G. So $\mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n)$ and G/N is simple. By [8, Lemma 2.5], $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(\mathrm{A}_n)$. Then, by [9, Theorem B], $G/N \cong \mathrm{A}_n$ so $N \neq 1$ since $G \ncong \mathrm{A}_n$.

Step 1: N is a minimal normal subgroup of G.

Suppose L is a nontrivial normal subgroup of G with $L < N$. Then by [8, Lemma 2.6], $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(G/L) \subseteq \mathrm{cod}(G)$. However, $\mathrm{cod}(G/N) = \mathrm{cod}(\mathrm{A}_n) = \mathrm{cod}(G)$, so equality must be attained in each inclusion. Thus, $\mathrm{cod}(G/L) = \mathrm{cod}(\mathrm{A}_n)$ which implies that $G/L \cong \mathrm{A}_n$ since G is a minimal counterexample. This is a contradiction since we also have $G/N \cong \mathrm{A}_n$, but $L < N$.

Step 2: N is the only nontrivial, proper normal subgroup of G.

Otherwise, we assume M is another proper nontrivial normal subgroup of G. If N is included in M, then $M = N$ or $M = G$ since G/N is simple, which is a contradiction. Then $N \cap M = 1$ and $G = N \times M$. Since M is also a maximal normal subgroup of G, we have $N \cong M \cong \mathrm{A}_n$. Choose $\psi_1 \in \mathrm{Irr}(N)$ and $\psi_2 \in \mathrm{Irr}(M)$ such that $\mathrm{cod}(\psi_1) = \mathrm{cod}(\psi_2) = \max(\mathrm{cod}(\mathrm{A}_n))$. Set $\chi = \psi_1 \cdot \psi_2 \in \mathrm{Irr}(G)$. Then $\mathrm{cod}(\chi) = (\max(\mathrm{cod}(\mathrm{A}_n)))^2 \notin \mathrm{cod}(G)$, which is a contradiction.

Step 3: χ is faithful, for each nontrivial $\chi \in \mathrm{Irr}(G|N) := \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

From the proof of [8, Lemma 2.5],

$$\mathrm{Irr}(G/N) = \{\hat{\chi}(gN) = \chi(g) \mid \chi \in \mathrm{Irr}(G) \text{ and } N \leq \ker(\chi)\}.$$

By the definition of $\mathrm{Irr}(G|N)$, it follows that if $\chi \in \mathrm{Irr}(G|N)$, then $N \nleq \ker(\chi)$. Thus, since N is the unique nontrivial, proper, normal subgroup of G, $\ker(\chi) = G$ or $\ker(\chi) = 1$. Therefore, $\ker(\chi) = 1$ for all nontrivial $\chi \in \mathrm{Irr}(G|N)$.

Step 4: N is an elementary abelian group.

Suppose that N is not abelian. Since N is a minimal normal subgroup, by [7, Theorem 4.3A(iii)], $N = S^n$, where S is a nonabelian simple group and $n \in \mathbb{Z}^+$. By [14, Lemma 4.2] and [11, Theorem 4.3.34], there is a nontrivial character $\chi \in \mathrm{Irr}(N)$ which extends to some $\psi \in \mathrm{Irr}(G)$. Now, $\ker(\psi) = 1$ by Step 3, so $\mathrm{cod}(\psi) = |G|/\psi(1) = |G/N| \cdot |N|/\chi(1)$. However, by assumption, $\mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n) = \mathrm{cod}(G/N)$. Thus, $\mathrm{cod}(\psi) \in \mathrm{cod}(G) = \mathrm{cod}(G/N)$, so $\mathrm{cod}(\psi) = |G/N|/\phi(1)$ for some $\phi \in \mathrm{Irr}(G/N)$. Hence, $|G/N|$ is divisible by $\mathrm{cod}(\psi)$ which contradicts the fact that $\mathrm{cod}(\psi) = |G/N| \cdot |N|/\chi(1)$, as $\chi(1) \neq |N|$. Thus, N must be abelian.

Now to show that N is elementary abelian, let a prime p divide $|N|$. Then N has a p-Sylow subgroup K, and K is the unique p-Sylow subgroup of N since N is abelian, so K is characteristic in N. Thus, K is a normal subgroup of G, so $K = N$ as N is minimal. Thus, $|N| = p^n$. Now, take the subgroup $N^p = \{n^p \mid n \in N\}$ of N, which is proper by Cauchy's theorem. Since N^p is characteristic in N, it must be normal in G, so N^p is trivial by the uniqueness of N. Thus, every element of N has order p and N is elementary abelian.

Step 5: $\mathbf{C}_G(N) = N$.

First note that since N is normal, $\mathbf{C}_G(N) \trianglelefteq G$. Additionally, since N is abelian by Step 4, $N \leq \mathbf{C}_G(N)$. By the maximality of N, we must have $\mathbf{C}_G(N) = N$ or $\mathbf{C}_G(N) = G$. If $\mathbf{C}_G(N) = N$, we are done.

If not, then $\mathbf{C}_G(N) = G$, so N must be in the centre of G. Then since N is the unique minimal normal subgroup of G by Step 2, $|N|$ must be prime. If not, there always exists a proper nontrivial subgroup K of N, and K is normal since it is contained in $\mathbf{Z}(G)$, contradicting the minimality of N. Hence, we have $N \leq \mathbf{Z}(G)$ which implies that $\mathbf{Z}(G) \cong N$. This is because N is a maximal normal subgroup of G so if not, we would have $\mathbf{Z}(G) = G$, implying G is abelian which is a contradiction. Thus, N is isomorphic to a subgroup of the Schur multiplier of G/N by [10, Corollary 11.20].

Now, we note that it is well known that for $n > 7$, the Schur multiplier of A_n is $\mathbb{Z}_2$, so $G \cong 2.\mathrm{A}_n$ [17]. From [3, Theorem 4.3], $2.\mathrm{A}_n$ always has a faithful irreducible character χ of degree $2^{\lfloor (n-2)/2 \rfloor}$. Recall that by Step 2, there is only one nontrivial proper normal subgroup of $G \cong 2.\mathrm{A}_n$. In particular, $N \cong \mathbb{Z}_2$ is the only nontrivial proper normal subgroup of G. Thus, $|\ker(\chi)| = 1$ or 2. Then $\mathrm{cod}(\chi) = |2.\mathrm{A}_n : \ker(\chi)|/\chi(1)$. If $|\ker(\chi)| = 1$, then $\mathrm{cod}(\chi) = n!/2^{\lfloor (n-2)/2 \rfloor}$, and if $|\ker(\chi)| = 2$, then $\mathrm{cod}(\chi) = (n!/2)/2^{\lfloor (n-2)/2 \rfloor} = n!/2^{\lfloor n/2 \rfloor}$. In either case, for any prime $p \neq 2$, $|\mathrm{cod}(\chi)|_p = |n!|_p = |\mathrm{A}_n|_p$. However, $\mathrm{cod}(\chi) \in \mathrm{cod}(\mathrm{A}_n)$ since $\mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n)$. Therefore, there is a character degree of A_n which is a power of 2.

However, from [13], for $n > 7$, A_n only has a character degree equal to a power of 2 when $n = 2^d + 1$ for some positive integer d. In this case, $2^d = n - 1 \in \mathrm{cd}(\mathrm{A}_n)$ so we need $|\mathrm{A}_n|/n - 1 = |2.\mathrm{A}_n|/2^{\lfloor (n-2)/2 \rfloor}$ or $|2.\mathrm{A}_n|/2^{\lfloor n/2 \rfloor}$. Hence,

$$\frac{1}{n-1} = \frac{2}{2^{\lfloor (n-2)/2 \rfloor}} = \frac{1}{2^{\lfloor (n-2)/2 \rfloor - 1}} \quad \text{or} \quad \frac{1}{2^{\lfloor n/2 \rfloor - 1}}$$

so $n - 1 = 2^{\lfloor (n-2)/2 \rfloor - 1}$ or $2^{\lfloor n/2 \rfloor - 1}$. However, the only integer solution to either of these equations occurs when $n = 9$ and $9 - 1 = 8 = 2^3 = 2^{\lfloor 9/2 \rfloor - 1}$. In this case, we check the ATLAS [6] to find that the codegree sets of A_9 and $2.A_9$ do not have the same order. This is a contradiction, so $\mathbf{C}_G(N) = N$.

Step 6. Let λ be a nontrivial character in $\mathrm{Irr}(N)$ and $\vartheta \in \mathrm{Irr}(I_G(\lambda)|\lambda)$, the set of irreducible constituents of $\lambda^{I_G(\lambda)}$, where $I_G(\lambda)$ is the inertia group of λ in G. Then $|I_G(\lambda)|/\vartheta(1) \in \mathrm{cod}(G)$. Also, $\vartheta(1)$ divides $|I_G(\lambda)/N|$ and $|N|$ divides $|G/N|$. Lastly, $I_G(\lambda) < G$, that is, λ is not G-invariant.

Let λ be a nontrivial character in $\mathrm{Irr}(N)$ and $\vartheta \in \mathrm{Irr}(I_G(\lambda)|\lambda)$. Let χ be an irreducible constituent of ϑ^G. By [10, Corollary 5.4], $\chi \in \mathrm{Irr}(G)$, and by [10, Definition 5.1], we have $\chi(1) = (|G|/|I_G(\lambda)|) \cdot \vartheta(1)$. Moreover, $\ker(\chi) = 1$ by Step 2, and thus

$\text{cod}(\chi) = |G|/\chi(1) = |I_G(\lambda)|/\vartheta(1)$, so $|I_G(\lambda)|/\vartheta(1) \in \text{cod}(G)$. Now, since N is abelian, $\lambda(1) = 1$, so we have $\vartheta(1) = \vartheta(1)/\lambda(1)$ which divides $|I_G(\lambda)|/|N|$, so $|N|$ divides $|I_G(\lambda)|/\vartheta(1)$. Moreover, $\text{cod}(G) = \text{cod}(G/N)$, and all elements in $\text{cod}(G/N)$ divide $|G/N|$, so $|N|$ divides $|G/N|$.

Next, we want to show $I_G(\lambda)$ is a proper subgroup of G. To reach a contradiction, assume $I_G(\lambda) = G$. Then $\ker(\lambda) \trianglelefteq G$. From Step 2, $\ker(\lambda) = 1$, and from Step 4, N is a cyclic group of prime order. Thus, by the Normaliser–Centraliser theorem, $G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \leq \text{Aut}(N)$ so G/N is abelian, which is a contradiction.

Step 7: Final contradiction.

From Step 4, N is an elementary abelian group of order p^m for some prime p and integer $m \geq 1$. By the Normaliser–Centraliser theorem, $\mathrm{A}_n \cong G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \leq \text{Aut}(N)$ and $m > 1$. Note that in general, $\text{Aut}(N) \cong \text{GL}(m, p)$. By Step 6, $|N|$ divides $|G/N|$, so $|N| = p^m$ divides $|\mathrm{A}_n|$ and $G/N \cong \mathrm{A}_n \lesssim \text{GL}(m, p)$. We prove by contradiction that this cannot occur.

First, we claim that if p^m divides $|\mathrm{A}_n|$ and $\mathrm{A}_n \lesssim (\text{GL}(m, p)$, then p must equal 2. To show this, we note that for $p > 2$, by [4], if p^m divides $|\mathrm{A}_n|$, then $m < n/2$. However, by [16, Theorem 1.1], if $n > 6$, the minimal faithful degree of a modular representation of A_n over a field of characteristic p is at least $n - 2$. Since embedding A_n as a subgroup of $\text{GL}(m, p)$ is equivalent to giving a faithful representation of degree m over a field of characteristic p, we have $m \geq n - 2$. This is a contradiction since $n/2 > n - 2$ implies $n < 4$. Therefore, $p = 2$.

Now, let $p = 2$. As above, from [4], we obtain $|n!|_2 \leq 2^{n-1}$. Thus, if 2^m divides $|\mathrm{A}_n|$, then $2^m \leq |\mathrm{A}_n|_2 \leq 2^{n-2}$ so $m \leq n - 2$. We will deal first with $n > 8$ and then treat the case $n = 8$ later. For $n > 8$, [15, Theorem 1.1] shows that the minimal faithful degree of a modular representation of A_n over a field of characteristic 2 is at least $n - 2$. Therefore, we must have $m \geq n - 2$, so we have equality, $m = n - 2$.

Let $\lambda \in \text{Irr}(N), \vartheta \in \text{Irr}(I_G(\lambda)|\lambda)$ and $T := I_G(\lambda)$. Then $1 < |G : T| < |N| = 2^{n-2}$ for $|G : T|$ is the number of all conjugates of λ. By Step 5, $|T|/\vartheta(1) \in \text{cod}(G)$ and moreover $|N|$ divides $|T|/\vartheta(1)$. Since $|N| = |N|_2 = |\mathrm{A}_n|_2$ and $\text{cod}(G) = \text{cod}(\mathrm{A}_n)$, it follows that $||T|/\vartheta(1)|_2 = |N|$. Thus, $||T/N|/\vartheta(1)|_2 = 1$ so the 2-parts of $|T/N|$ and $\vartheta(1)$ are equal. Thus, for every $\vartheta \in \text{Irr}(T \mid \lambda)$, we have $|\vartheta(1)|_2 = |T/N|_2$. However, $|T/N| = \sum_{\vartheta \in \text{Irr}(T|\lambda)} \vartheta(1)^2$. Hence, if $|\vartheta(1)|_2 = 2^k \geq 2$ for every $\vartheta \in \text{Irr}(T \mid \lambda)$, we would have $|T/N|_2 = 2^{2k}$, which contradicts the fact that $|\vartheta(1)|_2 = |T/N|_2$. Therefore, $|T/N|_2 = 1$. Thus, since $|G/N|_2 \geq |N| = 2^{n-2}$, we have $|G : T|_2 = |G/N : T/N|_2 \geq 2^{n-2}$, so $|G : T| \geq 2^{n-2} = |N|$, which is a contradiction.

Now we turn to the case $n = 8$. We have $p = 2$ and $m = 4, 5$ or 6. In this case, $\mathrm{A}_8 \cong \text{GL}(4, 2)$ and 2^6 divides $|\mathrm{A}_8|$. We look at each possibility for m in turn. If $m = 6$, then $|N|_2 = |\mathrm{A}_8|_2$. For this case, the same argument as above holds since $6 = 8 - 2$, and we reach a contradiction.

Second, let $m = 5$. As above, $|G : T| < |N| = 2^5$ and $|T|/\vartheta(1) \in \text{cod}(G)$ such that 2^5 divides $|T|/\vartheta(1)$. Further, $||T/N|/\vartheta(1)|_2 \leq 2$ so $|T/N|_2 \leq 4$ and $|G/N : T/N|_2 \geq 16$. Thus, 16 divides $|G/N : T/N|$ and $|G/N : T/N| < 32$. However, we may check the index of all subgroups of $G/N \cong \mathrm{A}_8$ using [6] and find that none of them satisfy these two properties.

Third, let $m = 4$. Then $G/N \cong \mathrm{A}_8 \cong \mathrm{GL}(4, 2)$ and $N = (\mathbb{Z}_2)^4$ so G is an extension of $\mathrm{GL}(4, 2)$ by N. We may computationally calculate the codegree set for any such group using MAGMA [5]. There are only four such nonisomorphic extensions and we find that none of them have the same codegree set as A_8. (The MAGMA code is available at https://github.com/zachslonim/Characterizing-Alternating-Groups-by-Their-Codegrees.) In every case, $|N| = p^m$ produces a contradiction, so $N = 1$ and $G \cong \mathrm{A}_n$.

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