$$\Sigma\, 1/b_i > c_2 \log n \ .$$

P. Erdős, Israel Institute of Technology

## SOLUTIONS

P 113. If $m > 4$ show that the integral part $n = [(m-1)!/m]$ is an even integer.

D.R. Rao, Secunderabad, India

Solution by A. Makowski, Warszawa, Poland.

If $m$ is prime then by Wilson's theorem $(m-1)! = -1 + km$ where $k$ is odd (since $(m-1)!$ is even). Thus $n = k-1$ is even. If $m = p^2$, then $(m-1)!$ contains as factors $p$ and $2p$, hence $m|(m-1)!$ with even quotient $n$. Otherwise we may write $m = ab$, $1 < a < b$ so again $m|(m-1)!$ Now $(m-1)!$ contains $2.3.4$ and to have $n$ odd we would require $a = 2$, $b = 4$; but then $m = 8$ and $n$ is a multiple of $6$.

Also solved by L. Carlitz, T.M. King and the proposer.

P 115. A set of polynomials $c_n(x)$ which appears in network theory is defined by,

$$c_{n+1}(x) = (x+2).c_n(x) - c_{n-1}(x) \qquad (n \geq 1)$$

with $c_0 = 1$ and $c_1 = (x+2)/2$.

Establish the following properties of $c_n(x)$:

(i) $c_n(x)$ satisfies the differential equation,

$$(x^2 + 4x)y'' + (x+2)y' - n^2 y = 0 \ .$$

(ii) The zeros of $c_n(x)$ are all real, negative and distinct, and these are

$$-4\sin^2\left\{\frac{(2k-1)\pi}{4n}\right\}, \qquad k = 1, 2, \ldots, n .$$

(iii) $c_n(x)$ is an orthogonal function over the interval $(-4, 0)$ with respect to the weighting function $\sqrt{-1/(x^2+4x)}$ .

M.N.S. Swamy, Nova Scotia Technical College

Solution by Louis Weisner, University of New Brunswick.

In the statement of the problem $c_2$ should be replaced by $c_1$ .

The Tchebichev polynomials $T_n(x)$ are defined by

$$T_n(x) = \cos n\theta , \ x = \cos\theta , \ n = 0, 1, 2, \ldots .$$

Thus $T_o(x) = 1$, $T_1(x) = x$ . From the trigonometric identity

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

we obtain the recurrence relations

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) , \qquad n = 1, 2, \ldots .$$

Comparing with the recurrence relations for $c_n(x)$ , we have $c_n(x) = T_n\left(\frac{x+2}{2}\right)$ , $n = 0, 1, 2, \ldots$ . The required properties of $c_n(x)$ are now readily derived from the well-known corresponding properties of $T_n(x)$ . See Polya-Szego, Aufgaben und Lehrsätze aus der Analysis, vol. 2, p.75.

Also solved by W.R. Allaway, D.R. Breach, S. Spital and the proposer.

P 115. Show that any set can be furnished with a compact Hausdorff topology.

B. Thorp, York University

Solution by M. Edelstein, Dalhousie University.

Let X be the given set and suppose $p \in X$ . The family of all subsets of $X - \{p\}$ together with those containing $p$ , whose



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complements are finite, (or empty), is readily seen to be a topology on X as desired.

Remark. Since the discrete topology is locally compact the statement of the problem follows also from the known fact that "every locally compact Hausdorff space can be given a weaker Hausdorff topology which makes it compact". (Cf. A. Wilansky: Functional Analysis, Blaisdell (1964), page 163).

Also solved by B. Thomson, J. Washenberger, J. Wilker and the proposer.



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