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ON SOME REVERSE INTEGRAL INEQUALITIES

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Abstract

We prove the higher integrability of nonnegative decreasing functions, verifying a reverse inequality, and we calculate the optimal integrability exponent for these functions.

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1. Introduction

The aim of this paper is to prove

THEOREM 1.1. *Let* $q > 1$, $C > 1$ *and* $g: (0, m) \to (0, +\infty)$ *be a decreasing function satisfying*

(1.1)
$$\frac{1}{b-a}\int_a^b g^q(x)\,dx \le Cg^{q-1}(b)\frac{1}{b-a}\int_a^b g(x)\,dx \qquad \text{for all } (a, b) \subseteq (0, m).$$

Then, for every $p \in [q, (Cq-1)/(C-1))$ *the function* g *belongs to* $L^p(0, m)$ *and there exists a constant* $K = K(C, p, q)$ *such that*

(1.2)
$$\frac{1}{b-a}\int_a^b g^p(x)\,dx \le K\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^p \qquad \text{for all } (a, b) \subseteq (0, m),$$

with $K(C, p, q) \to \infty$ *as* $p \to (Cq-1)/(C-1)$. *The result is sharp.*

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Since for every decreasing function $g(x)$ we have

$$\frac{1}{b-a}\int_a^b g^q(x)\,dx \geq \frac{g^{q-1}(b)}{b-a}\int_a^b g(x)\,dx \tag{1.3}$$

the inequality (1.1) is a reverse inequality, that is, an inequality which is the inverse of (1.3) but has a constant greater than 1 on the right side. Inequality (1.1) appears, for example, in the paper of Gehring [2], in his remarkable theorem showing higher integrability of a function verifying a reverse Hölder inequality. The upper bound for p is sharp since the function $g(x) = x^{-\beta}$, with $\beta = (C-1)/(Cq-1)$ satisfies (1.1) but fails to belong to $L^p(0, m)$ with $p = (Cq-1)/(C-1)$.

Theorem (1.1) is related to papers [4] and [6], where some sharp results are proved relative to two different reverse inequalities (see, respectively, Lemma 2.3 and Lemma 2.4 of the present paper).

2. Preliminaries

We begin with the classical Hardy's inequality [3].

LEMMA 2.1. *Let $g = g(t)$ belong to $L^q(0, m)$, $q > 1$. Then*

$$\int_0^m \left|\frac{1}{t}\int_0^t g(s)\,ds\right|^q dt \leq \left(\frac{q}{q-1}\right)^q \int_0^m |g(t)|^q\,dt.$$

In the sequel, we shall use the weighted Hardy's inequality [3].

LEMMA 2.2. *Let $f \geq 0$, $F(x) = \frac{1}{x}\int_0^x f(t)\,dt$, $q > 1$, and $\alpha < q-1$. Then*

$$\int_0^m x^\alpha F^q(x)\,dx \leq \left(\frac{q}{q-1-\alpha}\right)^q \int_0^m x^\alpha f^q(x)\,dx.$$

Let us now recall a sharp higher integrability result proved by Muckenhoupt [4].

LEMMA 2.3. *Let $h = h(t)$ be positive and decreasing on $(0, m)$ and assume that there exists $A > 1$ such that*

$$\frac{1}{t}\int_0^t h(s)\,ds \leq Ah(t) \qquad \text{for all } t \in (0, m). \tag{2.1}$$

Then, for every $r \in [1, A/(A-1))$, the function h belongs to $L^r(0, m)$ and

$$\frac{1}{m}\int_0^m h^r(s)\,ds \leq \frac{A}{A - r(A-1)}\left(\frac{1}{m}\int_0^m h(s)\,ds\right)^r. \tag{2.2}$$

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The result is sharp, because the function $f(x) = x^{-\beta}$ *with* $\beta = (A-1)/A$ *satisfies* (2.1) *but fails to belong to* $L^r(0, m)$ *with* $r = A/(A-1)$.

The following sharp result is proved in [6].

LEMMA 2.4. *Let* $B > 1$, *and let* $f\colon (0, m) \to (0, +\infty)$ *be a decreasing function satisfying*

(2.3)
$$\frac{1}{b-a}\int_a^b f^2(x)\,dx \le B\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \quad \text{for all } (a, b) \subseteq (0, m).$$

Set $\varepsilon(B) = \sqrt{B/(B-1)} - 1$. *Then the function* f *belongs to* $L^p(a, b)$ *for* $p \in [2, 2+\varepsilon)$ *and there exists a constant* $c = c(p, B)$ *such that*

$$\frac{1}{b-a}\int_a^b f^p(x)\,dx \le c\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^p \tag{2.4}$$

with $c(p, B) \to \infty$ *as* $p \to 2+\varepsilon(B)$. *The result is sharp, because the function* $f(x) = x^{-\beta}$ *with* $\beta = 1 - B + \sqrt{B^2 - B}$ *satisfies* (2.3) *but fails to belong to* $L^p(a, b)$ *with* $p = 2 + \varepsilon(B)$.

Finally, we quote another result from [3].

LEMMA 2.5. *Let* $f\colon (0, m) \to (0, +\infty)$ *be a decreasing function. Then for* $-1 < \beta < 0$, $(a, b) \subseteq (0, m)$, $q > 1$

$$\left(\int_a^b f^{q/(1+\beta)}\,dt\right)^{1+\beta} \le \int_a^b (t-a)^\beta f^q(t)\,dt.$$

3. A first result of higher integrability

Let $g\colon (0, m] \to (0, +\infty)$ be a nonnegative decreasing function. Then the following inequality holds.

$$\frac{g^{q-1}(t)}{t}\int_0^t g(x)\,dx \le \frac{1}{t}\int_0^t g^q(x)\,dx \tag{3.1}$$

for any $t \in (0, m]$ and for any $q > 1$.

The following theorem shows that if g satisfies an inequality which is reverse to (3.1), then g has higher integrability.

Theorem 3.1. *Let $C > 1$, $q > 1$. Then there exists a positive $\varepsilon = \varepsilon(q, C)$, such that every positive decreasing function g satisfying*

$$\frac{1}{t}\int_0^t g^q(x)\,dx \le \frac{Cg^{q-1}(t)}{t}\int_0^t g(x)\,dx \qquad \text{for all } t \in (0, m] \tag{3.1}$$

belongs to $L^p(0, m)$ for $p \in [q, q+\varepsilon]$, and we have

$$\frac{1}{m}\int_0^m g^p(x)\,dx \le \frac{(Cq)^{(p+q)/q}}{(q-1)^{p/q}[C(q-p)+p(1/q-1)]}\left(\frac{1}{m}\int_0^m g^q(x)\,dx\right)^{p/q}. \tag{3.2}$$

Proof. Let us integrate (3.1) between 0 and $y \in (0, m]$; using the Schwarz-Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}\int_0^y \frac{1}{t}\int_0^t g^q(x)\,dx\,dt &\le C\int_0^y \frac{g^{q-1}(t)}{t}\int_0^t g(x)\,dx\,dt \\ &\le C\left(\int_0^y (g^{q-1}(t))^{q/(q-1)}dt\right)^{(q-1)/q}\left(\int_0^y\left(\frac{1}{t}\int_0^t g(x)\,dx\right)^q dt\right)^{1/q} \\ &\le C\frac{q}{q-1}\left(\int_0^y g^q(x)\,dx\right)^{(q-1)/q}\left(\int_0^y g^q(x)\,dx\right)^{1/q}.\end{aligned} \tag{3.3}$$

Therefore, if we set $h(t) = \frac{1}{t}\int_0^t g^q(x)\,dx$, we have proved that

$$\frac{1}{y}\int_0^y h(t)\,dt \le C\left(\frac{q}{q-1}\right)h(y) \quad \text{for all } y \in (0, m]. \tag{3.4}$$

We can now apply Lemma 2.3 to $h(t)$ and we get

$$\frac{1}{m}\int_0^m h^r(t)\,dt \le \frac{Cq}{Cq - r(Cq-q+1)}\left(\frac{1}{m}\int_0^m h(t)\,dt\right)^r, \qquad r \in \left[1, \frac{Cq}{Cq-q+1}\right). \tag{3.5}$$

Since $g(x)$ is decreasing, we have $g^q(t) \le \frac{1}{t}\int_0^t g^q(x)\,dx$ and it follows from (3.5) that

$$\frac{1}{m}\int_0^m g^{rq}(t)\,dt \le \frac{Cq}{Cq - r(Cq-q+1)}\left(\frac{1}{m}\int_0^m h(t)\,dt\right)^r. \tag{3.6}$$

Since

$$\frac{1}{m}\int_0^m h(t)\,dt \le \frac{Cq}{(q-1)m}\int_0^m g^q(t)\,dt,$$

we finally obtain

$$\frac{1}{m}\int_0^m g^{rq}(t)\,dt \le \frac{(Cq)^{r+1}}{(q-1)^r[Cq - r(Cq-q+1)]}\left(\frac{1}{m}\int_0^m g^q(t)\,dt\right)^r, \tag{3.7}$$

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as desired. This higher integrability result could also be deduced from that proved in [5, Proposition 1], from the fact that g is decreasing; however, the constant in (3.2) is better than that obtained in [5].

4. The integrability exponent

We shall now calculate the "optimal" integrability exponent $p(C, q)$ of a function satisfying the assumptions of Theorem 3.1. With regard to this, we recall that in [6] the best integrability exponent has been calculated for a function g satisfying the reverse Hölder inequality

$$\frac{1}{b-a}\int_a^b g^2\,dx \le B\left(\frac{1}{b-a}\int_a^b g\,dx\right)^2 \tag{4.1}$$

(see Lemma 2.4).

Since g is decreasing, inequality (3.1) for $q = 2$ implies (4.1) with the same constant C. Therefore, we can expect for the functions verifying (3.1) a value of the optimal exponent higher than that one calculated in [6]. Moreover, we are able to calculate this exponent also in the case $q > 2$.

With this aim, we state the following Hardy type inequality.

LEMMA 4.1. *Let* $g\colon (a, b) \to (0, +\infty)$, $q > 1$, $\alpha < q - 1$. *Then,*

$$\int_a^b (x-a)^\alpha g^{q-1}(x)\left(\frac{1}{x-a}\int_a^x g(s)\,ds\right)dx \le \frac{q}{q-1-\alpha}\int_a^b (x-a)^\alpha g^q(x)\,dx. \tag{4.2}$$

PROOF. Using the Schwarz-Hölder inequality and Lemma 2.2 we get

$$\begin{aligned}
&\int_a^b (x-a)^\alpha g^{q-1}(x)\left(\frac{1}{x-a}\int_a^x g(s)\,ds\right)dx \\
&\quad= \int_a^b (x-a)^{(q-1)\alpha/q}(x-a)^{\alpha/q} g^{q-1}(x)\left(\frac{1}{x-a}\int_a^x g\,ds\right)dx \\
&\quad\le \left(\int_a^b \left((x-a)^{(q-1)\alpha/q} g^{q-1}(x)\right)^{q/(q-1)} dx\right)^{(q-1)/q} \\
&\qquad\times \left(\int_a^b \left((x-a)^{\alpha/q}\left(\frac{1}{x-a}\int_a^x g\,ds\right)\right)^q dx\right)^{1/q} \\
&\quad\le \frac{q}{q-1-\alpha}\int_a^b (x-a)^\alpha g^q(x)\,dx,
\end{aligned}$$

that is, we have (4.2).

LEMMA 4.2. *Let* $g\colon (0, m) \to (0, +\infty)$ *satisfy*

(4.3)
$$\frac{1}{b-a}\int_a^b g^q(x)\,dx \le \frac{Cg^{q-1}(b)}{b-a}\int_a^b g(x)\,dx \qquad \text{for all } (a, b) \subset (0, m)$$

and $g \in L^\infty(0, m)$. *Set* $\gamma(\alpha) = 1 + \alpha Cq/(q-1-\alpha)$, *for* $1 - q/(Cq-1) < \alpha < 0$. *Then*

(4.4)
$$\gamma(\alpha)\int_a^b (x-a)^\alpha g^q(x)\,dx \le (b-a)^\alpha \int_a^b g^q(x)\,dx \qquad \text{for all } (a, b) \subset (0, m).$$

PROOF. By (4.3), using the Fubini theorem, we get

$$\begin{aligned}(4.5)\qquad &\int_a^b (x-a)^\alpha g^{q-1}(x)\left(\frac{1}{x-a}\int_a^x g(t)\,dt\right)dx \\ &\ge \frac{1}{C}\int_a^b (x-a)^{\alpha-1}\int_a^x g^q(t)\,dt\,dx \\ &= \frac{1}{C}\int_a^b g^q(t)\int_t^b (x-a)^{\alpha-1}\,dx\,dt \\ &= \frac{1}{\alpha C}\int_a^b g^q(t)[(b-a)^\alpha - (t-a)^\alpha]\,dt.\end{aligned}$$

By Lemma (4.1), from (4.5) it follows that

$$\frac{q}{q-1-\alpha}\int_a^b (x-a)^\alpha g^q(x)\,dx \ge \frac{1}{\alpha C}\int_a^b g^q(x)[(b-a)^\alpha - (x-a)^\alpha]\,dx,$$

and from this, since $\alpha < 0$, we deduce

$$\left(\frac{Cq}{q-1-\alpha} + 1\right)\int_a^b (x-a)^\alpha g^q(x)\,dx \le (b-a)^\alpha \int_a^b g^q(x)\,dx.$$

We eliminate the assumption $g \in L^\infty(0, m)$ in Lemma 4.2, by an approximation argument. First, we prove the following

LEMMA 4.3. *Let* $g = g(t)$ *be a non-negative, decreasing function satisfying*

$$(4.6)\qquad \int_a^b g^q(x)\,dx \le Cg^{q-1}(b)\int_a^b g(x)\,dx \qquad \text{for all } (a, b) \subset (0, m).$$

Then the convolution $G(x) = \eta * g(x) = \int_R \eta(y)g(x-y)\,dy$ *of* g *with a mollifier* η *satisfies* (4.6) *with the same constant* C.

PROOF. Let $x - y = z$; then, from (4.6) we get

$$\int_a^b g^q(x-y)\,dx = \int_{a-y}^{b-y} g^q(z)\,dz \le C g^{q-1}(b-y)\int_{a-y}^{b-y} g(z)\,dz = C g^{q-1}(b-y)\int_a^b g(x-y)\,dx. \tag{4.7}$$

Moreover, from [1] we have

$$\left(\int_R G^q(x)\,dx\right)^{1/q} \le \int_R \eta(y)\left(\int_a^b g^q(x-y)\,dx\right)^{1/q} dy. \tag{4.8}$$

From (4.7), (4.8) and the Schwarz-Hölder inequality it follows that

$$\begin{aligned}
&\left(\int_a^b G^q(x)\,dx\right)^{1/q} \\
&\le C^{1/q}\int_R \eta(y)\left(g^{q-1}(b-y)\int_a^b g(x-y)\,dx\right)^{1/q} dy \\
&= C^{1/q}\int_R \eta^{q-1/q}(y)\eta^{1/q}(y)\left(g^{q-1}(b-y)\int_a^b g(x-y)\,dx\right)^{1/q} dy \\
&\le C^{1/q}\left(\int_R \eta(y) g(b-y)\,dy\right)^{q-1/q}\left(\int_R \eta(y)\int_a^b g(x-y)\,dx\,dy\right)^{1/q},
\end{aligned}$$

and from this the result follows.

PROOF OF THEOREM 1.1. We can write (4.4) for $G_h = \eta_h * g$ and, passing to limit as $h \to \infty$, we get

$$\gamma(\alpha)\int_a^b (x-a)^\alpha g^q(x)\,dx \le (b-a)^\alpha \int_a^b g^q(x)\,dx.$$

From this and Lemma (2.5) it follows that

$$\left(\int_a^b g^{q/1+\alpha}(x)\,dx\right)^{1+\alpha} \le \frac{(b-a)^\alpha}{\gamma(\alpha)}\int_a^b g^q(x)\,dx \quad \text{for } \alpha > \frac{1-q}{Cq-1}.$$

In conclusion, we have obtained that $g \in L^p(0, m)$ with $q \le p < (Cq-1)/(C-1)$. The function $g(x) = x^{-\beta}$ with $\beta = (C-1)/(Cq-1)$ satisfies (1.1) but fails to belong to $L^p(0, m)$ for $p = (Cq-1)/(C-1)$ and so the result is sharp.

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