J. Austral. Math. Soc. 19 (Series A) (1975), 358–370.

SYSTEMS OF DERIVATIONS ON TOPOLOGICAL ALGEBRAS OF POWER SERIES

HENRY J. SCHULTZ

(Received 15 June 1973, revised 18 December 1973)

Communicated by J. B. Miller

1. Introduction

If $D_0, D_1, \cdots$ are linear maps from an algebra A to an algebra B, both over the complexes, then $\{D_0, D_1, \cdots\}$ is a *system of derivations* if for all a, b in A and for all nonnegative integers k, we have

$$D_k(ab) = \sum_{i=0}^{k} C(k,i)D_i(a)D_{k-i}(b) \tag{1.1}$$

where $C(k,i)$ is the binomial coefficient $k!/i!(k-i)!$. By (1.1) we see that D_0 must be a homomorphism and in case $D_0 = I$, where I is the identity map, D_1 is a derivation and, for $k \geqq 2$, the D_k are higher derivations in the sense of Jacobson (1964), page 191. Gulick (1970), Theorem 4.2, proved that if A is a commutative regular semi-simple F-algebra with identity and $\{D_0, D_1, \cdots\}$ is a system of derivations from A to $B = C(S(A))$, the algebra of all continuous functions on the spectrum of A, where $D_0 x = x$, then the D_k are all continuous. Carpenter (1971), Theorem 5, shows that the regularity condition is unnecessary and Loy (1973) generalizes this a bit further. One of the many interesting features of systems of derivations is that they help determine analytic structure in Banach algebras (see for example, Miller (to appear)).

In Section 2 we will characterize the systems of derivations on the algebra of all power series over the complex numbers.

In Section 3 we show that given a system of derivations $\{D_0, D_1, \cdots\}$ on certain F-algebras of power series, we have continuity of the maps D_k, for $k \geqq 0$, under a mild restriction on D_0. This generalizes Loy's (1969) result for single derivations and the proof is a modification of his proof.

In Section 4 we will use the results of Sections 2 and 3 to characterize essentially all systems of derivations on certain radical Banach algebras of power series considered by Grabiner (1967), (1971), (1973), (1974), (to appear). generalizing his characterization of derivations in (1974).

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

A characterization of systems of derivations is made in Gulick (1970), Corollary 5.12, where $A = C^N(U) = \{f\colon U \to C,\ U \text{ open in } R\colon f^{(k)} \text{ is in } C(U),\ 0 \leqq k \leqq N\}$ for some N where $N \geqq 1$, $B = C^1(U)$ and $D_0 = I$. An interesting representation theorem for higher derivations may be found in Miller (1967). The author wishes to express his thanks to Professor Sandy Grabiner who directed the author's thesis (of which this paper is a part) and to the referee for his simplification of the proof of Theorem 3.1.

We now establish some preliminary notations and formulas. Throughout this paper A will denote an algebra of power series over the complex numbers without the identity and which contains the indeterminate z. A' will denote the algebra formed from A by adjoining the identity. If R and S are sets, then $R\backslash S$ will denote the complement of S in R. If $\{D_0, D_1, \cdots\}$ is a system of derivations on A', we have the following important formulas which are a special case of Gulick (1970), Theorem 3.4.

$$(1.2) \qquad D_k(p(z)) = \sum_{j=1}^{k} D(k,j)p^{(j)}(D_0(z))$$

where $k \geqq 1$, $p(z)$ is any polynomial in A', $p^{(j)}$ is the j^{th} derivative of p and

$$(1.3) \qquad D(k,j) = k! \sum_{\sigma(k,j)} \prod_L \left(\frac{1}{n_q!}\left(\frac{1}{m_q!}D_{m_q}(z)\right)^{n_q}\right)$$

where the sum runs over the collection $\sigma(k,j)$ of all sets of integers

$$L = \{n_1, \cdots, n_{r(L)}, m_1, \cdots, m_{r(L)}\}$$

such that

$$1 \leqq m_1 < \cdots < m_{r(L)} \leqq k, \quad \sum_{q=1}^{r(L)} n_q m_q = k \text{ and } \sum_{q=1}^{r(L)} n_q = j$$

where, if L is in $\sigma(k,j)$, $\Pi_L(\Phi(m_q, n_q))$ is the product over q from 1 to $r(L)$ of $\Phi(m_q, n_q)$ where the m_q and n_q are in L.

2. Systems of Derivations on $C[[z]]$

Throughout this section we will assume that $A' = C[[z]]$, the algebra of all formal power series $\sum_{i\geqq 0} a_i z^i$ with complex coefficients where A' is given the usual Fréchet topology determined by coordinatewise convergence (that is, the seminorms $\{\|\cdot\|_n\colon \|\sum_{i\leqq 0} a_i z^i\|_n = \sup_{0\geqq j\geqq n} |a_j|\}_{n=0}^{\infty}$ generate the topology). We note that A' is a topological algebra with this topology. Since each A'/z^nA', $0 \leqq n < \infty$, is finite dimensional, we have a norm on each A'/z^nA' which induces a seminorm on A' for each $n \geqq 1$. It is easy to see that the usual Fréchet topology is the same as that generated by these seminorms (compare Scheinberg (1970) for this and other questions concerning A).

DEFINITION 2.1. *Suppose f, g belong to A' and h belongs to A and let $f = a_0 + a_1 z + \cdots$. Also suppose R and S are two linear operators on A'. We define:*

(*A*) $T_h(f) = f \circ h$, *formal power series composition.*

(*B*) $Df = a_1 + 2a_2 + \cdots + na_n z^{n-1} + \cdots$.

(*C*) $(gD)f = g(Df)$.

(*D*) $(S \circ R)f = S(R(f))$.

(*E*) $S^j = S \circ \cdots \circ S$, *the operator S being repeated j times.*

(*F*) $p_n(f) = a_n$, $0 \leqq n < \infty$.

(*G*) *If $g \neq 0$, then order(g) is the least integer k such that $p_k(g) \neq 0$. If $g = 0$, then order(g) $= \infty$.*

It is clear that for all h in A and g in A', T_h is an endomorphism of A', gD is a derivation of A' and T_h and gD are continuous in the usual Fréchet topology on A'.

Any endomorphism T of A' is continous. To see this we notice that a series in A' is invertible if and only if it does not belong to A (i.e., A is a *radical* algebra). From this we can easily show that order$(T(z)) \geqq 1$; since if $p_0(T(z)) = \delta \neq 0$, we would have $T(\delta - z)$ in A and not invertible in A' whereas $\delta - z$ is invertible in A'. Since T is an endomorphism, $T(\delta - z)$ is invertible in A' which is a contradiction. Now, for all n, we see that $T(z^n A')$ is contained in $T(z)^n A'$ which is contained in $z^n A'$ since order$(T(z)) \geqq 1$ and hence T is continuous. (Scheinberg (1970), page 324, uses this type of argument in cases where we are assured that order$(T(z)) \geqq 1$).

Due to the special nature of our topology on $C[[z]]$, we can modify an argument of Johnson (1969), page 9, without using the closed graph theorem as he does, to obtain the following lemma:

LEMMA 2.2. *Suppose $\{D_0, D_1, \cdots\}$ is a system of derivations on $A' = C[[z]]$. Then all D_k, $0 \leqq k < \infty$, are continuous.*

PROOF. We know already that D_0 is continuous. We proceed by induction on the D_k. Let k be some positive integer and let $g = g_0 + g_1 z + \cdots$ be an arbitrary element of A'. Then $D_k(g) = \sum_{n=0}^{\infty} f_n(g) z$, where all of the f_n are linear functionals on A'. We claim that all the f_n are continuous. To see this, we fix an arbitrary non-negative integer n and pick an arbitrary h in A'. We now obtain:

$$\begin{aligned} f_n(z^{n+k+1}h) &= p_n D_k(z^{n+k+1}h) \\ &= p_n\left(k! \sum_{\substack{i_1+\cdots+i_{n+k+2}=k \\ i_j \geqq 0}} \frac{D_{i_1}(z)}{i_1!} \cdots \frac{D_{i_{n+k+1}}(z)}{i_{n+k+1}!} \frac{D_{i_{n+k+2}}(h)}{i_{n+k+2}!}\right) \end{aligned}$$

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

by repeated application of formula (1.1). In each summand of the last expression at least $n+1$ of the $i_1, \cdots, i_{n+k+1}$ must be zero. However, we know that order $(D_0(z)) \geqq 1$ and so it must be the case that order$(D_k(z^{n+k+1}h)) > n$, which implies that $f_n(z^{n+k+1}h) = 0$. Hence

$$f_n(g) = g_0 f_n(1) + \cdots + g_{k+n} f_n(z^{k+n})$$

and so f_n is continuous. Since n was arbitrary, all the $f_n = p_n D_k$ are continuous. Since the p_n are all continuous and our topology on A' is that of coordinatewise convergence, it is clear that D_k is continuous and this completes the induction.

Notice that in the preceding proof the crucial facts used were that A is closed under division (i.e., f in A and fg in A imply g is in A' (Grabiner (1971), p. 653) and that order$(D_0(z) \geqq 1$ (which as we have seen, is true if A is a radical algebra). This means that Lemma 2.2 holds on any subalgebra B' of A' if B is closed under division and is a radical algebra.

We now characterize the systems of derivations on A'.

THEOREM 2.3. *The collection of all systems of derivations* $\{D_0, D_1, \cdots\}$ *of* $A' = C[[z]]$ *is in a one-to-one correspondence with the collection of sequences of power series* $\{g_0, g_1, \cdots\}$ *where* g_0 *runs over* A *and, for* $i \geqq 1$, *the* g_i *run over* A'. *The correspondence is determined by* (2.4) *and* (2.5) *below*:

(2.4)
$$\{D_0, \dots, D_k, \cdots\} \to \{g_0, \cdots, g_k, \cdots\}, \text{ where}$$
$$g_k = D_k(z), \qquad 0 \leqq k < \infty$$

(2.5)
$$\{g_0, \cdots, g_k, \cdots\} \to \{D_0, \cdots, D_k, \cdots\}, \text{ where}$$
$$D_0 = T_{g_0} \text{ and}$$
$$D_k = \sum_{j=1}^{k} k! \sum_{\sigma(k,j)} \prod_{L} \left(\frac{1}{n_q!}\left(\frac{1}{m_q!} g_{m_q}\right)^{n_q}\right)(T_{g_0} \circ D^j), \qquad 1 \leqq k < \infty$$

where the notation is as in formulala (1.3).

PROOF. Suppose $\{D_0, D_1, \cdots\}$ is a system of derivations of A'. Since D_0 is an endomorphism of A', we have $D_0(p(z)) = T_{D_0(z)}(p(z))$ for all polynomials $p(z)$ in A' and, as we saw above, we also have that D_0 is continuous in the usual Fréchet topology on A'. The polynomials are dense in A' in this topology and so $D_0 = T_{D_0(z)}$ on all of A'. Lemma 2.2 tells us that, for all $k \geqq 1$, D_k is continuous, and formulas (1.2) and (1.3) tell us how each D_k is defined on the polynomials of A'. Combining this information we can conclude that, for $k \geqq 1$, D_k is given on all of A' by the following formula:

(2.6)
$$D_k = \sum_{j=1}^{k} k! \sum_{\sigma(k,j)} \prod_{L} \left(\frac{1}{n_q!}\left(\frac{1}{m_q!} D_{m_q}(z)\right)^{n_q}\right)(T_{D_0(z)} \circ D^j)$$

where the notation is as in formula (1.3). So, with $\{D_0, D_1, \cdots\}$ we associate the sequence of power series $\{D_0(z), D_1(z), \cdots\}$.

Conversely, given a sequence of power series $\{g_0, g_1, \cdots\}$, we know that T_{g_0} is an endomorphism and see immediately that $g_i = D_i(z)$ for $0 \leqq i < \infty$ and that each D_k is continuous. Hence, we have only to show that each D_k satisfies formula (1.1) where $a = p(z)$ and $b = q(z)$ are polynomials, since the polynomials are dense in A'. But the linearity of each D_k requires that we need only show formula (1.1) holds for monomials $a = z^m$ and $b = z^n$, $m \geqq 1$, $n \geqq 1$. Now, letting $s = m + n$ so that $s \geqq 2$, we may apply formula (2.5) for D_k, $k \geqq 1$, to z^s where $g_i = D_i(z)$ for $0 \leqq i < \infty$. Then we reverse the argument of Gulick (1970), pages 473–74, to obtain $D_k(z^s) = \sum_{j=0}^{k} C(k,j) D_j(z^m) D_{k-j}(z^n)$ for all $k \geqq 1$ and the proof is completed.

Looking again at the proof of Theorem 2.3 we see that if B' is any subalgebra of $C[[z]]$ for which B is closed under division and radical, then any system of derivations of B' must be of the form (2.5) where g_0 belongs to B and g_i belongs to B' for all $i \geqq 1$. This conclusion also holds (by Wilansky (1964), page 204)) if B' is an F-algebra of power series continuously embedded in $C[[z]]$ (with the Fréchet topology) such that B is closed under division and radical.

3. Contiuuity of F-algebras of Power Series

Throughout this section, A' will be an algebra of powe: series with a complete metrizable locally convex topology determined by a sequence of seminorms $\{ \| \cdot \|_n \}$ which makes it a topological algebra under which the coefficient projections p_j are all continuous. In this section, we do not require the polynomials to be dense in A' for this topology and we will not be able to use the technique of Johnson (1969), page 9, to obtain continuity as we did in the previous section since his technique requires that the algebra be closed under division (see Grabiner (1971), page 653)) and the algebras we wish to consider do not usually have this property. Conversely, the techniques used in this section will not apply to $A' = C[[z]]$, the algebra of all power series with complex coefficients, since $C[[z]]$ fails (for example, by note (1) at the end of this section) to satisfy condition (E) which will come into play in the following paragraph.

Loy (1969), Theorem 1, showed that if A' satisfied the condition which we shall call condition (E), that there exists a sequence $\{\varepsilon_n\}_{n \geqq 0}$ of positive numbers such that the family $\{\varepsilon_n^{-1} p_n\}_{n \geqq 0}$ is equicontinuous, then any derivation D_1 (or in our language, a map D_1 in a system of derivations of the form $\{I, D_1, D_2, \cdots\}$) on A' is continuous. Note that if A' is a Banach algebra, then condition (E) always holds, where we let $\varepsilon_n = \| p_n \|$ for $n \geqq 0$. We now have the following generalization of Loy's result.

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

THEOREM 3.1. *Suppose A' satisfies condition (E) and suppose $\{D_0, D_1, \cdots\}$ is a system of derivations on A' for which order $(D_0(z)) \geqq 1$. Then D_i is continuous for $i \geqq 0$.*

PROOF. The idea of the proof is similar to that of Loy (1969), pages 420–421, but the details are more complicated. Since order$(D_0(z)) \geqq 1$ we have that D_0 is an endomorphism of A' for which the dimension of the range of D_0 is greater than one, and so by Loy (preprint), Theorem 10, (compare Johnson (1967), Theorem 9.1), D_0 is continuous. We proceed by induction and assume $D_0, D_1, \cdots, D_{h-1}$ are continuous. Suppose D_h is not continuous. Then, since $\{p_n\}_{n\geqq 0}$ is a separating family of continuous linear functionals on A', the closed graph theorem implies that the discontinuity of D_h gives us a (least) $k \geqq 0$ such that $p_k D_h$ is discontinuous.

Let U be a neighborhood of zero such that for all x in U, $|p_j D_h(x)| \leqq 1$ for $0 \leqq j \leqq k-1$. We use the equicontinuity of $\{\varepsilon_n^{-1} p_n\}_{n \geqq 0}$ to produce a neighborhood of zero V such that for all x in V, $|p_n(x)| \leqq \varepsilon_n$ for $n \geqq 0$. Since multiplication on A' is jointly continuous, there exist sequences of positive numbers $\{M_n\}_{n\geqq 1}$ and $\{\delta_n\}_{n \geqq 1}$ so that if $\|x\|_i < \alpha$ and $\|y\|_i < \beta$ for $1 \leqq i \leqq M_n$ where $\alpha\beta < \delta_n$, then $\|xy\|_j < 1$ for $1 \leqq j \leqq n$. Let order $(D_0(z)) = N \geqq 1$. For simplicity, we allow sums over void index sets and let their value be zero.

We now define inductively a sequence $\{x_n\}_{n \geqq 1}$ in A' so that the following conditions hold:

(1) x_n is in U.

(2) $\sum_{j=1}^{h} C(h,j) D_j(z^n) D_{h-j}(x_n)$ is in V.

(3) $\|x_n\|_i < 2^{-n}\delta_n \min\{\|z^j\|_m^{-1} : 1 \leqq j \leqq n, 1 \leqq m \leqq n\}, 1 \leqq i \leqq M_n$.

(4) $|p_{nN}(D_0(z)^n)|\,|p_k D_h(x_n)| \geqq (2(nN+k)+h+1)\varepsilon_{nN+k}$
$+ \sum_{i=1}^{n-1}\sum_{j=0}^{nN} |p_j(D_0(z)^i)|\,|p_{nN+k-j}(D_h(x_i))| + \sum_{i=1}^{nN+k+h+1}\sum_{j=nN+1}^{nN+k} |p_j(D_0(z)^i)|$

Now, by (3), if $m > j$, then $\|z^{m-j}x_m\|_i < 2^{-m}$ for $1 \leqq i \leqq m$. Hence, for each j, $\sum_{m>j} z^{m-j}x_m$ converges in A' to some element y_j. We have $y_0 = \sum_{m \geqq 1} z^m x_m$. Furthermore, in the following argument, we assume $n = QN + k$ for some integer $Q \geqq 1$ and define $h(n) = n + h + 1$. From the definition of the y_j we have

$$p_n(D_h(y_0)) = p_n\left(D_h\left(z^{h(n)} y_{h(n)} + \sum_{i=1}^{h(n)} z^i x_i\right)\right)$$

Moreover, using the fact that D_0 is an endomorphism with order$(D_0(z)) \geqq 1$ and using formulas (1.1), (1.2) and (1.3) we obtain

$$p_n(D_h(z^{h(n)} y_{h(n)})) = p_n\left(\sum_{j=0}^{h} C(h,j) D_j(z^{h(n)}) D_{h-j}(y_{h(n)})\right) = 0$$

Hence,

$$\begin{aligned} p_n(D_h(y_0)) &= p_n\left(\sum_{i=1}^{h(n)} D_h(z^i x_i)\right) \\ &= p_n\left(\sum_{i=1}^{h(n)} D_0(z)^i D_h(x_i)\right) + p_n\left(\sum_{i=1}^{h(n)} \sum_{j=1}^{h} C(h,j) D_j(z^i) D_{h-j}(x_i)\right) \\ &= \sum_{i=1}^{h(n)} \sum_{j=0}^{n} p_j(D_0(z)^i) p_{n-j}(D_h(x_i)) + \sum_{i=1}^{h(n)} p_n\left(\sum_{j=1}^{h} C(h,j) D_j(z^i) D_{h-j}(x_i)\right) \end{aligned}$$

Now we notice that if $j < Ni$, we have $p_j(D_0(z)^i) = 0$, and so

$$\begin{aligned} p_n(D_h(y_0)) = {} & p_{QN}(D_0(z)^Q) p_k(D_h(x_Q)) + \sum_{i=1}^{Q-1} \sum_{j=0}^{QN} p_j(D_0(z)^i) p_{n-j}(D_h(x_i)) \\ & + \sum_{i=1}^{h(n)} \sum_{j=QN+1}^{n} p_j(D_0(z)^i) p_{n-j}(D_h(x_i)) + \sum_{i=1}^{h(n)} p_n\left(\sum_{j=1}^{h} C(h,j) D_j(z^i) D_{h-j}(x_i)\right) \end{aligned}$$

So, using (1) and (2), we have

$$\begin{aligned} \left|p_n D_h(y_0)\right| \geqq {} & \left|p_{QN}(D_0(z)^Q)\right| \left|p_k D_h(x_Q)\right| \\ & - \sum_{i=1}^{Q-1} \sum_{j=0}^{QN} \left|p_j(D_0(z)^Q)\right| \left|p_{n-j}(D_h(x_i))\right| \\ & - \sum_{i=1}^{h(n)} \sum_{j=QN+1}^{n} \left|p_j(D_0(z)^i)\right| - (QN+k+h+1)\varepsilon_{QN+k} \geqq n\varepsilon_n \end{aligned}$$

by condition (4).

Now, choose $\lambda > 0$ such that $\lambda D_h(y_0)$ is in V. Then for all n such that $n = QN + k$, $Q = 1, 2, \cdots$, we have $|\lambda| n\varepsilon_n \leqq |p_n(\lambda D_h(y_0))| \leqq \varepsilon_n$. Hence, since $\lambda > 0$, we have $1/n \geqq |\lambda| > 0$ for all $n = QN + k$ where $Q = 1, 2, \cdots$. This contradiction proves the theorem.

NOTES: (1) Loy (1971), Theorem 2, shows that A' satisfies condition (E) if and only if there exists a sequence $\{\varepsilon_n\}_{n \geqq 0}$ of positive numbers such that $\varepsilon_n p_n(x) \to 0$ for each x in A'.

(2) The most interesting case, where $D_0 = I$, satisfies order$(D_0(z)) = 1$.

(3) Suppose A is a radical algebra. Then by the same argument we used after Definition 2.1 we can conclude that order$(D_0(z)) \geqq 1$ holds for any endomorphism D_0 on A'.

(4) Without modifying the proof in the above theorem we could have allowed A' to be an algebra of power series over some commutative Banach algebra B (compare Loy (1970), page 373).

4. Systems of Derivations

Throughout this section we assume that A is a radical Banach algebra of power series continuously embedded in $C[[z]]$ with its usual Fréchet topology. We will also assume that the polynomials are dense in the norm topology on A. On A' we define $\| \lambda + f \| = |\lambda| + \|f\|$, where f belongs to A. These and more general algebras have been considered by Grabiner (1967), (1971), (1973), (1974), (to appear) and the characterizations we obtain in Theorems 4.3 and 4.6 generalize that obtained in (1974). For spaces of power series we now establish (Grabiner (to appear), Definitions 2.7 and 3.2):

Definition 4.1. *(A)* $A^{(0)} = A'$ *and, for* $n \geqq 1$, $A^{(n)}$ *is the set of all* f *belonging to* A *such that order*$(f) \geqq n$.

(B) If j *is a non-negative integer,* $S_{-j}(A)$ *is the Banach space of all power series* f *with zero constant term such that* $z^j f$ *belongs to* A. *The norm* $\| \cdot \|'$ *on* $S_{-j}(A)$ *is given by* $\|f\|' = \|z^j f\|$. *Note that* $S_{-0}(A) = A$; $i > j$ *implies* $S_{-j}(A)$ *is contained in* $S_{-i}(A)$; *and, for all* $i,j \geqq 0$, *we have* $S_{-(i+j)}(A) = S_{-i}(S_{-j}(A))$.

For the rest of this section we assume that $S_{-j}(A)$ is a radical algebra for $j \geqq 1$, and with this assumption, we will characterize essentially all systems of derivations on A'. This blanket assumption is made for simplicity even though some of the following results hold with fewer restrictions. It will be seen in Remark (1) at the end of this section that this is actually a very natural assumption.

Suppose we are given a map V of A' which is the restriction of a map W, where W is one of the maps of $C[[z]]$ given in Definition 2.1 (A), (B), or (C). Given a positive integer j, we can consider V to be defined on $S_{-j}(A)$ by letting V be the restriction of W to $S'_{-j}(A)$. In what follows this fact will be used without explanation.

Before we proceed to the main theorem, we will need the following lemma due to Grabiner (1967), (1974).

Lemma 4.2. *If* g *is a formal power series, then the following conditions are equivalent, where* j *is a positive integer.*

(A) g *belongs to* $S'_{-j}(A)$.

(B) gz^j *belongs to* A'.

(C) gf *belongs to* A' *for all* f *in* $A^{(j)}$.

(D) gf *belongs to* A' *for some* f *belonging to* $A^{(j)}$ *with order* $(f) = j$.

Proof. The equivalence of (A) and (B) is Definition 2.1 (B). We now prove (B) implies (C). Suppose gz^j belongs to A'. If we are given an arbitrary f in $A^{(j)}$, we can write $f = z^j f'$ where f' belongs to $S'_{-j}(A)$. Since $S'_{-j}(A)$ is an algebra, gf' belongs to $S'_{-j}(A)$ and hence $gf = z^j g f'$ belongs to A'. (C) implies (D) is trivial, so we conclude by proving (D) implies (B). Let gf belong to A' and let

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

$f = z^i f'$ where f' where f' belongs to $S'_{-j}(A)$ and has a non-zero constant term. Since we assume that $S_{-j}(A)$ is a radical algebra, f' is invertible in $S'_{-j}(A)$. Now, $gf = z^j h$, where h belongs to $S'_{-j}(A)$, and so $gz^j = (gf)(f')^{-1} = z^i h(f')^{-1}$. As $S'_{-j}(A)$ is a radical algebra, $h(f')^{-1}$ belongs to $S'_{-j}(A)$ and hence gz^j belongs to A' and the proof is completed.

THEOREM 4.3. *Suppose A is a radical Banach algebra of power series continuously embedded in $C[[z]]$ so that the polynomials are dense in A' in the norm topology. Further, assume that $S_{-j}(A)$ is a radical algebra for all $j \geqq 1$ and that $z^2 Df$ belongs to A' for all f in A'. Suppose D_0 is a fixed automorphism of A'. Then the class of all systems of derivations $\{D_0, D_1, \cdots\}$ is in a one-to-one correspondence with the class of sequences $\{g_0, g_1, \cdots\}$ where $g_0 = D_0(z)$ is a fixed element in $A \backslash A^{(2)}$ and the $D_i(z) = g_i$ are allowed to run over $A^{(2)}$ for $i \geqq 1$. The correspondence is as in Theorem 2.3.*

PROOF. For the first, and most difficult, half of the proof, we will show that, for a fixed automorphism D_0 of A', any system of derivations has the form indicated in formula (2.5), with $D_0(z) = g_0$ belonging to $A \backslash A^{(2)}$ and $D_i(z) = g_i$ belonging to $A^{(2)}$ for $i \geqq 1$.

D_0 is an endomorphism and so $D_0 = T_{D_0(z)}$ on the polynomials of A'. We claim that $D_0(f) = T_{D_0(z)}(f)$ for any element f in A'. To see this we notice that since the polynomials are dense in A', there exists a sequence $\{p_k\}_{k \geqq 1}$ of polynomials which converge to f in the norm topology on A'. As D_0 is an automorphism, a result of Loy (preprint), Theorem 10 (compare Johnson (1967), Theorem 9.1) implies D_0 is continuous on A' and so $\{D_0(p_k)\}_{k \geqq 1}$ converges to $D_0(f)$ in the norm topology on A' and hence, since A' is continuously embedded in $C[[z]]$, converges to $D_0(f)$ in the usual Fréchet topology restricted to A'. However, $T_{D_0(z)}$ is continuous on $C[[z]]$ with the usual Fréchet topology and since $T_{D_0(z)} = D_0$ on the polynomials the claim is proved. (A consequence of this is that D_0 is actually an isomorphism on $C[[z]]$). Since z is quasinilpotent and D_0 is bounded, it must be that $D_0(z)$ is quasinilpotent and so belongs to A, the radical of A'. Hence order$(D_0(z)) \geqq 1$. Further, as D_0 is onto, we must have $p_1(D_0(z)) = \varepsilon$ for some non-zero ε and so order$(D_0(z)) = 1$ implying that $g_0 = D_0(z)$ belongs to $A \backslash A^{(2)}$.

The fact that order$(D_0(z)) = 1$ implies we may apply Theorem 3.1 to conclude that D_k is continuous for $k \geqq 1$. Formulas (1.2) and (1.3) show that, for $k \geqq 1$,

$$D_k = \sum_{j=1}^{k} k! \sum_{\sigma(k,j)} \prod_{L} \left(\frac{1}{n_q!}\left(\frac{1}{m_q!} D_{m_q}(z)\right)^{n_q}\right)(T_{D_0(z)} \circ D^j) \tag{4.4}$$

holds on the polynomials of A'. For all $k \geqq 1$, the right side of (4.4) is continuous on $C[[z]]$. By the argument used at the beginning of the previous

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

paragraph we see that (4.4) holds on all of A', this being the form of formula (2.5) as required.

To complete the first part of the proof we need only show that $D_k(z)$ belongs to $A^{(2)}$ for all $k \geqq 1$. We proceed by induction on k. For $k = 1$, $D_1 = D_1(z)(D_0 \circ D)$. $D_1(f)$ belongs to A' for all f in A'. If we can show that $(D_0 \circ D)(f')$ belongs to $S'_{-2}(A) \backslash S'_{-1}(A)$ for some f' belonging to A', we may apply Lemma 4.2 to conclude that $D_1(z)$ belongs to $A^{(2)}$ which is what we desire. To do this, we show first that there exists an f' in A' so that $D(f')$ belongs to $S'_{-2}(A) \backslash S'_{-1}(A)$. Clearly, if f belongs to A', the hypothesis that $z^2 D(f)$ belongs to A' implies that $D(f)$ belongs to $S'_{-2}(A)$. Now suppose that $D(A')$ is contained entirely in $S'_{-1}(A)$. Then $zD(A')$ is contained in A' and since $\{I, zD, (zD)^2, \cdots\}$ is a system of derivations of A', zD is continuous by Theorem 3.1. However, $zD(z^n) = nz^n$ for all $n \geqq 1$ and this contradicts the continuity of zD. Hence there exists some f' in A' so that $D(f')$ belongs to $S'_{-2}(A) \backslash S'_{-1}(A)$.

Since $D_0 = T_{D_0(z)}$ is an isomorphism of A' (and also of $C[[z]]$), $D_0(z^2 D(f'))$ $= D_0(z)^2 D_0(D(f'))$ belongs to A'. Also order $(D_0(z)^2) = 2$ and so, by Lemma 4.2 ((D) implies (A)), $D_0(Df'))$ belongs to $S'_{-2}(A)$.

We now show that $D_0(D(f'))$ does not belong to $S'_{-1}(A)$ by showing that D_0 is an automorphism of $S'_{-1}(A)$. Suppose g belongs to $S'_{-1}(A)$, and hence zg belongs to A'. Then $D_0(zg) = D_0(z)D_0(g)$ belongs to A' where order $(D_0(z)) = 1$ and so, by Lemma 4.2, $D_0(g)$ belongs to $S'_{-1}(A)$. In the same way $D_0^{-1}(S'_{-1}(A))$ is contained in $S'_{-1}(A)$ and so D_0 is an automorphism of $S'_{-1}(A)$. This says that $D_0(D(f'))$ belongs to $S'_{-2}(A) \backslash S'_{-1}(A)$, hence $D_1(z)$ belongs to $A^{(2)}$ and the first step of the induction is completed.

We now assume $D_1(z), \cdots, D_{k-1}(z)$ all belong to $A^{(2)}$. We may use formula (4.4) to conclude that the following formula holds for all f in A'.

$$
\text{(4.5)} \quad \begin{aligned} D_k(f) = \left(\sum_{j=2}^{k} k! \sum_{\sigma(k,j)} \prod_L \left(\frac{1}{n_q!}\right) \frac{1}{m_q!} D_{m_q}(z) \right)^{n_q} \Big) (D_0 \circ D^j) \Big)(f) \\ + (D_k(z)(D_0 \circ D))(f) \end{aligned}
$$

We now claim that, if f belongs to A', then $D^j(f)$ belongs to $S'_{-2j}(A)$ for all $j \geqq 1$. This holds for $j = 1$ so, proceeding by induction, we assume that $j > 1$ and that $D^{j-1}(A')$ is contained in $S'_{-(2j-2)}(A)$. Since $D^j(A') = D(D^{j-1}(A'))$, it remains only to show $D(S'_{-(2j-2)}(A))$ is contained in $S'_{-2j}(A)$. Let f belong to $S'_{-(2j-2)}(A)$. Then $(z^2(Df)z^{2j-2} = z^2 D(fz^{2j-2}) - (2j-2)fz^{2j-1}$ and since both terms on the right hand side of the equality belong to A', it must be that $(z^2 Df)z^{2j-2}$ belongs to A' and so $D(\mathrm{f})$ belongs to $S'_{-2j}(A)$. This completes the induction implying that $D^j(A')$ is contained in $S'_{-2j}(A)$ for $j \geqq 1$.

Now, for all $i \geqq 1$, $D_0(S'_{-i}(A))$ is contained in $S'_{-i}(A)$. This is true since, if g belongs to $S'_{-i}(A)$, then $D_0(z^i g) = D_0(z)^i D_0(g)$ belongs to A' where

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

$\text{order}(D_0(z)^i) = i$. Hence, Lemma 4.2 ((D) implies (A)) says that $D_0(g)$ belongs to $S'_{-i}(A)$. We may conclude that $(D_0 \circ D^j)(A')$ is contained in $S'_{-2j}(A)$.

For each L belonging to $\sigma(k,j)$ where j is an integer such that $2 \leqq j \leqq k$, $\Pi_L(1/n_q!(1/m_q!D_{m_q}(z))^{n_q})$ belongs to $A^{(2j)}$. This is because $\sum_{q=1}^{r(L)} n_q = j$ and also since, by the induction hypothesis, each $D_{m_q}(z)$ belongs to $A^{(2)}$ where $1 \leqq q \leqq r(L)$. Hence, by Lemma 4.2(C), the first of the two summands on the right hand side of (4.5) is in A' and so $(D_k(z)(D_0 \circ D))(A')$ is contained in A'. Now, using the same argument as in the first step of the induction, $D_k(z)$ must belong to $A^{(2)}$, completing the induction and with it, the first half of the proof.

Conversely, given a sequence of power series $\{g_0, g_1, \cdots\}$, where g_0 is a fixed element of $A \backslash A^{(2)}$ so that $D_0 = T_{g_0}$ is an automorphism of A' and g_i belongs to $A^{(2)}$ for $i \geqq 1$, suppose that $\{D_0, D_1, \cdots\}$ is a system of maps where the D_k are defined by formula (2.5). We want to show that $\{D_0, D_1, \cdots\}$ is a system of derivations of A'. Since, by Theorem 2.3, $\{D_0, D_1 \cdots\}$ satisfies the Leibniz formula (1.1), we need only that $D_j(A')$ is contained in A' for all j.

$D_0(A')$ is contained in A' by hypothesis. We proceed by induction and assume that $D_0(A'), D_1(A'), \cdots, D_{k-1}(A')$ are all contained in A'. We then have:

$$D_k(A') = \left(\sum_{j=1}^{k} k! \sum_{\sigma(k,j)} \prod_L \left(\frac{1}{n_q!} \left(\frac{1}{m_q!} g_{m_q} \right)^{n_q} \right) (T_{g_0} \circ D^j) \right)(A').$$

However, as was seen earlier, $(T_{g_0} \circ D^j)(A')$ is contained in $S'_{-2j}(A)$ for all j and also, for each L in $\sigma(k,j)$, $1 \leqq j \leqq k$, $\Pi_L(1/n_q!(1/m_q!g_{m_q})^{n_q})$ belongs to $A^{(2j)}$. Combining these two facts, Lemma 4.2(C) implies that $D_k(A')$ is contained in A'. This completes the proof of the theorem.

We proceed immediately to our second major theorem.

THEOREM 4.6. *Suppose A is a radical Banach algebra of power series continuously embedded in* C[[z]] *so that the polynomials are dense in A' in the norm topology. Further, assume that $S_{-j}(A)$ is a radical algebra or $j \geqq 1$ and that $(z^2D)(f)$ converges in A' for all f in A'. If D_0 is a fixed endomorphism of A' of the form T_{g_0} for some g_0 in $A^{(2)}$, then the class of all systems of derivations $\{D_0, D_1, \cdots\}$ is in a one-to-one correspondence with the class of all sequences of power series $\{g_0, g_1, \cdots\}$ where g_i belongs to A' for $i \geqq 1$. The correspondence is as in Theorem* 2.3.

PROOF. The proof is similar to that of Theorem 4.3, except that in the case of Theorem 4.3 we had only that $T_{g_0}(S'_{-k}(A))$ was contained in $S'_{-k}(A)$ for each $k \geqq 1$. However, in this case, $T_{g_0}(S'_{-k}(A))$ is contained in A'. We argue as follows. Let $f = a_0 + a_1 z + \cdots$ belong to $S'_{-k}(A)$. Then $T_{g_0}(f)$

https://doi.org/10.1017/S1446788700031566 Published online by Cambridge University Press

$= f \circ g_0 = a_0 + a_1 g_0 + \cdots + a_{k-1} g_0^{k-1} + h \circ g_0$ where h is a member of $S_{-k}^{(k)}(A)$. Hence, $h = z^k h'$ where h' belongs to $S'_{-2k}(A)$, and since $a_0 + \cdots + a_{k-1}(g_0)^{k-1}$ is in A', we need only show that $h \circ g_0$ belongs to A'. $h \circ g_0 = (g_0)^k (h' \circ g_0)$ belongs to A' by Lemma 4.2 since $(g_0)^k$ belongs to $A^{(2k)}$ and $h' \circ g_0$ is in $S'_{-2k}(A)$. ($h' \circ g_0 = T_{g_0}(h')$ belongs to $S'_{-2k}(A)$ since T_{g_0} is an endomorphism of $S'_{-2k}(A)$. In Theorem 4.3 we were able to argue that T_{g_0} being an automorphism of A' implied T_{g_0} was also an automorphism of $S'_{-1}(A)$. We could use the same arguments to conclude that T_{g_0} is also an automorphism of $S'_{-j}(A)$ for any $j > 1$. These arguments do not work in Theorem 4.6 where T_{g_0} is an endomorphism of A' with g_0 belonging to $A^{(2)}$ since the fact that order $(T_{g_0}(z)) = N > 1$ for some N only allows Lemma 4.2 to guarantee that $T_{g_0}(S'_{-j}(A))$ is contained in $S'_{-jN}(A)$ for $j \geqq 1$. However, it is easy to see that if A satisfies condition (B) of Remark (1) below (and in particular, satisfies the hypotheses of Theorem 4.6), then $S_{-j}(A)$ also satisfies condition (B) for all $j \geqq 1$ (where A is replaced by $S_{-j}(A)$). For those Banach algebras of power series A which satisfy condition (B), Grabiner (1974) has shown that for every g in $A^{(2)}$, T_g is an endomorphism of A'. If we replace A' by $S'_{-2k}(A)$, we are done.) The rest of the proof is similar to that of Theorem 4.3 and is omitted.

DEFINITION 4.7. (Grabiner (1971), page 643). *If* $\{c_n\}_{n \geqq 1}$ *is a sequence of positive numbers, then* $K\langle c_n \rangle$ *is the set of all power series* $f = \sum_{n \geqq 1} \lambda_n z^n$ *for which* $\|f\| = \sum_{n \geqq 1} |\lambda_n| c_n < \infty$. $K\langle c_n \rangle$ *is a Banach space under this norm.*

REMARKS. (1) Grabiner (1974) has shown that if A is a Banach algebra of power series with $\|z^n\| = c_n$, then either condition (A) or (B) below will imply that $S_{-j}(A)$ is a radical algebra for $j \geqq 0$, and will also imply the hypotheses on $z^2 D$ for Theorems 4.3 and 4.6. We further note that (B) is strictly weaker than (A).

(A) $\{c_{n+k}/c_n\}$ is eventually non-increasing for some positive integer k and $\sum_{n \geqq 0} n \|p_n\| |c_{n+1}|$ converges.

(B) $S_{-j}(K\langle c_n \rangle)$ is an algebra for $j \geqq 1$ and $z^2 Df$ is convergent in A' for all f belonging to A' (i.e., the partial sums of $z^2 Df$ converge in norm to $z^2 Df$).

(2) Theorems 4.3 and 4.6 characterize all the systems of derivations $\{D_0 = T_{g_0}, D_1, D_2, \cdots\}$ on A' where either T_{g_0} is an automorphism (and so g_0 is in $A \backslash A^{(2)}$) or T_{g_0} is an endomorphism of A' such that g_0 belongs to $A^{(2)}$. Now suppose T_{g_0} is an endomorphism (but not an automorphism) of A' where g_0 belongs to $A \backslash A^{(2)}$. In this case it is easy to see that for each sequence of power series $\{g_0, g_1, \cdots\}$ where g_i is in $A^{(2)}$ for $i \geqq 1$ there corresponds a system of derivations $\{D_0, D_1, \cdots\}$ by formula (2.5), but, unlike the two previous cases, T_{g_0} does not seem to have nice enough properties to allow complete determination of the systems of derivations in this case.

References

R. L. Carpenter (1971), 'Continuity of systems of derivations on *F*-algebras', *Proc. Amer. Math. Soc.* **30**, 141–146.

S. Grabiner (1967), *Radical Banach algebras and formal power series*, thesis, (Harvard, 1967).

S. Grabiner (1971), 'A formal power series operational calculus for quasi-nilpotent operations', *Duke Math. J.* **38**, 641–658.

S. Grabiner (1973), 'A formal power series operational calculus for quasi-nilpotent operations II', *J. Math. Anal. Appl.* **43**, 170–192.

S. Grabiner (1974), 'Derivations and automorphisms of Banach algebras of power series', *Mem. Amer. Math. Soc.* **146**.

S. Grabiner (to appear), 'Weighted shifts and Banach algebras of power series', *Amer. J. Math.*

F. Gulick (1970), 'Systems of derivations', *Trans. Amer. Math. Soc.* **149**, 465–488.

N. Jacobson (1964), *Lectures in abstract algebra* Vol. III, *Theory of fields and Galois theory*, (Van Nostrand, Princeton, N. J. 1964).

B. E. Johnson (1967), 'Continuity of linear operators commuting with continuous linear operators', *Trans. Amer. Math. Soc.* **128**, 88–102.

B. E. Johnson (1969), 'Continuity of derviations on commutative algebras', *Amer. J. Math.* **91**, 1–10.

R. J. Loy (1969), 'Continuity of derivations on topological algebras of power series', *Bull, Austral. Math. Soc.* **1**, 419–442.

R. J. Loy (1970), 'Uniqueness of the complete norm topology and continuity of derivations of Banach algebras', *Tohoku Math. J.* **22**, 371–378.

R. J. Loy (1971), 'Uniqueness of the Frechet space topology on certain topological algebras', *Bull. Austral. Math. S:c.* **4**, 1–7.

R. J. Loy (1973), 'Continuity of higher derivations', *Proc. Amer. Math. Soc.* **37**, 505–510.

R. J. Loy (preprint), 'Banach algebras of power series'.

J. B. Miller (1970), 'Higher derivations on Banach algebras', *Amer. J. Math.* **92**, 301–333.

J. B. Miller (to appear), 'Analytic structure and higher derivations on commutative Banach algebras', *Aequationes Mathematicae*.

S. Scheinberg (1970), 'Power series in one variable', *J. Math. Anal. Appl.* **31**, 321–333.

A. Wilansky (1964), *Functional Analysis*, (Blaisdell, New York 1964).

Claremont Graduate School
Claremont, California 91711
U.S.A.